Orthogonal Drawings of Graphs and Their Relatives

Part 1 - Topology-shape-metrics

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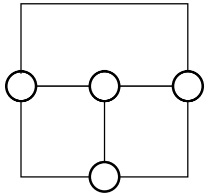
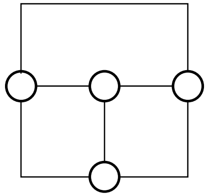
Summary

- Part 1.1 – The topology-shape-metrics approach
- Part 1.2 – Engineering the topology-shape-metrics approach
- Part 1.3 – Ortho-polygon drawings



Part 1.1
The Topology-Shape-Metrics Approach



Topology-shape-metrics

- Approach to compute an orthogonal drawing of a graph G = (V, E)
  - *C. Batini, E. Nardelli, R. Tamassia*: A Layout Algorithm for Data Flow Diagrams. IEEE Trans. Software Eng. 12(4): 538-546 (1986)
  - *R. Tamassia*: On Embedding a Graph in the Grid with the Minimum Number of Bends. SIAM J. Comput. 16(3): 421-444 (1987)
  - *R. Tamassia, G. Di Battista, C. Batini*: Automatic graph drawing and readability of diagrams. IEEE Trans. Systems, Man, and Cybernetics 18(1): 61-79 (1988)

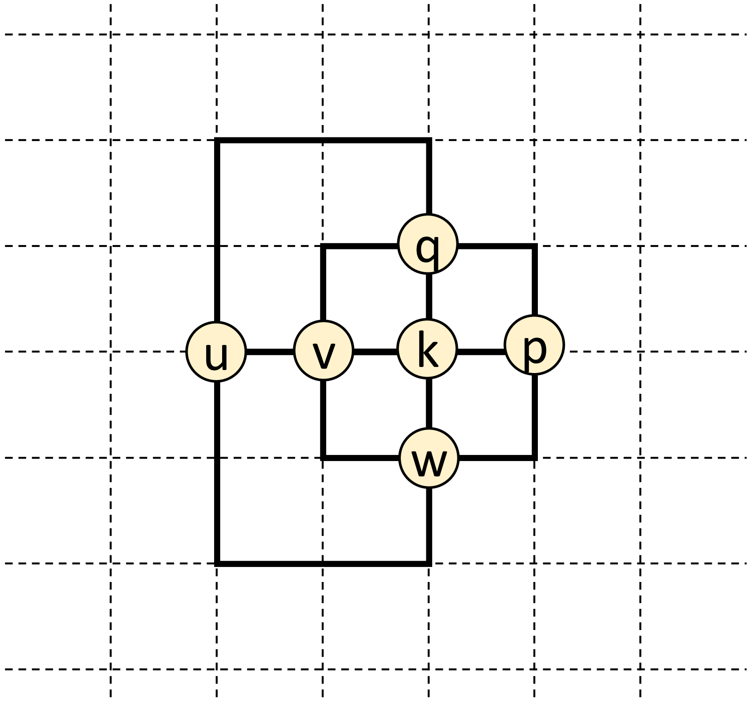
Topology-shape-metrics

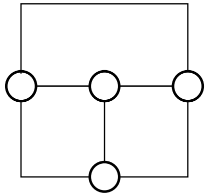
Input: 4-graph G=(V,E)

V = {u, v, w, k, p, q}
E = {(u, q), (u, v), (u, w), (v, q), (v, k), (v, w),
(q, p), (q, k), (k, p), (k, w), (w, p)}

TSM ⇒

Output: orthogonal drawing Γ of G





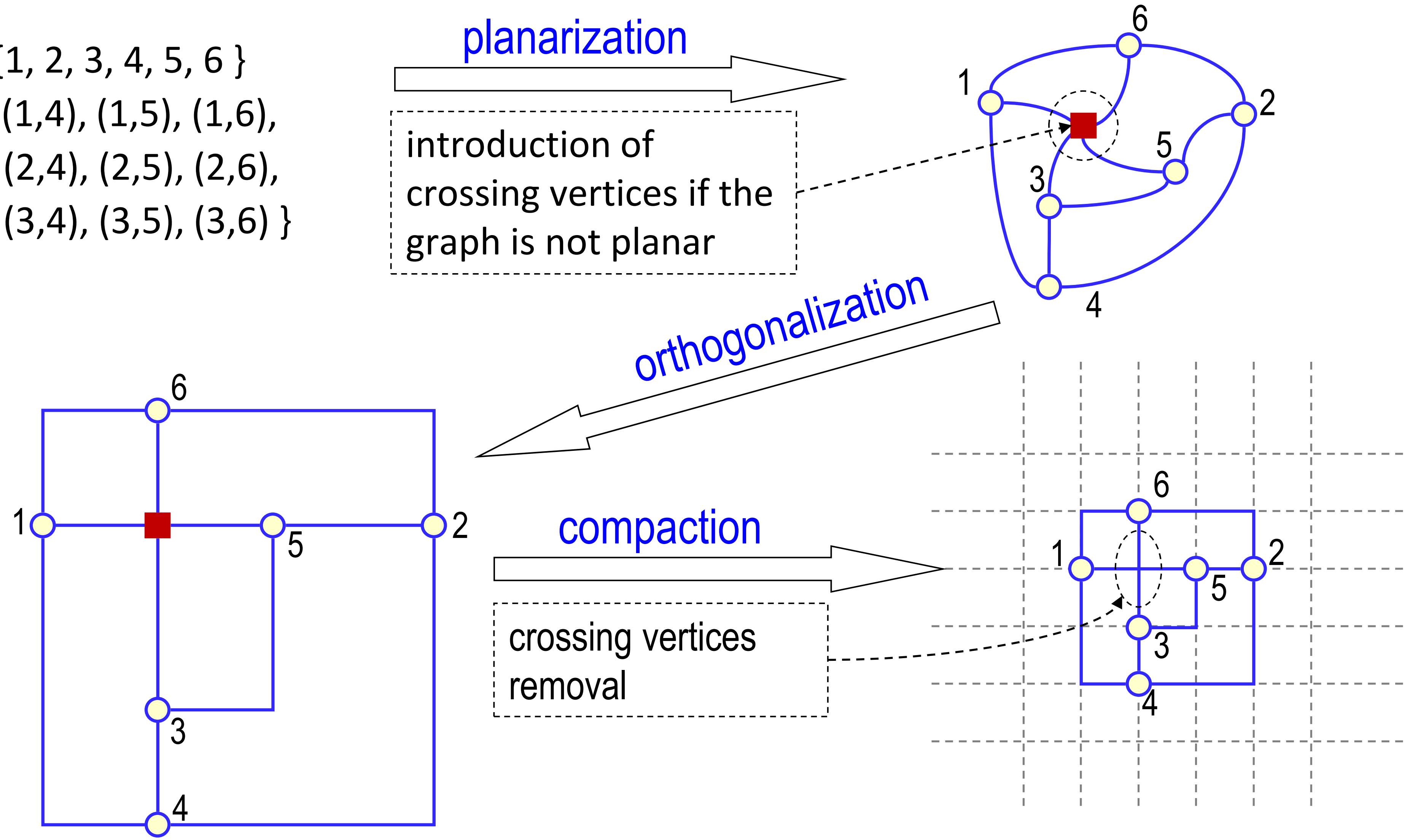
Topology-shape-metrics

- Topology (embedding): set of (internal and external) faces, with possible crossing vertices
- Shape (orthogonal representation): vertex angles and edge bends
- Metrics (orthogonal drawing): vertex and bend coordinates

- These abstraction levels make it possible to design a drawing strategy in three phases:
  - planarization ⇒ compute a topology (embedding)
  - orthogonalization ⇒ compute a shape (orthogonal representation)
  - compaction ⇒ compute a metrics (final drawing)

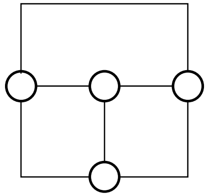
Topology-shape-metrics: Illustration

V = {1, 2, 3, 4, 5, 6 }
E = {(1,4), (1,5), (1,6),
(2,4), (2,5), (2,6),
(3,4), (3,5), (3,6) }



Planarization

- **Objective**: Compute an embedding of G with few crossings
  - G *planar* ⇒ the planarization algorithm computes a planar embedding
    - *J. Hopcroft and R. E. Tarjan*: Efficient planarity testing, Journal of the Association for Computing Machinery, 21 (4): 549–568 (1974)
    - *K. S. Booth, G. Luecker*: Testing for the Consecutive Ones Property, Interval Graphs, and Graph Planarity Using PQ-Tree Algorithms. J. Comput. Syst. Sci. 13(3): 335-379 (1976)
    - *J. M. Boyer, W.J. Myrvold*: On the cutting edge. Simplified O(n) planarity by edge addition, J. of Graph Alg. and Appl. 8 (3): 241–273 (2004)
  - G *non-planar* ⇒ the planarization algorithm computes an embedding with "small" number of crossings, i.e., an embedded planar graph **G'** obtained by replacing crossings with dummy vertices (**crossing vertices**)



Planarization: Crossing minimization

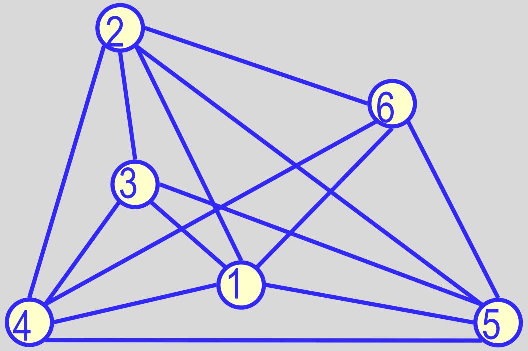
- Minimizing the number of edge crossings is NP-complete
  - *M. Garey, D. S. Johnson*. Crossing number is NP-complete. SIAM Journal on Algebraic and Discrete Methods. 4 (3): 312–316 (1983)
- Determining the maximum planar subgraph is also NP-complete
- A simple planarization heuristic can work in two steps:
  - Step 1: compute a *maximal* planar embedded subgraph
  - Step 2: insert the remaining edges one by one trying to minimize the number of crossings

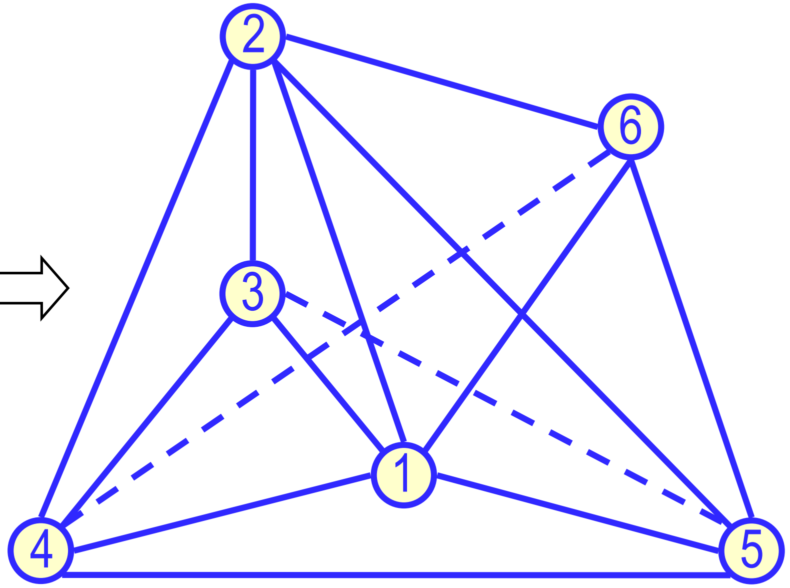
Planarization heuristic: Step 1

Input graph G = (V, E)

V = {1, 2, 3, 4, 5, 6}

E = {(1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5) (4, 5), (4, 6), (5, 6)}





Maximal planar subgraph G' = (V', E') of G

V' = {1, 2, 3, 4, 5, 6}

E' = {(1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 4), (4, 5), (5, 6)}

- (1, 2) ⇒ planar
- (1, 3) ⇒ planar
- (1, 4) ⇒ planar
- (1, 5) ⇒ planar
- (1, 6) ⇒ planar
- (2, 3) ⇒ planar
- (2, 4) ⇒ planar
- (2, 5) ⇒ planar
- (2, 6) ⇒ planar
- (3, 4) ⇒ planar
- (4, 5) ⇒ planar
- (3, 5) ⇒ non-planar
- (4, 6) ⇒ non-planar
- (5, 6) ⇒ planar

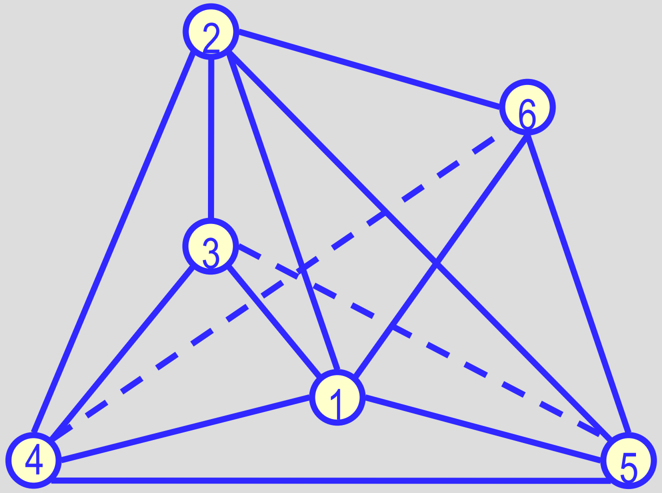
Planarization heuristic: Step 1

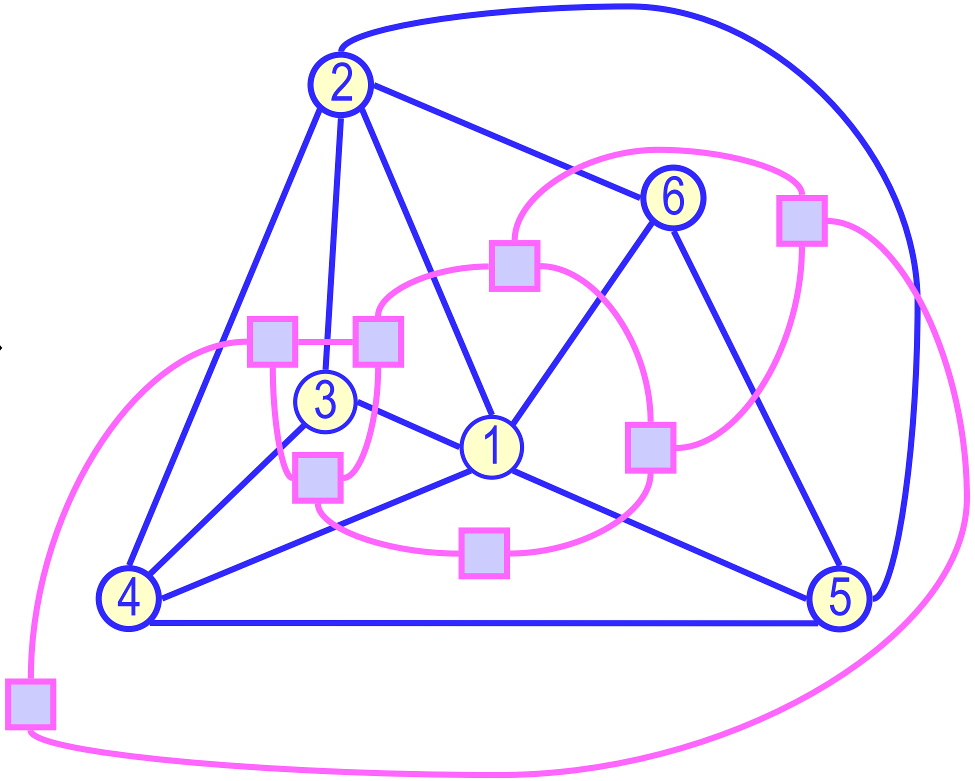
maximal planar subgraph G' = (V', E') of G

V' = {1, 2, 3, 4, 5, 6}

E' = {(1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 4), (4, 5), (5, 6)}

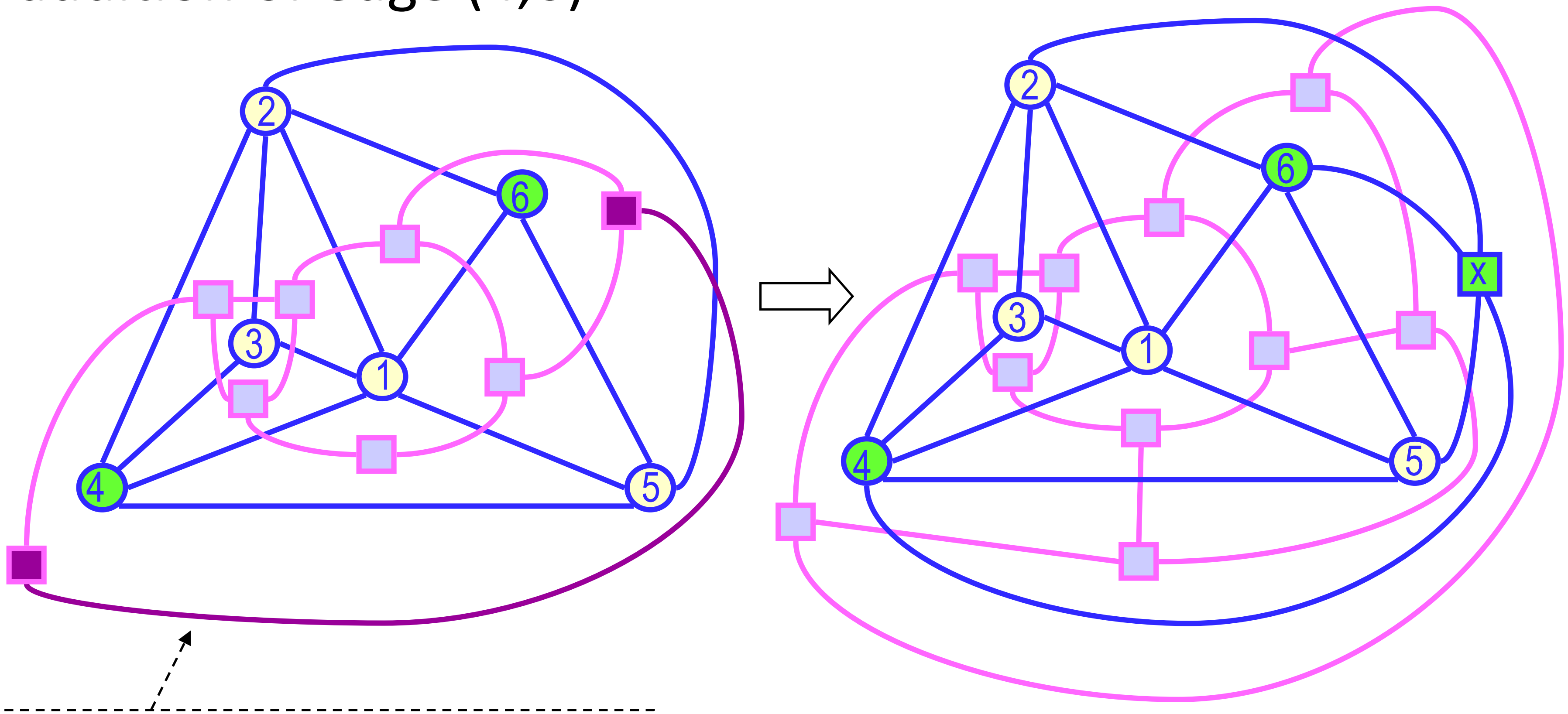
non-planar edges of G: (3, 5) e (4, 6)





Planarization heuristic: Step 2

addition of edge (4,6)

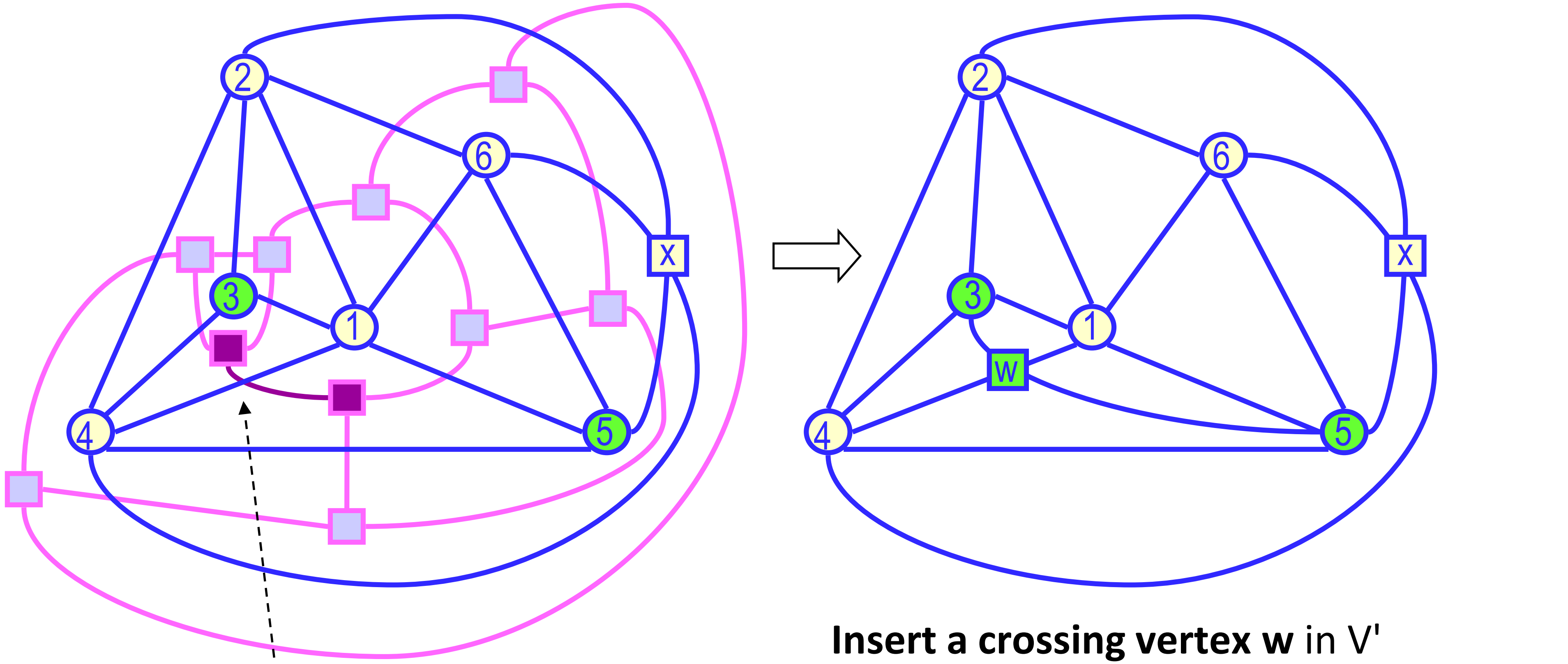


Shortest path on the dual graph of G' between two faces incident to vertices **4 e 6**, respectively.

- **Insert a crossing vertex x** in V' and update the dual graph of G’

Planarization heuristic: Step 2

addition of edge (3,5)



Shortest path on the dual graph of G' between two faces incident to vertices **3 and 5**, respectively

Insert a crossing vertex w in V'

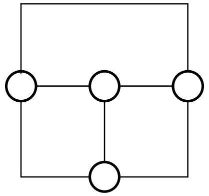
Planarization: Further references

- *M. Jünger, P. Mutzel*: Maximum Planar Subgraphs and Nice Embeddings: Practical Layout Tools. Algorithmica 16(1): 33-59 (1996)
- *C. Gutwenger, P. Mutzel, R. Weiskircher*: Inserting an Edge into a Planar Graph. Algorithmica 41(4): 289-308 (2005)
- *M. Chimani, C. Gutwenger*: Advances in the Planarization Method: Effective Multiple Edge Insertions. J. Graph Algorithms Appl. 16(3): 729-757 (2012)
- *C. Buchheim, M. Chimani, C. Gutwenger, M. Jünger, P. Mutzel*: Crossings and Planarization. In Handbook of Graph Drawing and Visualization, Roberto Tamassia (Ed.). Chapman and Hall/CRC, 43–85 (2013).

Planarization: Open problem

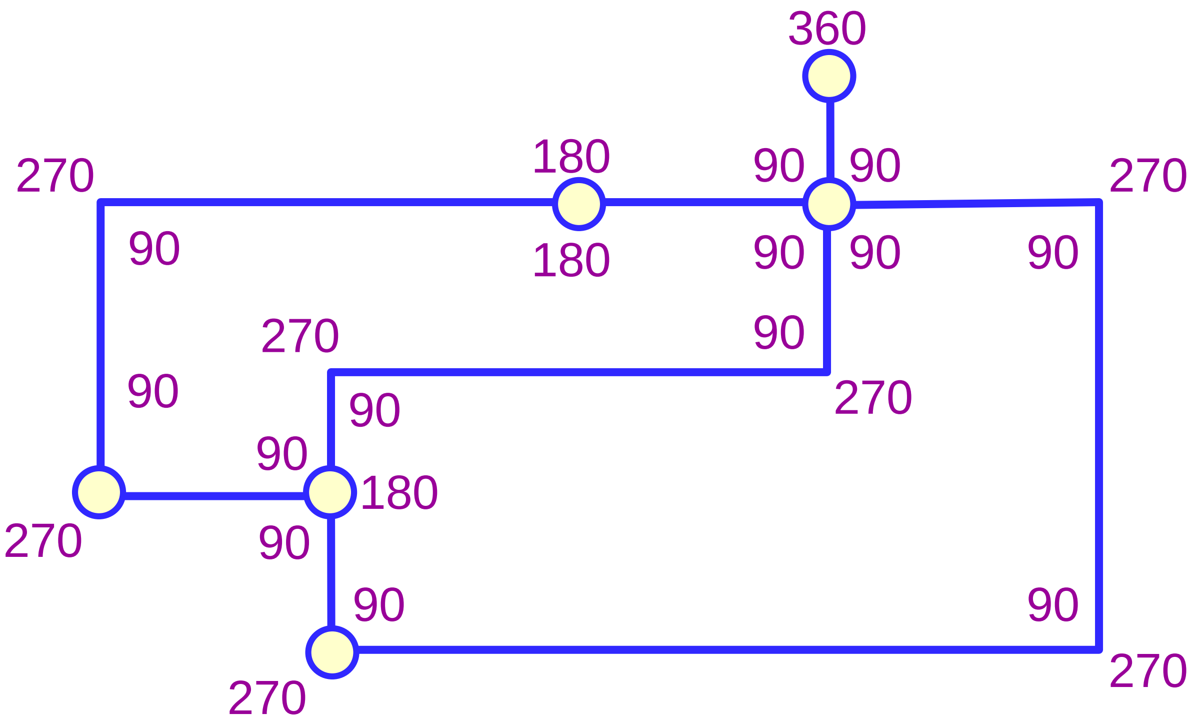
- **Problem 1** Design planarization heuristics that compute embeddings with "few" crossings per edge

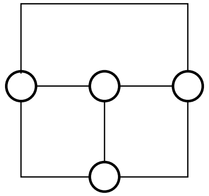
- **Remark**: Deciding whether a graph is k-planar (i.e., it has a drawing with at most k crossings per edge) is NP-hard
  - *A. Grigoriev and H. L. Bodlaender*: Algorithms for graphs embeddable with few crossings per edge. Algorithmica 49, 1 (2007)
  - *V. P. Korzhik and B. Mohar*: Minimal obstructions for 1-immersions and hardness of 1-planarity testing. J. Graph Theory 72, 1 (2013)



Orthogonalization: Shape

- **Objective**: Compute a shape of G with few bends
  - shape (orthogonal representation): described by the *angles at each vertex* and by the ordered *sequence of bends along each edge*

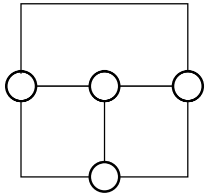




Orthogonalization: Bend minimization

- **Theorem** [Tamassia 1987] Given an embedded planar 4-graph G=(V,E), there exists a polynomial-time algorithm that computes an embedding preserving *orthogonal representation* of *G* with *minimum number of bends*

- **Proof idea**
  - orthogonal representations of G ⇔ integer feasible flows in a suitable network N(G)
  - cost of the flow = number of bends of the orthogonal representation
  - computation of a bend-minimum orthogonal representation of G ⇔ computation of a min-cost flow in N(G)



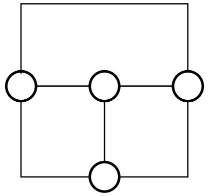
\begin{flow network}

Flow network: Basic definitions

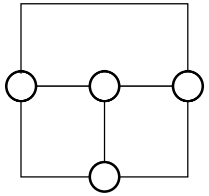
- flow network: directed graph $N = (U, A)$
  - every node $v \in U$ is associated with an amount of flow $b(v)$
    - $b(v) > 0 \Rightarrow$ v is a producer (it produces $|b(v)|$ units of flow)
    - $b(v) < 0 \Rightarrow$ v is a consumer (it consumes $|b(v)|$ units of flow)
    - $b(v) = 0 \Rightarrow$ v is a neutral node
  - it must be $\Sigma_{v \in U}\ b(v) = 0$
  - every arc $e \in A$ is associated with three non-negative integers:
    - $l(e)$ = lower capacity of e
    - $u(e)$ = upper capacity of e
    - $c(e)$ = cost of e

Flow network: Basic definitions

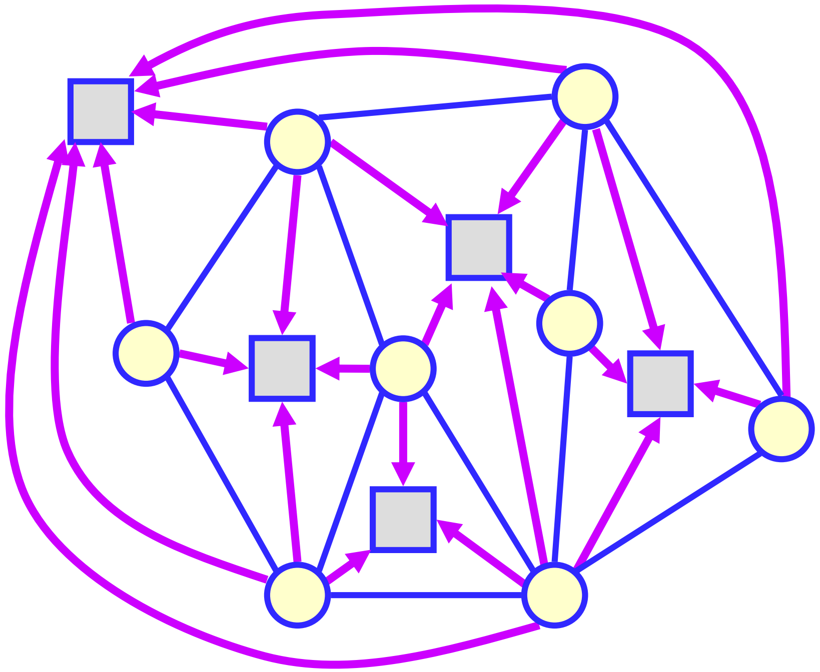
- feasible flow in N: a function $x: A \to \mathbf{N}$ such that:
  - $\forall e \in A \quad l(e) \leq x(e) \leq u(e)$
  - $\forall v \in U \quad \sum_{e \in out(v)} x(e) - \sum_{e \in in(v)} x(e) = b(v)$
- cost of x: $C(x) = \sum_{e \in A} c(e)\, x(e)$
- min-cost flow in N: feasible flow of minimum cost



\end{flow network}



Orthogonalization: Flow network – part I

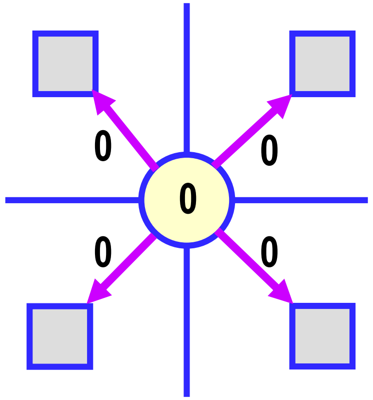


- nodes of N(G) ⇔ vertices and faces of G
- arc (v, f) in N(G) ⇔ angle at v in face f

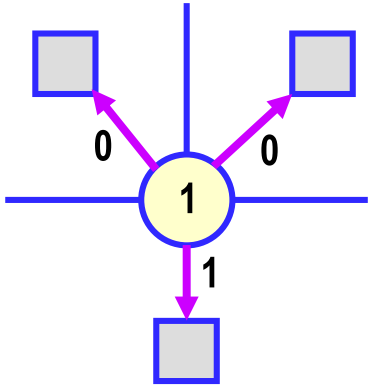
- flows on these arcs represent the values of the corresponding angles
- the flow originates from vertices (producers) and move towards faces (consumers)

Orthogonalization: Flow network – part I

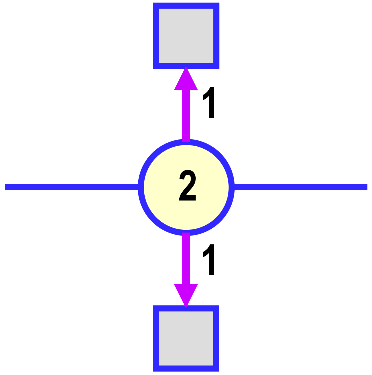
- flow and angles
  - k units of flow ⇔ (k+1)90° angle
  - a vertex v produces 4-deg(v) units of flow



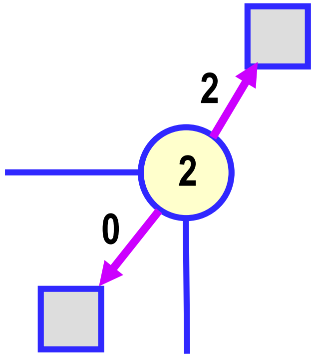
vertex of deg. 4 produces flow 0



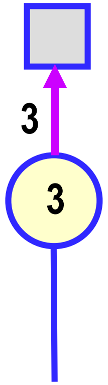
vertex of deg. 3 produces flow 1



vertex of deg. 2 produces flow 2



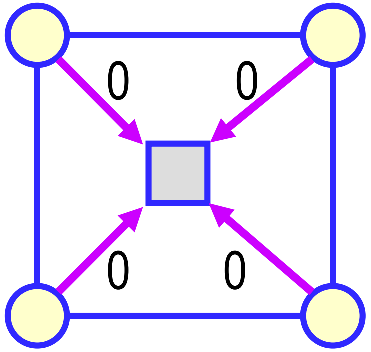
vertex of deg. 2 produces flow 2



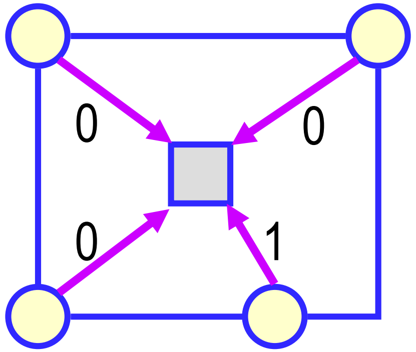
vertex of deg. 1 produces flow 3

Orthogonalization: Flow network – part I

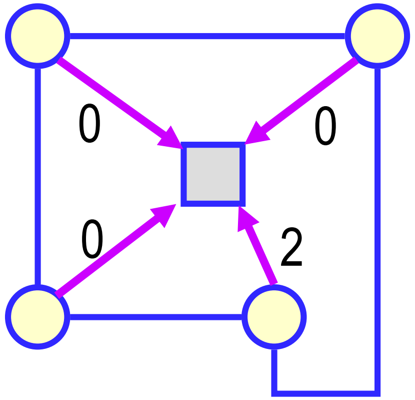
- flow, angles, and face capacities
  - cap(f) = capacity of a face f ⇔ how many units of flow it can consume without generating bends on its boundary



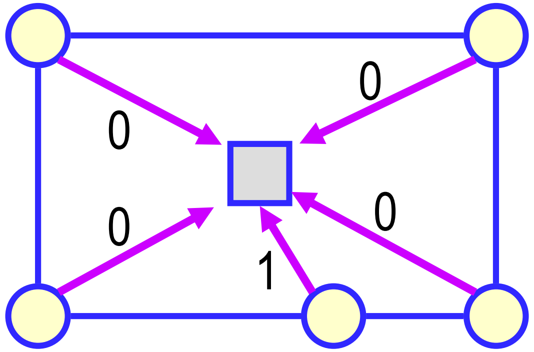
face of deg 4
with 0 bends



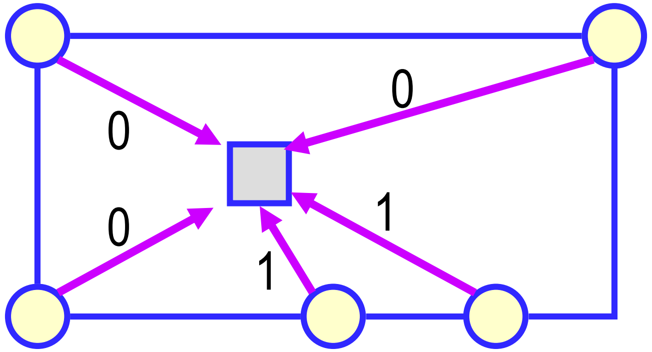
face of deg 4
with 1 bend



face of deg 4
with 2 bends



face of deg 5
with 0 bends

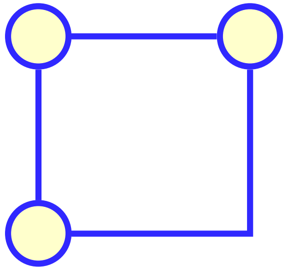


face of deg 5
with 1 bend

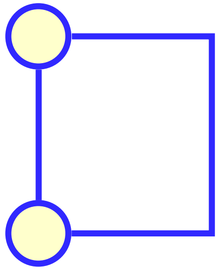
Orthogonalization: Flow network – part I

- General rule for an *internal* face f
  - cap(f) = deg(f) - 4

- Implications:
  - if f receives k > cap(f) units of flow ⇒ f generates k - cap(f) bends on its boundary, each forming a 90° angle inside f
  - deg(f) < 4 ⇒ cap(f) is negative ⇒ f produces (4 – deg(f)) units of flow



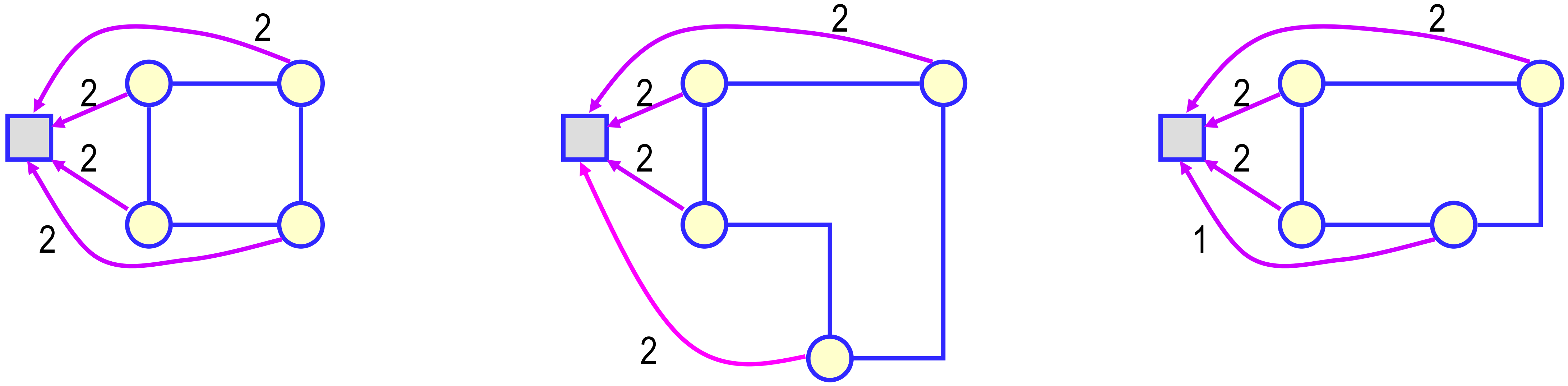
deg(f) = 3



deg(f) = 2

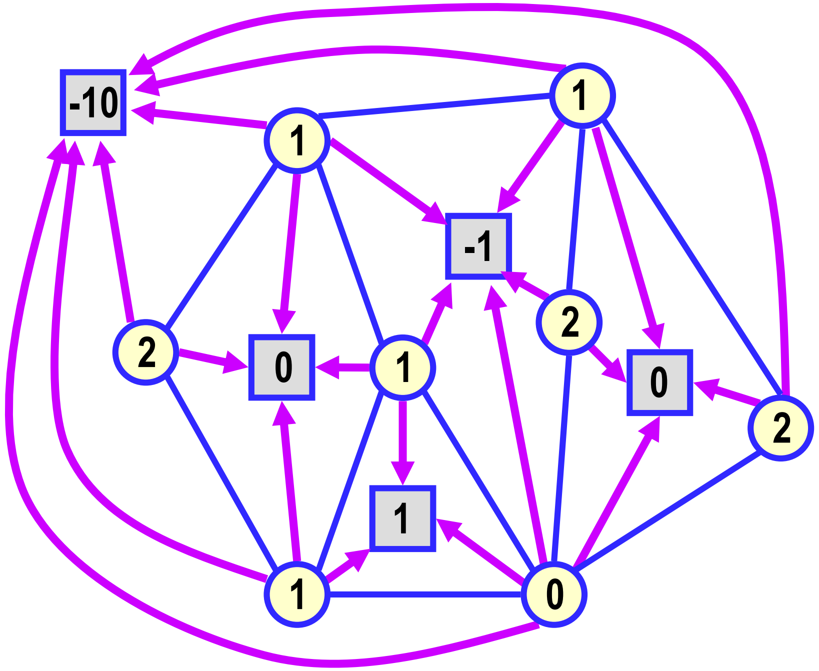
Orthogonalization: Flow network – part I

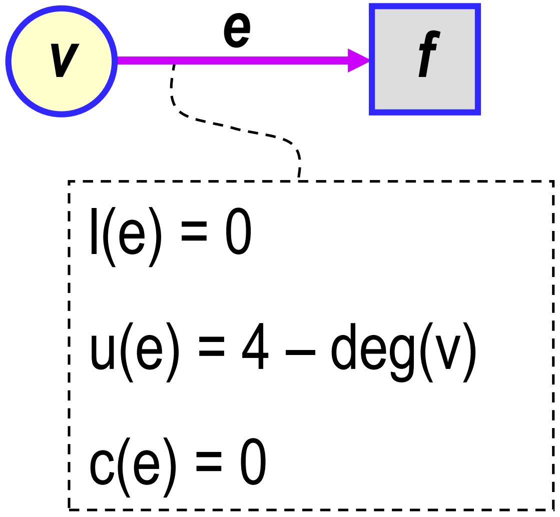
- for the external face h: cap(h) = deg(h) + 4



Orthogonalization: Flow network – part I

- flow, angles, and face capacities – **summarizing**
  - a *vertex* v produces 4 – deg(v) units of flow
  - an *internal face* f *of degree* > 3 consumes deg(f) – 4 units of flow
  - an *internal face* f *of degree* $\leq$ 3 produces 4 – deg(f) units of flow
  - the *external face h* consumes deg(h) + 4 units of flow



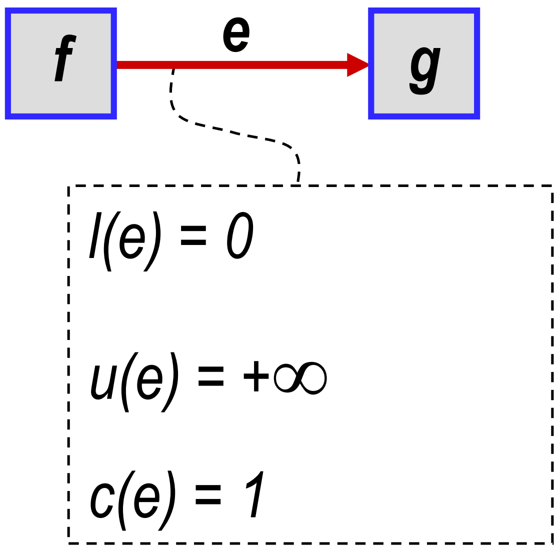


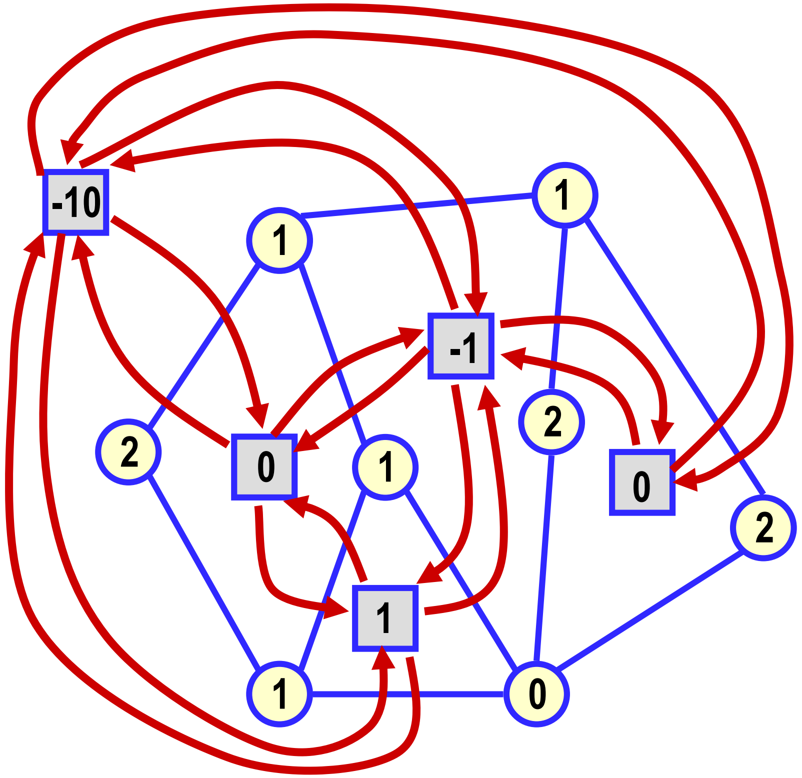
Orthogonalization: Flow network – part II

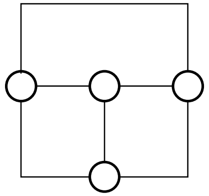
- How to model bends in the flow network? If a face f receives more than cap(f) units of flow, it must forward the excess to an adjacent face:
  - insert face-to-face arcs in N(G) to allow flow exchange between adjacent faces
  - k units of flow on an arc (f, g) correspond to k bends along an edge shared by f and g; each bend forms an angle of 90° inside f and of 270° inside g
  - face-to-face arcs have cost 1, so that the number of bends equals the total flow cost

Orthogonalization: Flow network – part II

- face-to-face arcs

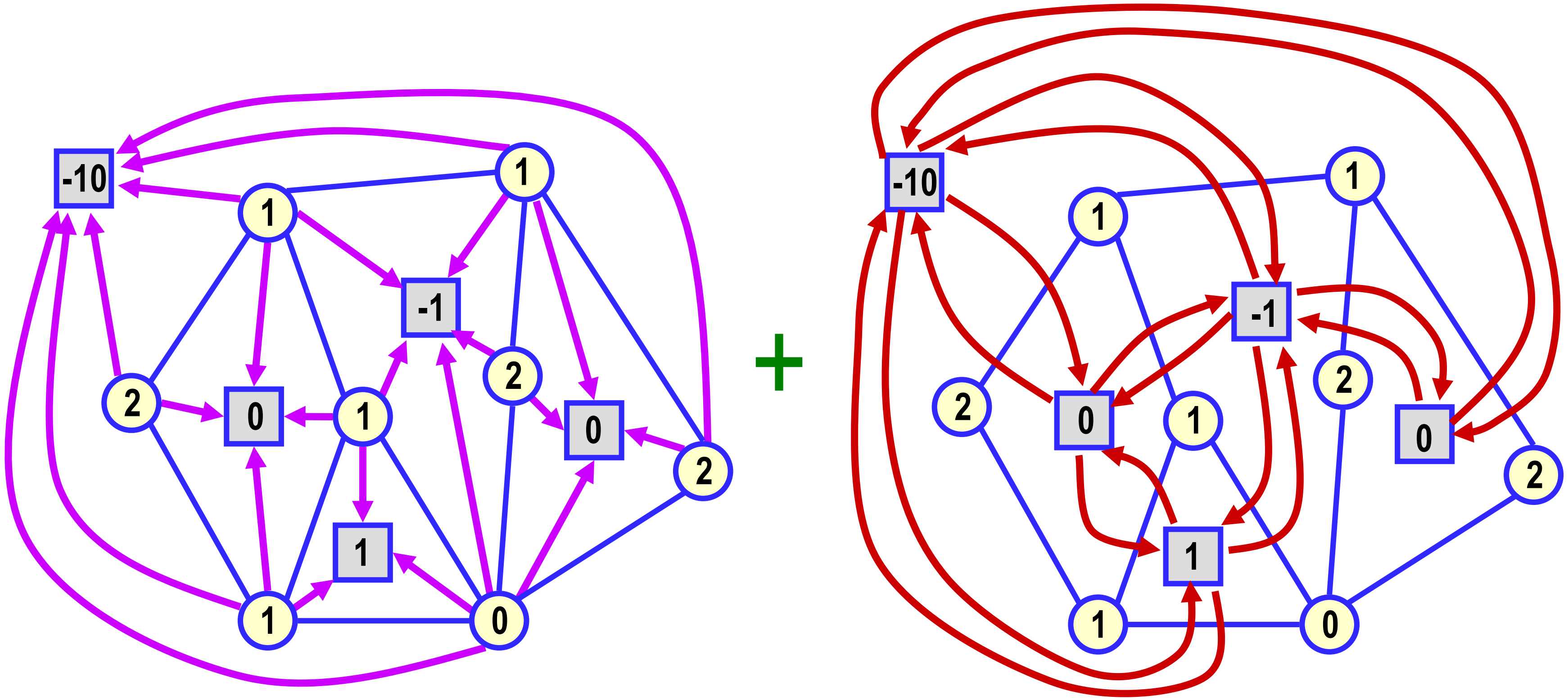






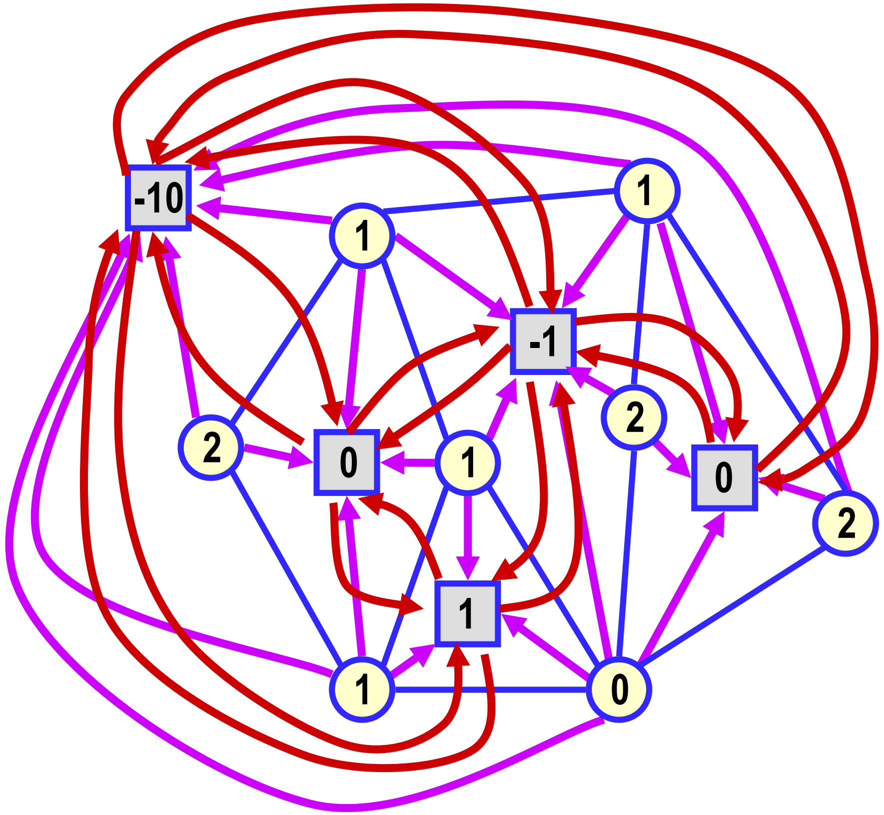
Orthogonalization: Flow network

- Flow network: **putting all together**



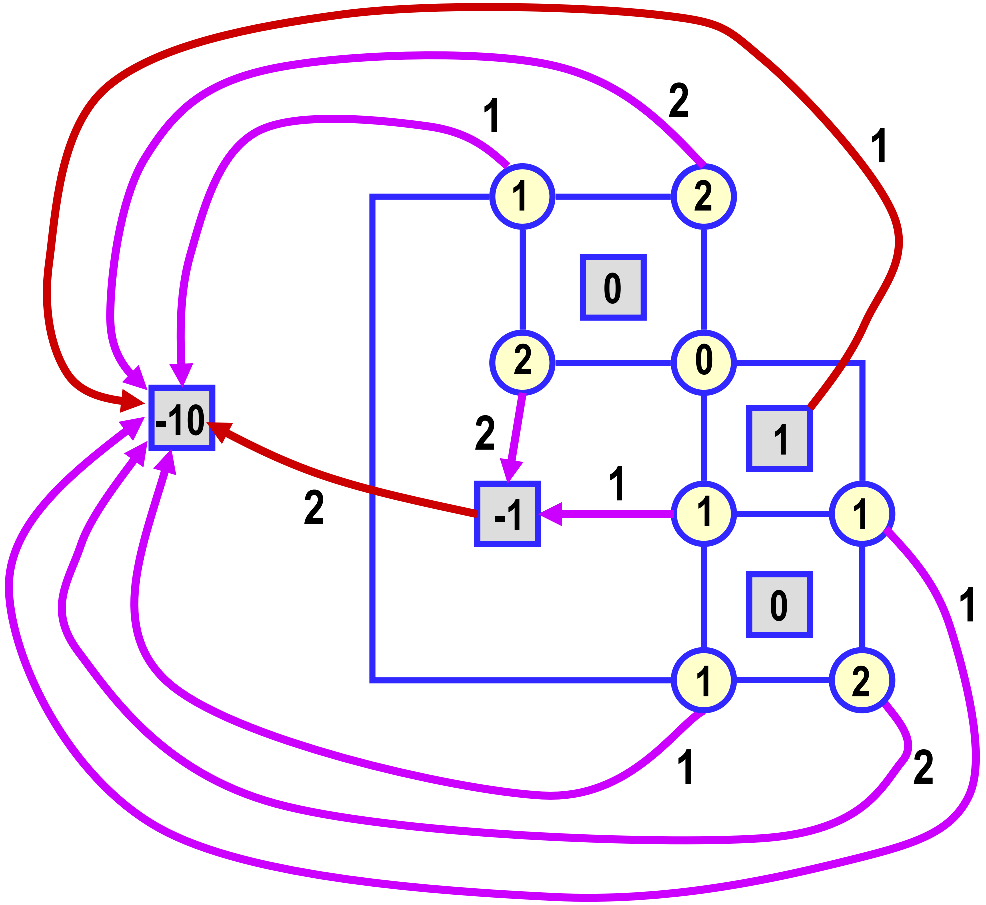
Orthogonalization: Flow network

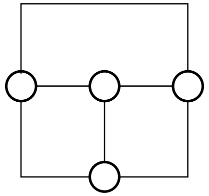
- Final flow network N(G)



Orthogonalization: Flow and shape

- Example of flow and its corresponding shape
  - only arcs with non-zero flow are shown





Orthogonalization: Flow and shape

- Why an integer feasible flow always exists in N(G)

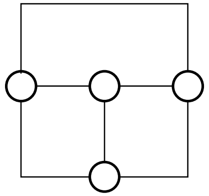
1) produced flow – consumed flow = 0

$\sum_{v \in V} (4 - \deg(v)) + \sum_{f \text{ int}:\deg(f) \leq 3} (4 - \deg(f)) - \sum_{f \text{ int}:\deg(f) > 3} (\deg(f) - 4) - (\deg(h) + 4)\ ) =$

$4|V| - 2|E| - \sum_{f \in F} (\deg(f) - 4) - 8 =$

$4\,(|V| - |E| + |F| - 2) = 0$ (by Euler's formula)

2) face-to-face arcs allow unbounded flow exchange



Orthogonalization: Computational cost

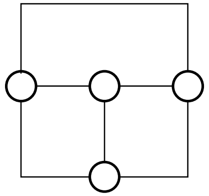
- Computing a min-cost flow of O(n) given value in N(G)
  - $O(n^2 \log n)$ [Tamassia 1987]
  - $O(n^{7/4} \log n)$ [Garg and Tamassia 1996]
  - $O(n^{3/2})$ [Cornelsen and Karrenbauer 2011]

- **Open Problem.** Is there an $o(n^{3/2})$-time algorithm for the bend-minimization problem of *plane* 4-graphs?

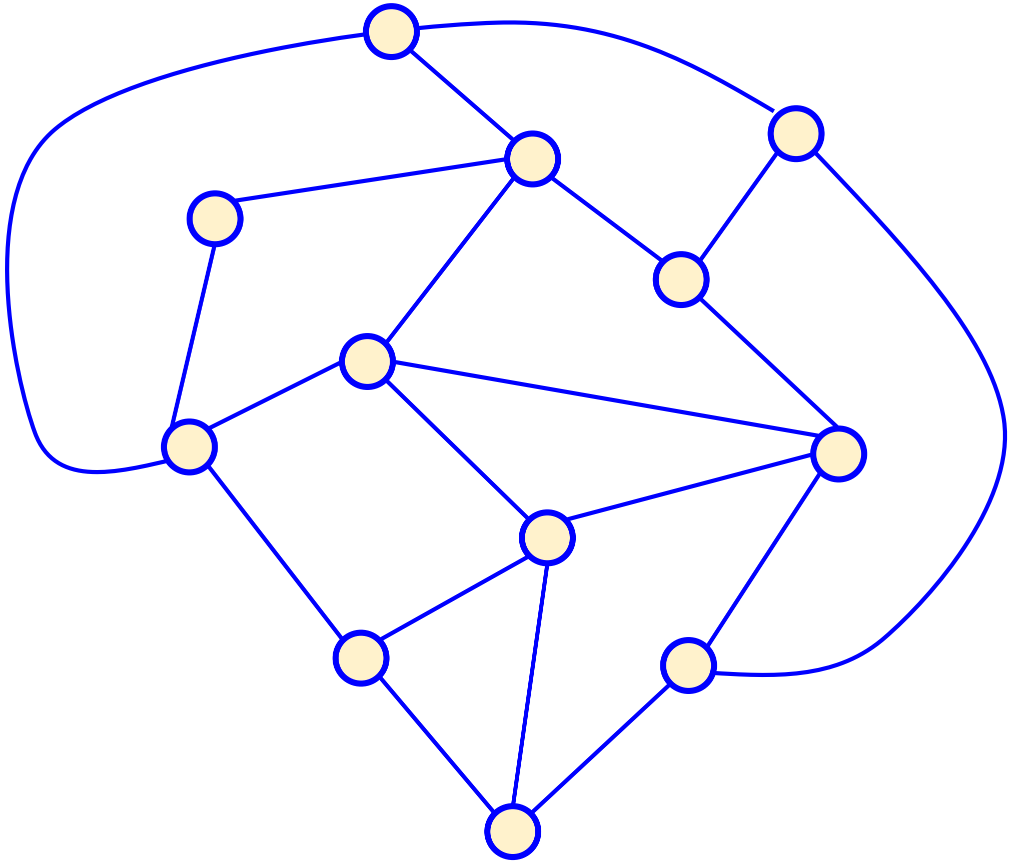
Orthogonalization: Exercise

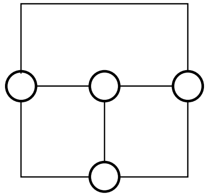
Exercise (partial answer). Prove the following

Theorem (unpublished). Let G be an embedded planar 4-graph with n vertices and all internal faces of degree less than 5. There exists an O(n)-time algorithm that computes an embedding-preserving bend-minimum orthogonal representation of G

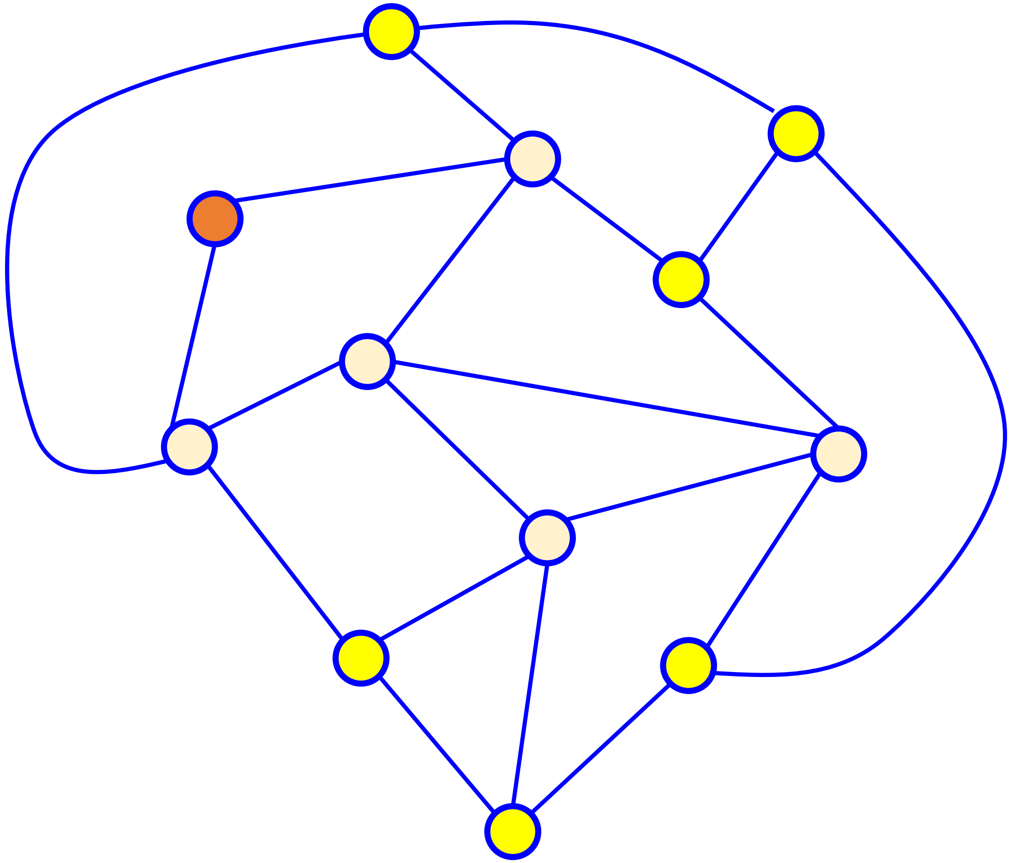


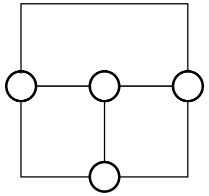
Orthogonalization: Solution



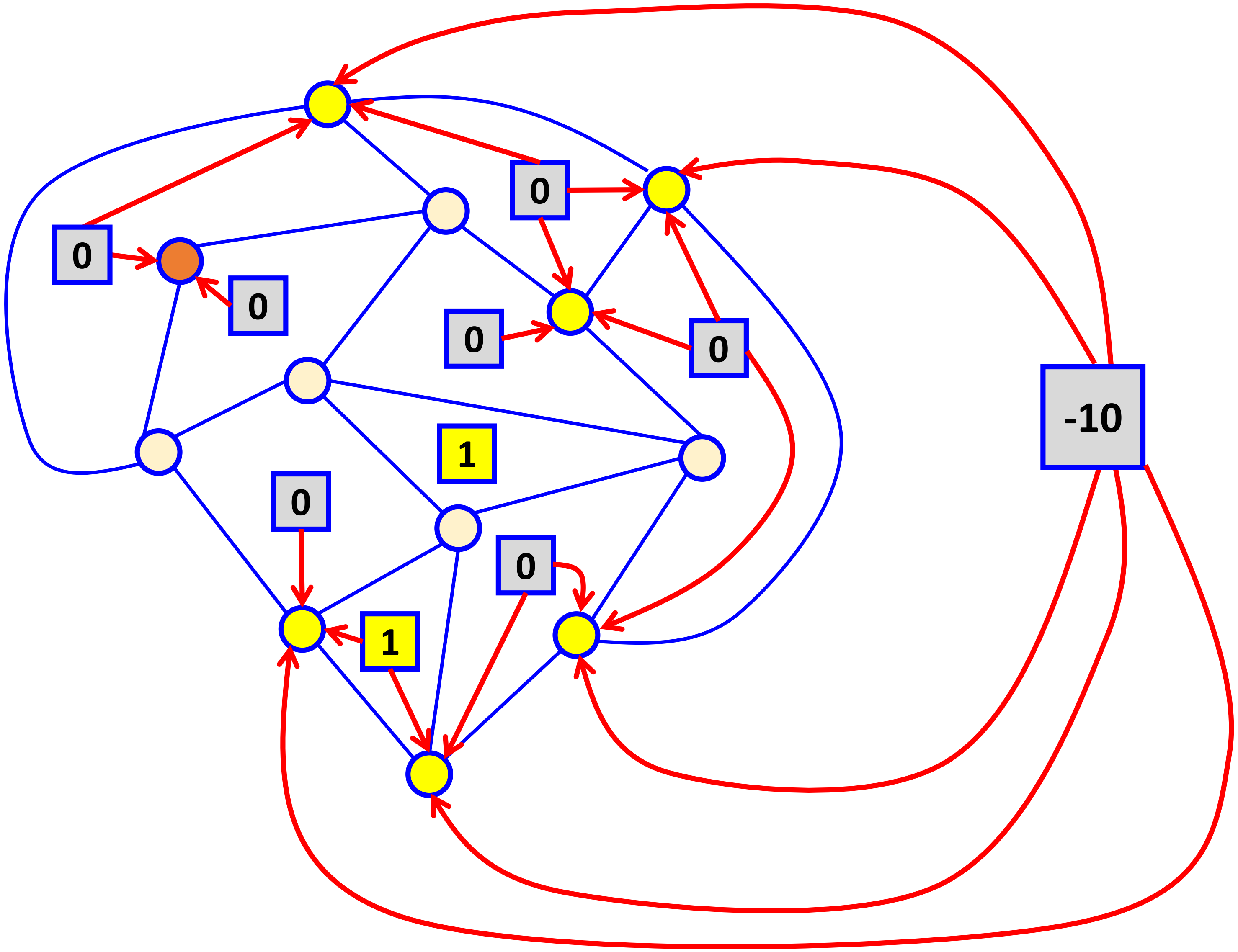


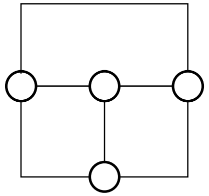
Orthogonalization: Solution



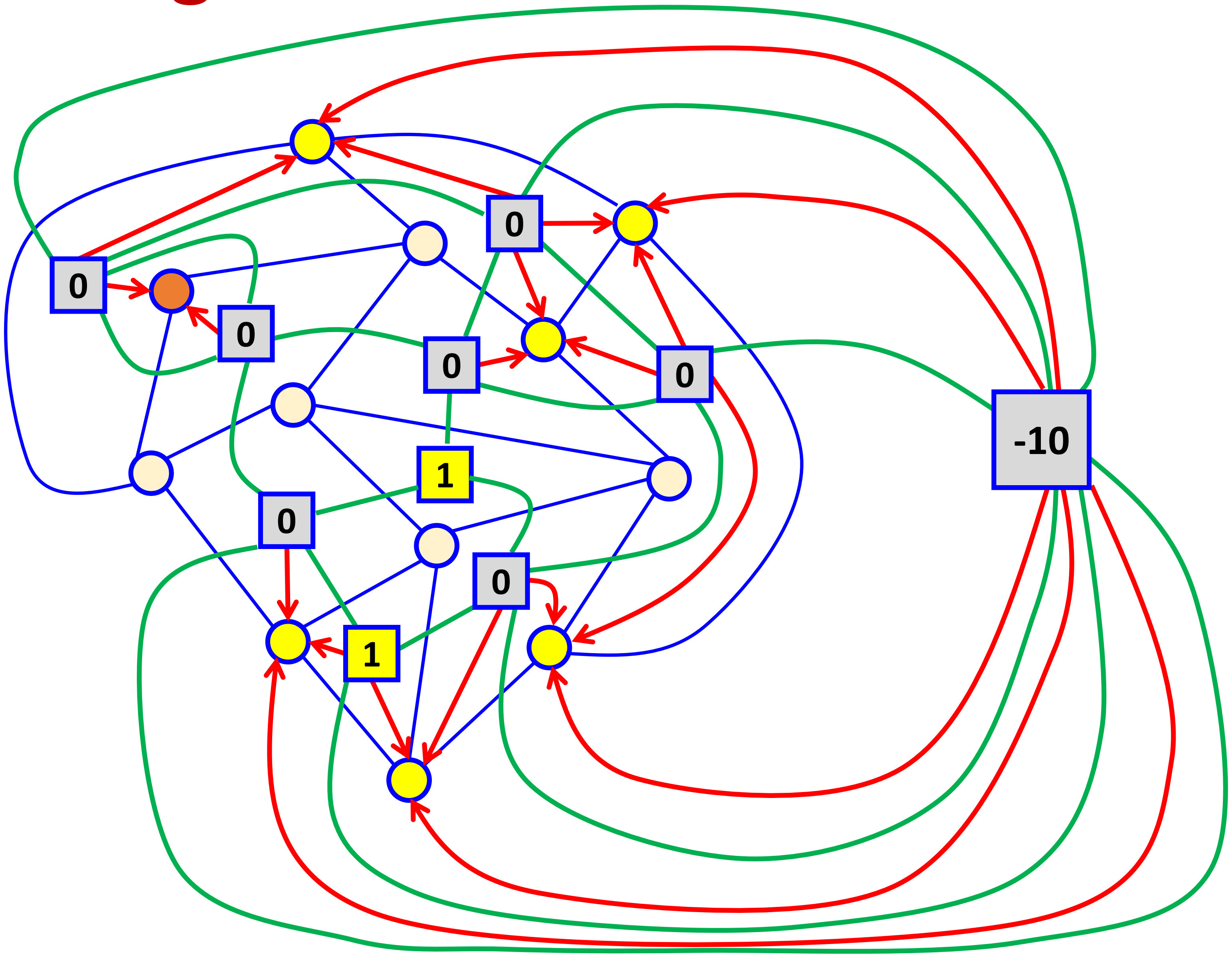


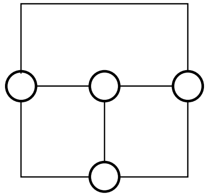
Orthogonalization: Solution



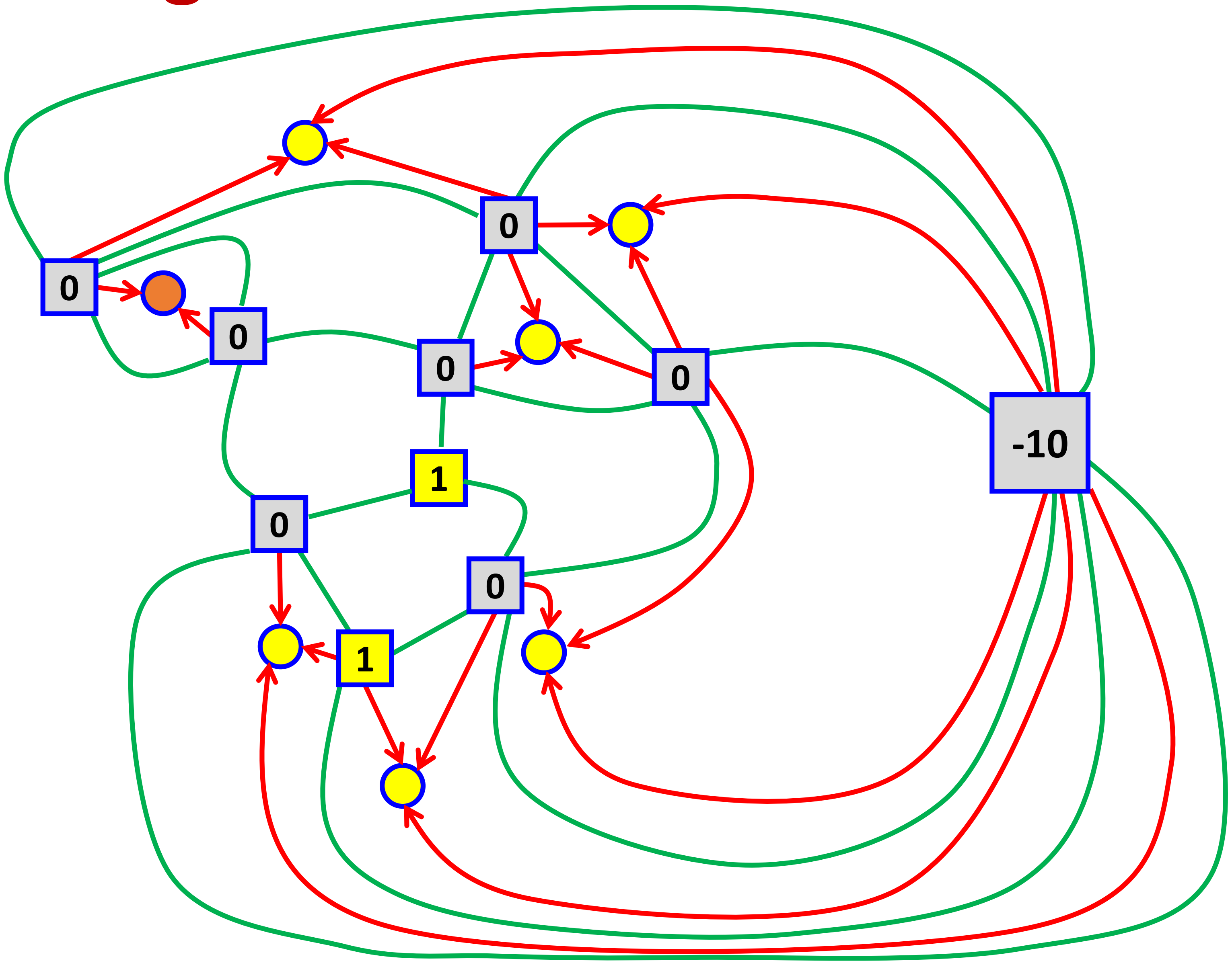


Orthogonalization: Solution

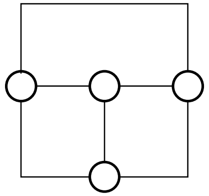




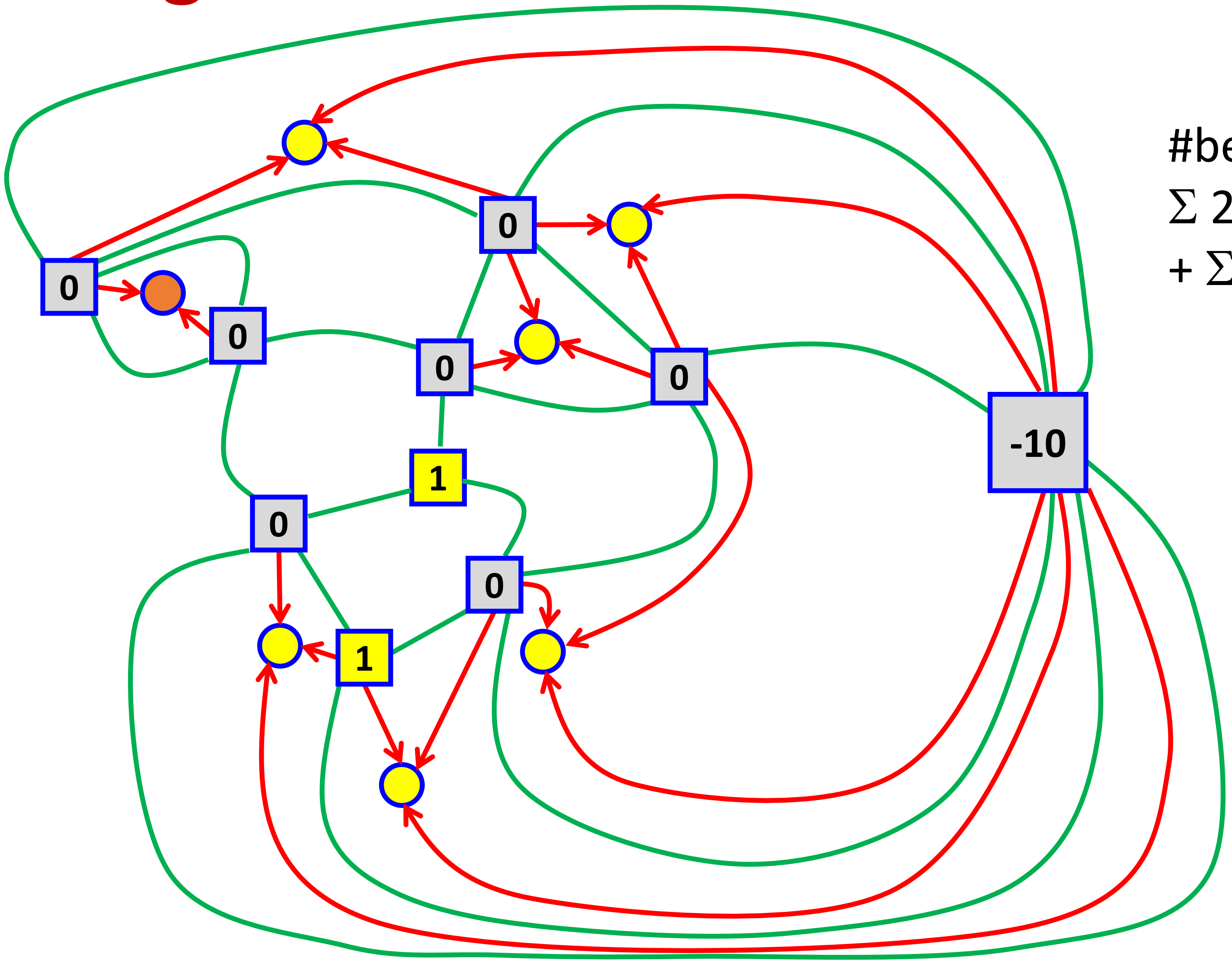
Orthogonalization: Solution



run a BFS visit from the external face



Orthogonalization: Solution

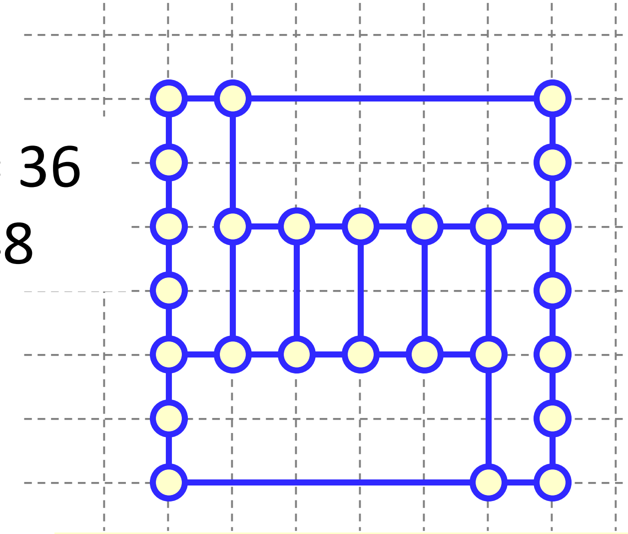


$\#\text{bends} = \sum (sp(\circ) - 1) + \sum 2(sp(\bullet) - 1) + \sum sp(\square) = 1 + 2 + 3 = 6$

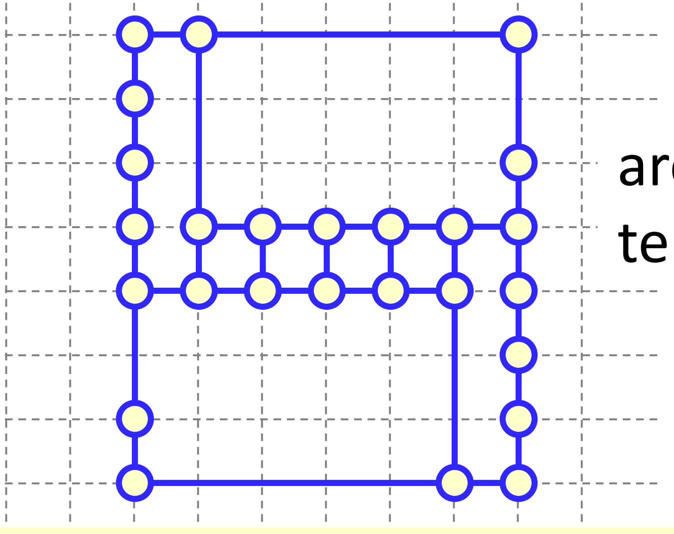
Compaction

- **Objective**: Assign vertex and bend coordinates such that the final drawing has either small area or small total edge length
  - for some orthogonal representations it is impossible to minimize both these parameters together

area = 36
tel = 48

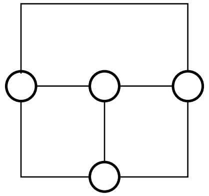


minimum area



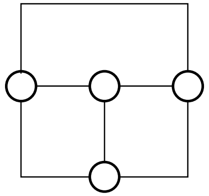
area = 42
tel = 47

minimum total edge length



Compaction: Complexity

- Minimizing the area (or the total edge length) of an orthogonal representation is NP-hard
  - M. Patrignani: On the complexity of orthogonal compaction. Comput. Geom. 19(1): 47-67 (2001)
- The problem is polynomial-time solvable if all faces are rectangles
  - this result is generalized to a larger class of orthogonal representations called *turn-regular* (see later)
    - S. S. Bridgeman, G. Di Battista, W. Didimo, G. Liotta, R. Tamassia, L. Vismara: Turn-regularity and optimal area drawings of orthogonal representations. Comput. Geom. 16(1): 53-93 (2000)

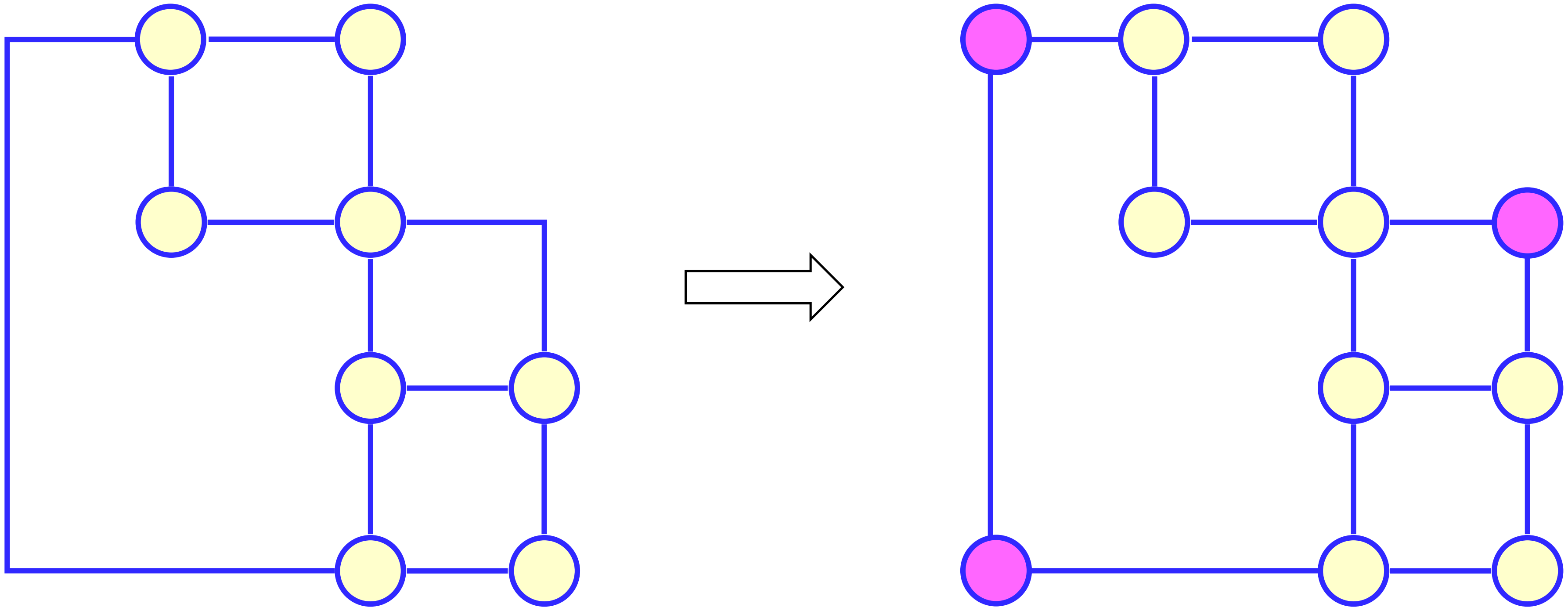


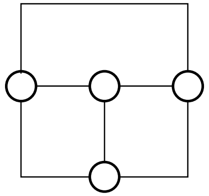
Compaction: General strategy

1. Transform the shape into a rectangular shape
   a) replace every bend with a dummy vertex
   b) add dummy edges and vertices until all faces are rectangles
2. Compute vertex coordinates
3. Remove all dummy edges and vertices

Compaction: Step 1

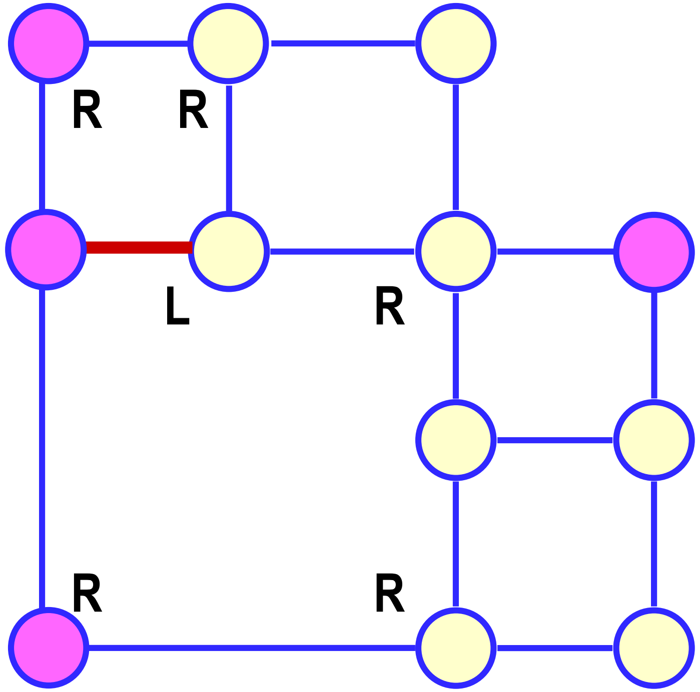
a) replace every bend with a dummy vertex





Compaction: Step 1

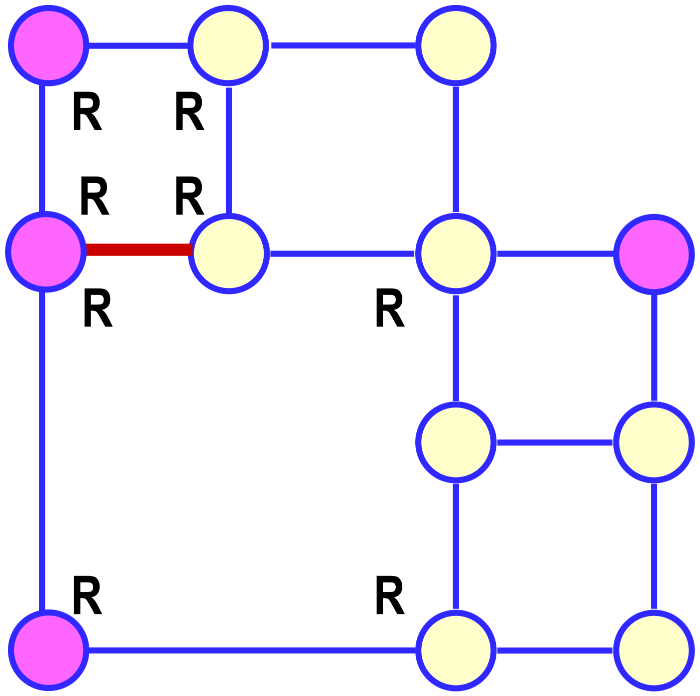
b) add dummy edges and vertices until all faces are rectangles



split recursively each *internal face* every time a subsequence *RRL* is found while walking *clockwise*

Compaction: Step 1

b) add dummy edges and vertices until all faces are rectangles

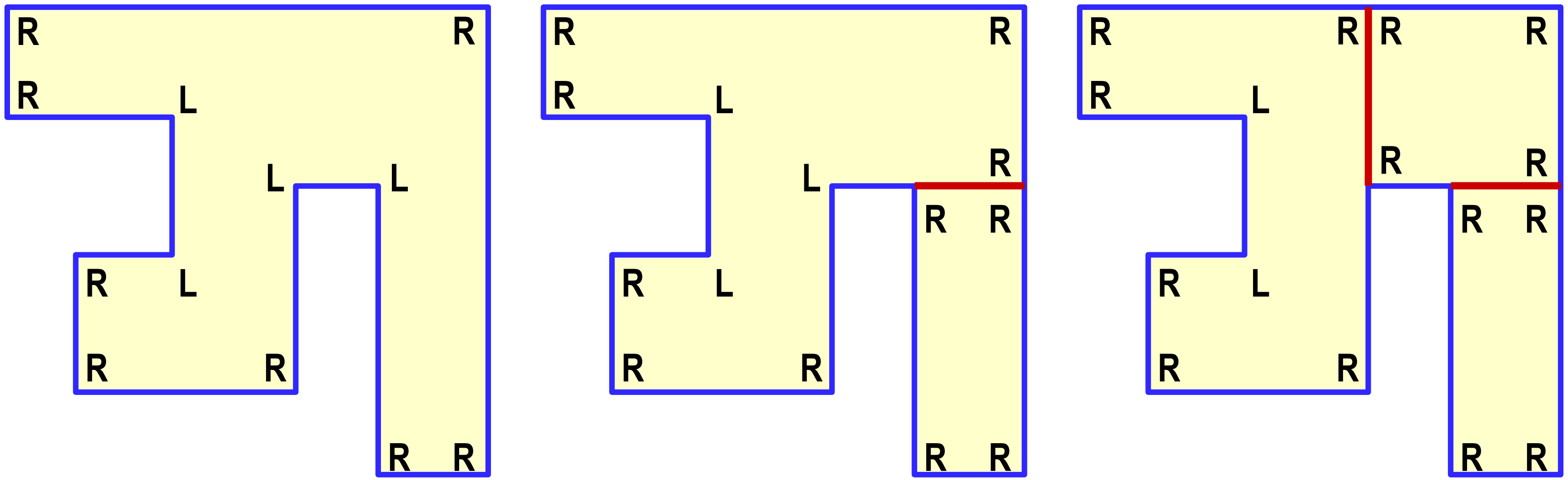


split recursively each *internal face* every time a subsequence *RRL* is found while walking *clockwise*

Compaction: Step 1

b) add dummy edges and vertices until all faces are rectangles

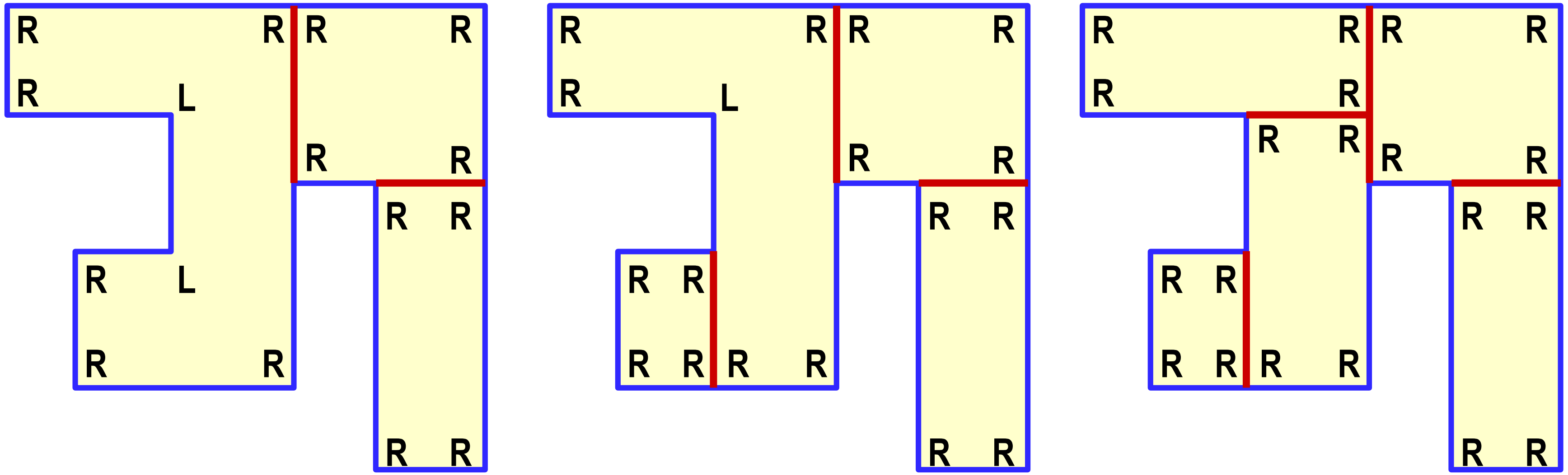
… a more complex example

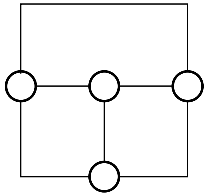


Compaction: Step 1

b) add dummy edges and vertices until all faces are rectangles

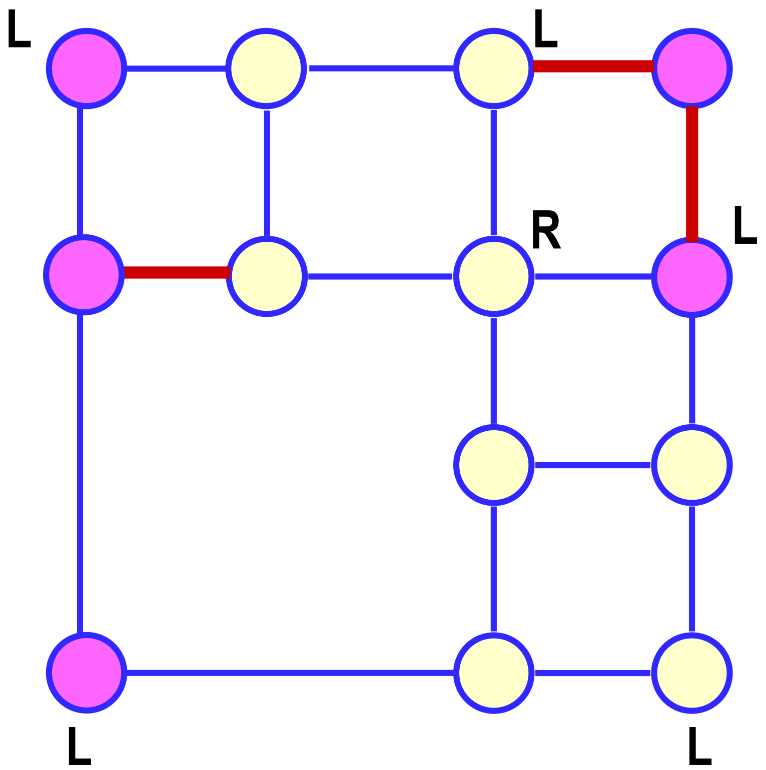
… a more complex example



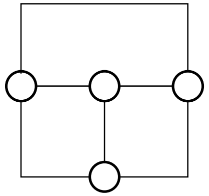


Compaction: Step 1

b) add dummy edges and vertices until all faces are rectangles

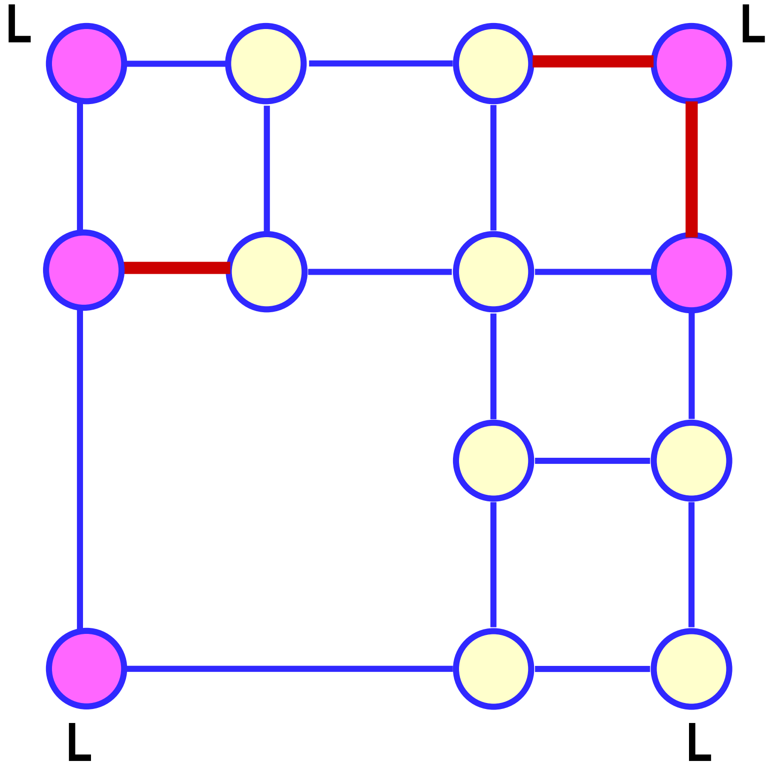


split recursively the *external face* every time a subsequence *LRL or LRR* is found while walking *counterclockwise*



Compaction: Step 1

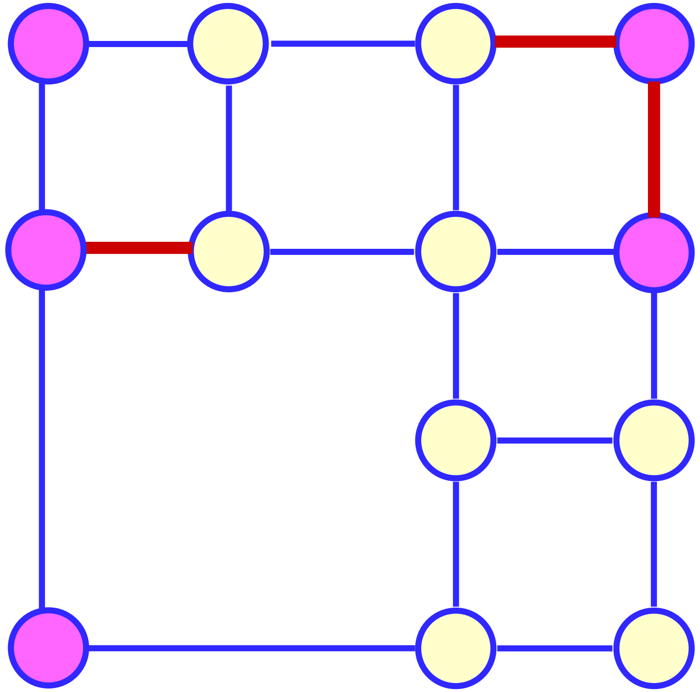
b) add dummy edges and vertices until all faces are rectangles



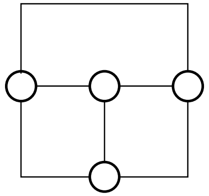
split recursively the *external face* every time a subsequence *LRL or LRR* is found while walking *counterclockwise*

Compaction: Step 2

- Compute vertex coordinates

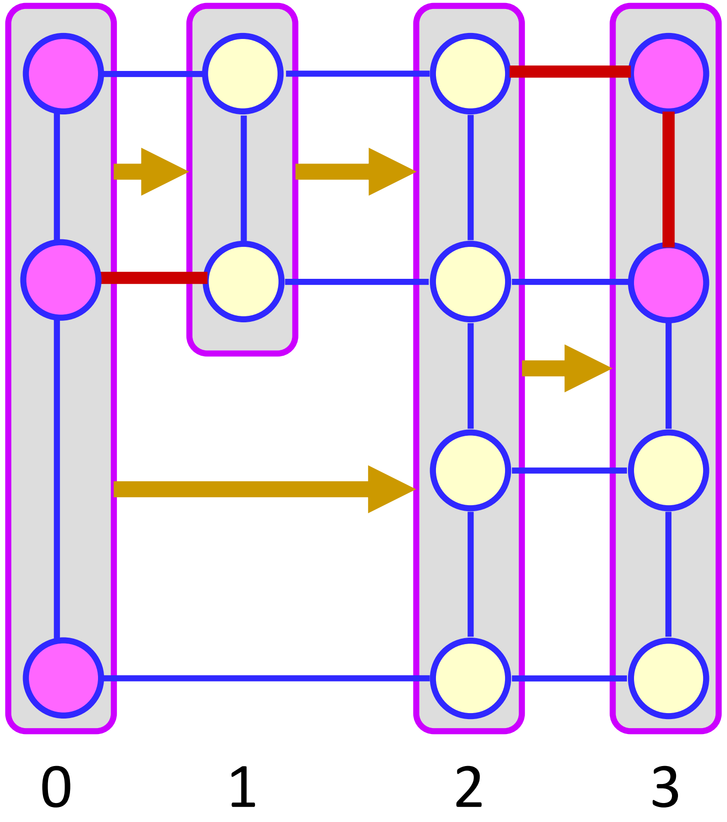


- assign the x-coordinates so that the width is minimized
- assign the y-coordinates so that the height is minimized
- for a rectangular shape this leads to minimum area



Compaction: Step 2

- Compute vertex coordinates

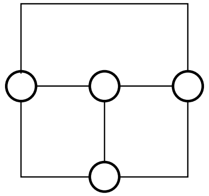


- Find the x-coordinates so that the width is minimized

– create super-nodes that group the vertices in the same vertical chain

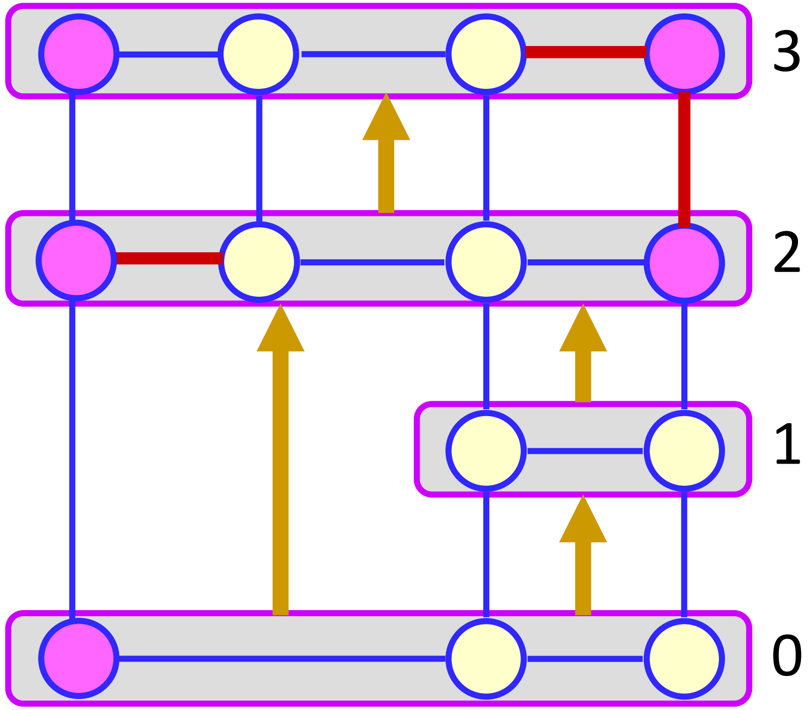
– connect two super-nodes with a left-to-right directed edge if the corresponding chains are connected in the shape

– assign to chains the x-coordinates computed by an *optimal topological numbering* of their super-nodes



Compaction: Step 2

- Compute vertex coordinates

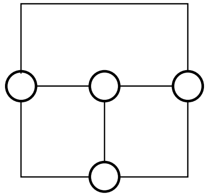


- Find the y-coordinates so that the height is minimized

− uses a super-node that groups the vertices in the same horizontal chain

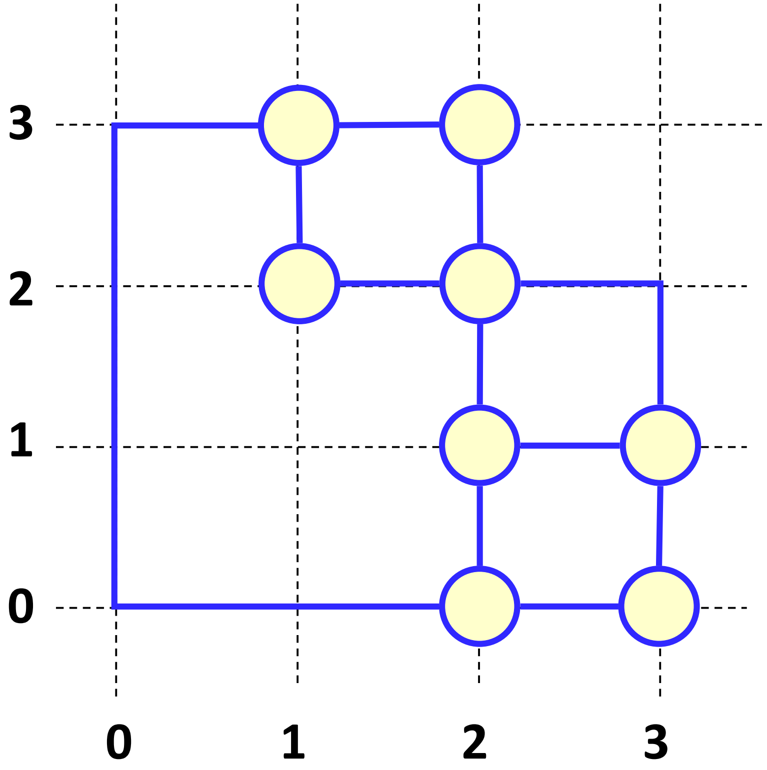
− connect two super-nodes with a bottom-to-top directed edge if the corresponding chains are connected in the shape

− assign to chains the y-coordinates computed by an *optimal topological numbering* of their super-nodes

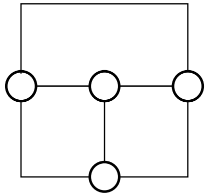


Compaction: Step 3

- Remove dummy edges and vertices

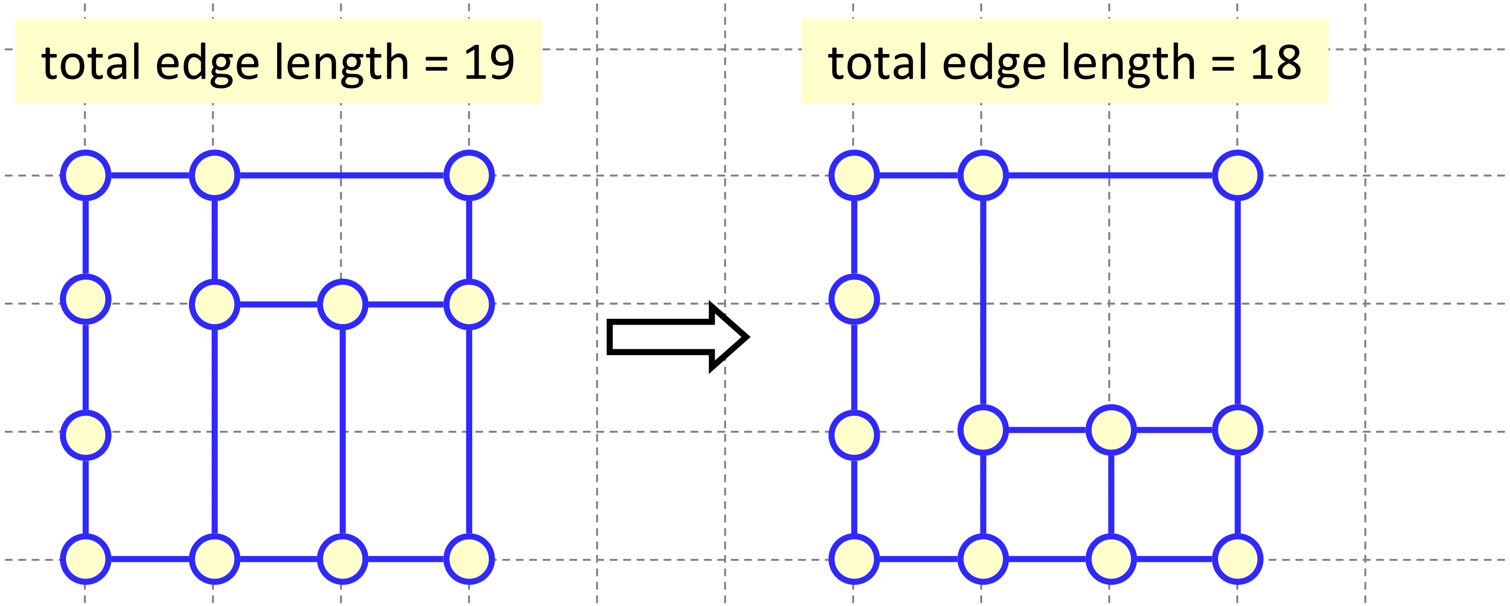


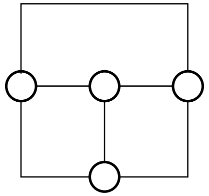
the described algorithm works in O(n) time



Compaction: Total edge length

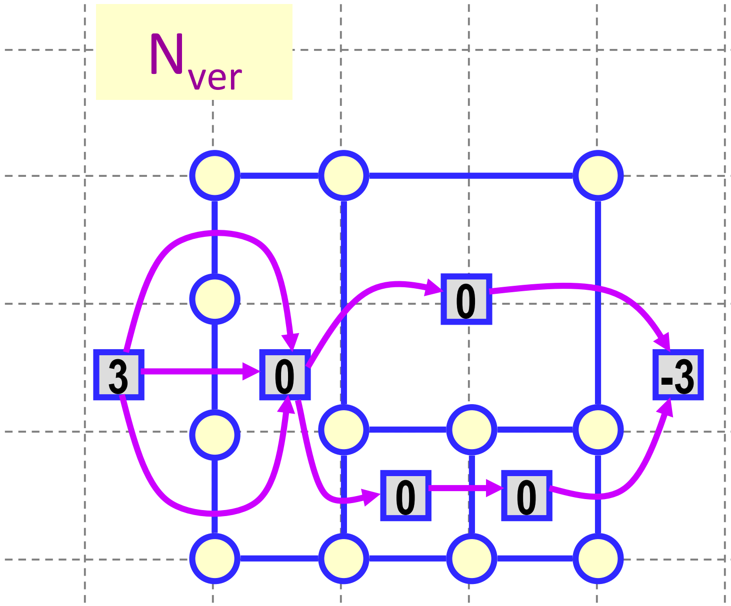
For a rectangular shape of given width and height, it is possible to minimize the total edge length within its dimensions in polynomial time

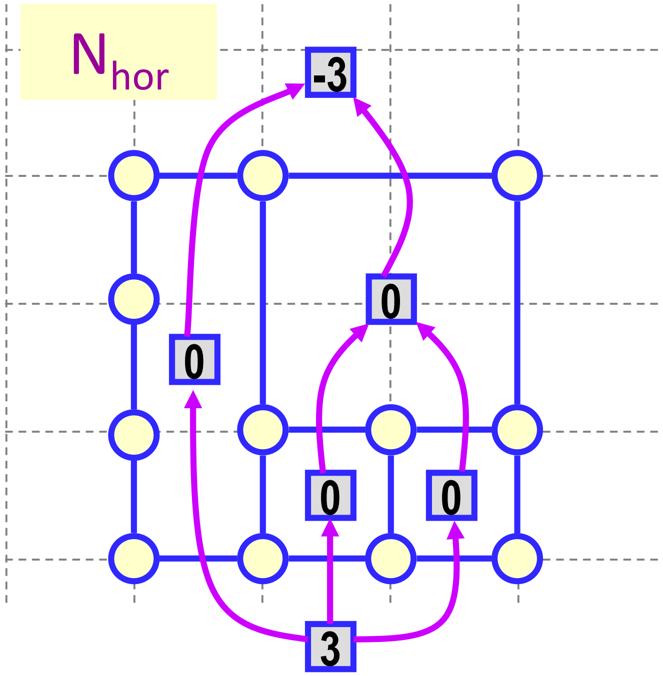




Compaction: Total edge length

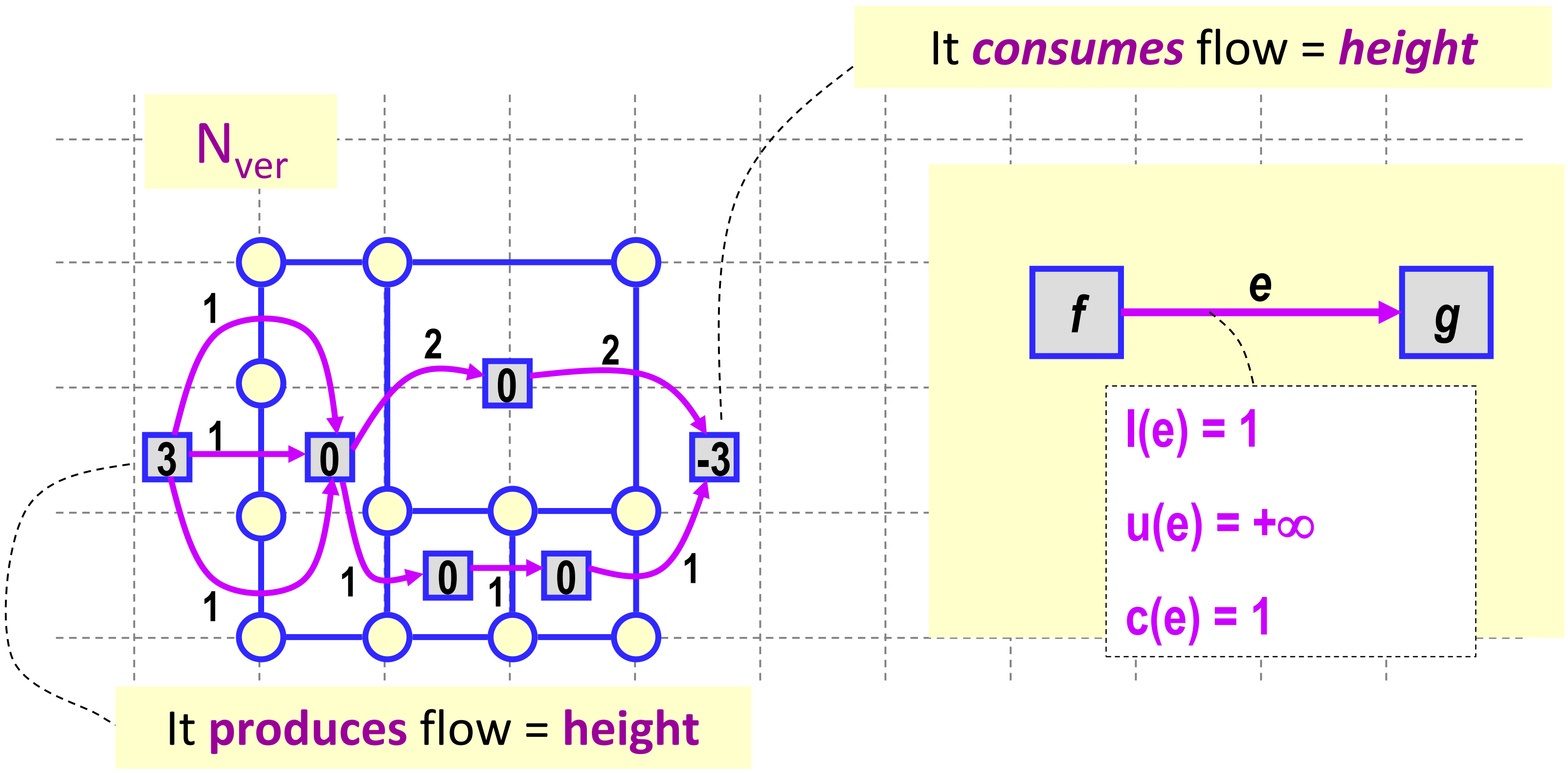
- use two flow networks, one for the vertical compaction (N_{ver}) and the other for the horizontal compaction (N_{hor})

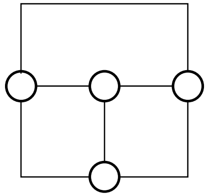




Compaction: Total edge length

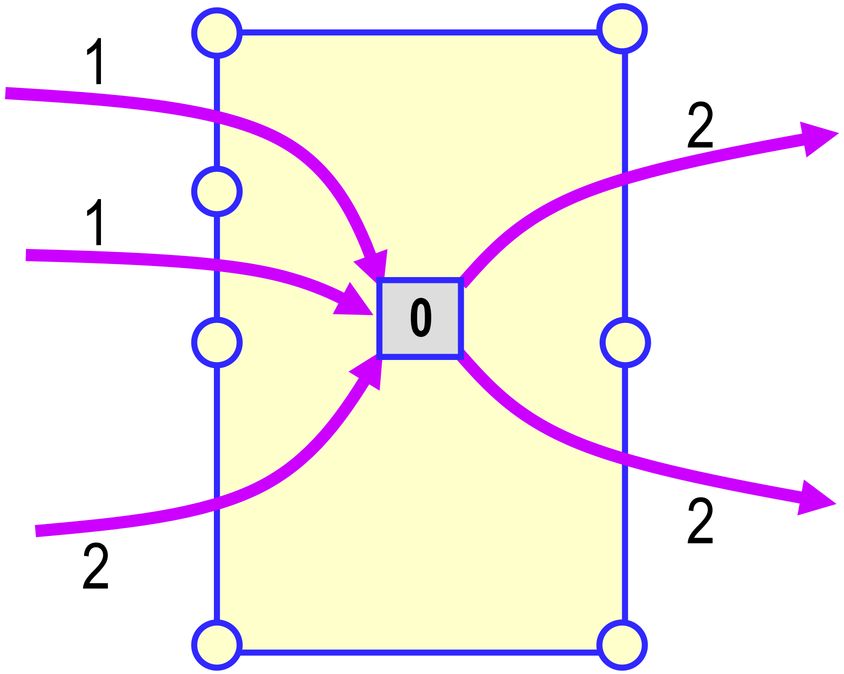
- The flow on each arc corresponds to the length of the corresponding edge of the orthogonal shape





Compaction: Total edge length

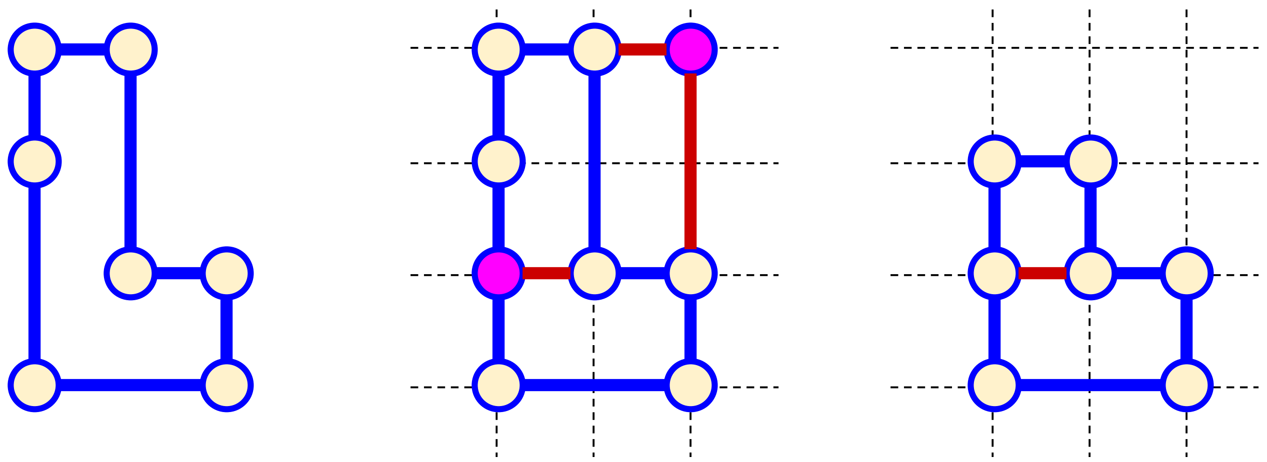
the fact that the node of the network associated with each internal face is a neutral node guarantees the consistency of the face dimensions



Compaction: Further issues

Observation: The dummy vertices and edges added in the rectangularization phase represent a constraint, which may strongly affect the result of the compaction algorithm

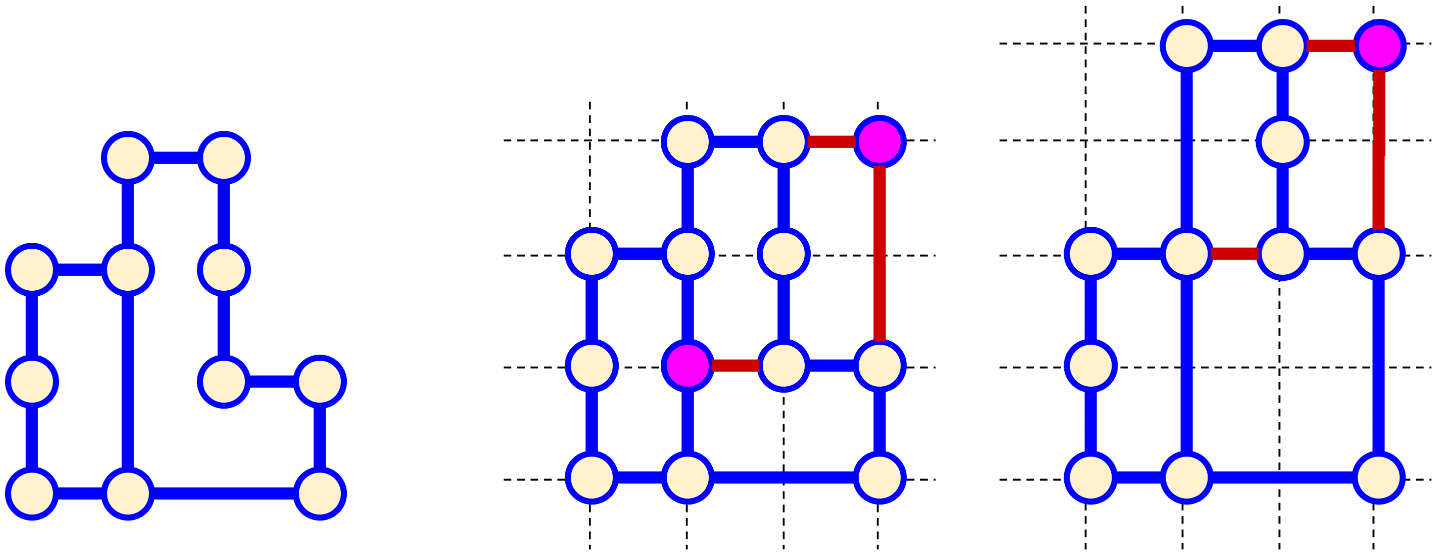
Example 1



Compaction: Further issues

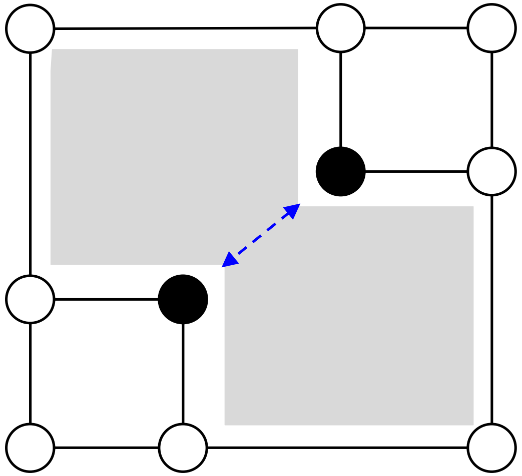
Observation: The dummy vertices and edges added in the rectangularization phase represent a constraint, which may strongly affect the result of the compaction algorithm

Example 2

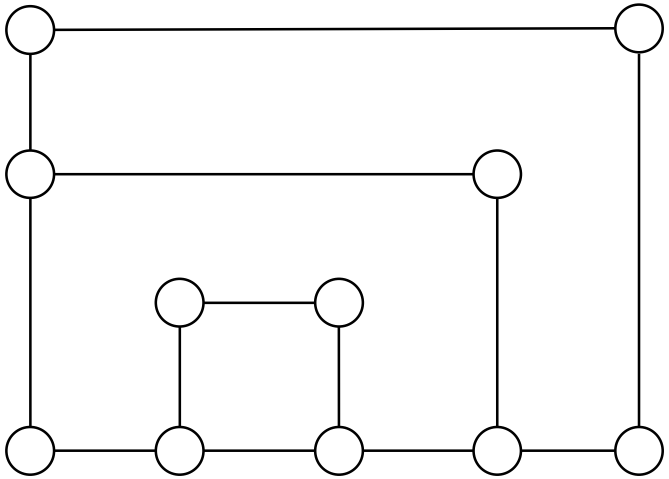


Compaction: Turn-regularity

A planar orthogonal representation is turn-regular if it has no pairs of *kitty-corners* (opposing reflex vertices) inside a face



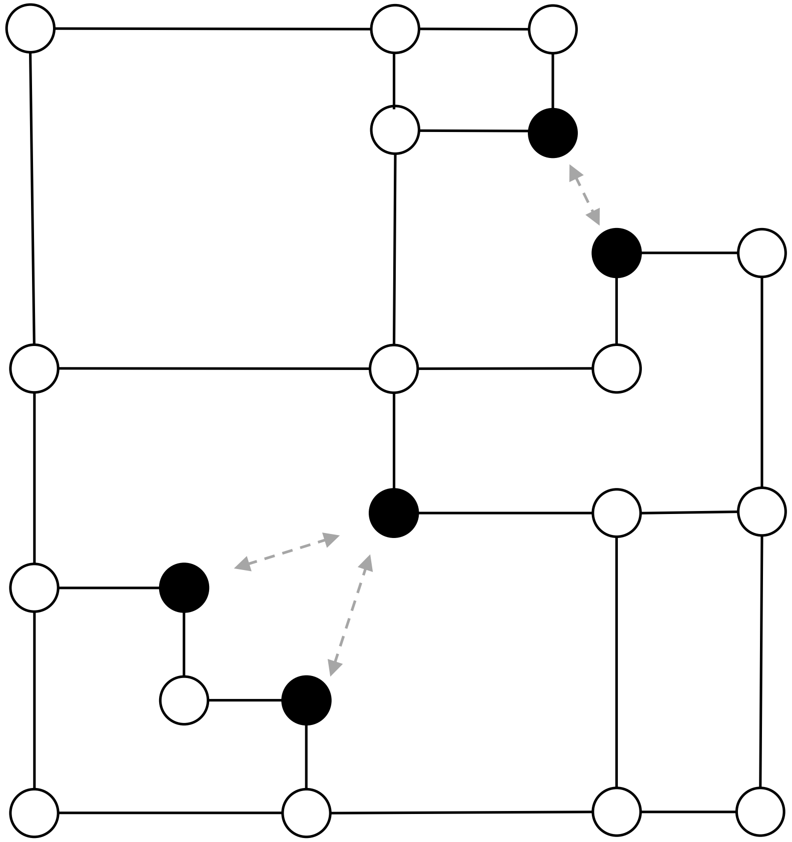
not turn-regular



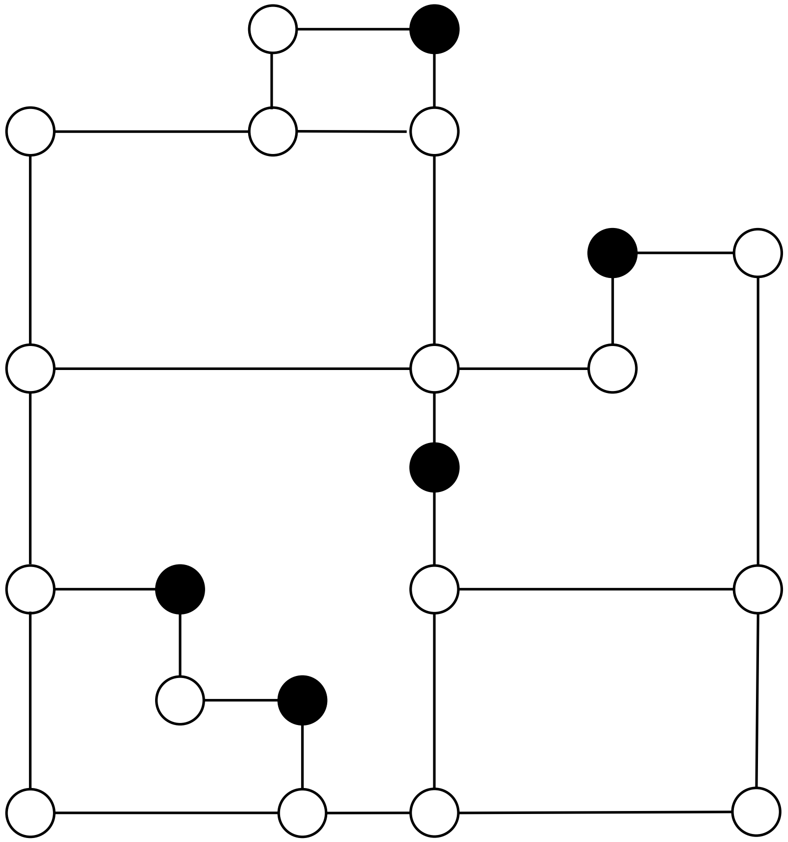
turn-regular

Compaction: Turn-regularity

Two orthogonal representations of the same plane graph



not turn-regular



turn-regular

Compaction: Turn-regularity

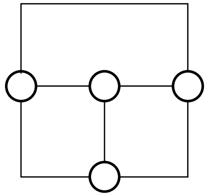
Theorem. Let H be an orthogonal representation of an embedded planar 4-graph with n vertices. It is possible to test in O(n) time whether H is turn-regular. In the positive case, an orthogonal drawing of H of minimum area can be computed in O(n) time.

S. S. Bridgeman, G. Di Battista, W. Didimo, G. Liotta, R. Tamassia, L. Vismara: Turn-regularity and optimal area drawings of orthogonal representations. Comput. Geom. 16(1): 53-93 (2000)

Part 1.2
Engineering the Topology-Shape-Metrics Approach

Practical considerations

- Real graphs typically contain high-degree vertices (with degree larger than 4)
- Many applications usually need to customize a generic drawing algorithm by imposing some drawing constraints
  - vertices represented as boxes of prescribed sizes
  - specific edges that cannot cross or that cannot bend
  - …
- In the following we briefly discuss the above issues

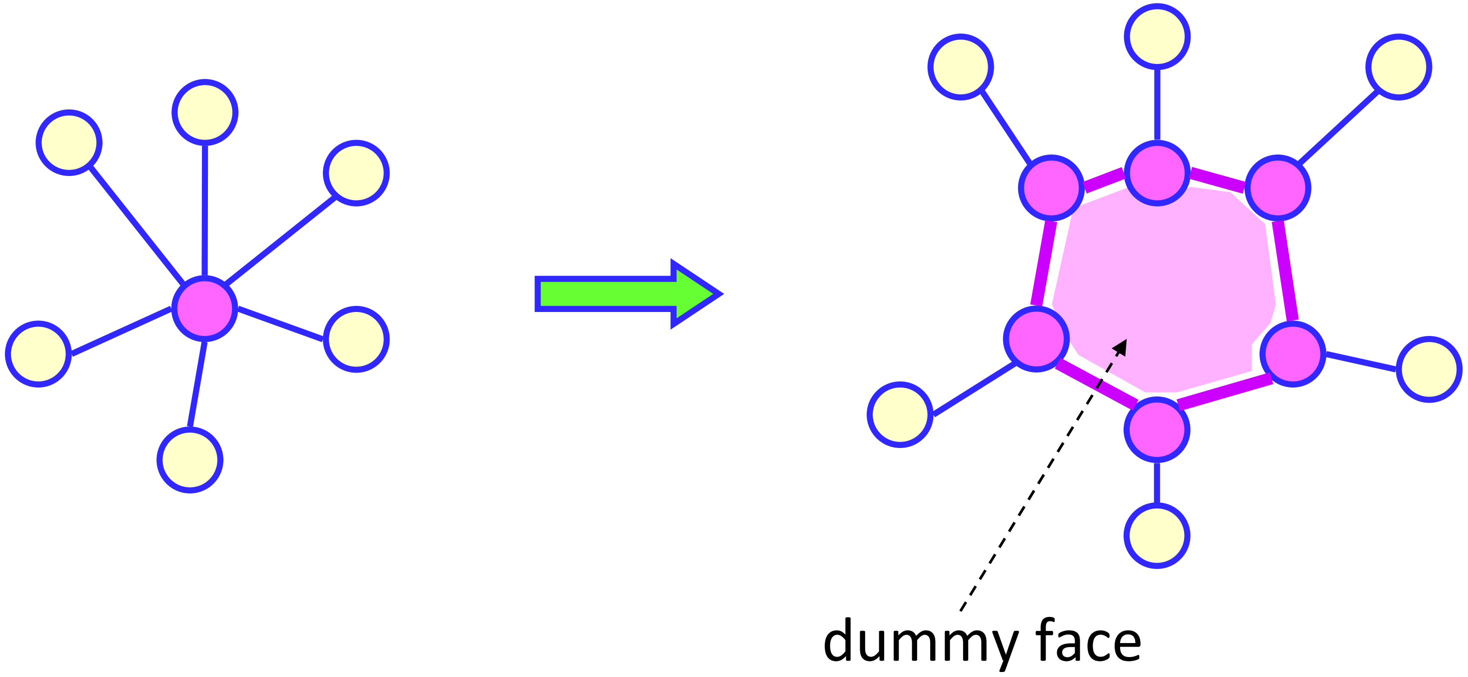


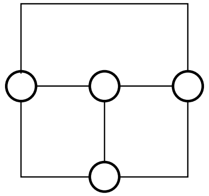
High-degree vertices: First strategy

- After the planarization step, replace each high-degree vertex with a dummy face, having all vertices of degree 3
- Apply the topology-shape-metrics approach with some constraints that guarantee that each dummy face is drawn as a rectangle
- In the final drawing dummy faces will be shown as boxes

High-degree vertices: First strategy

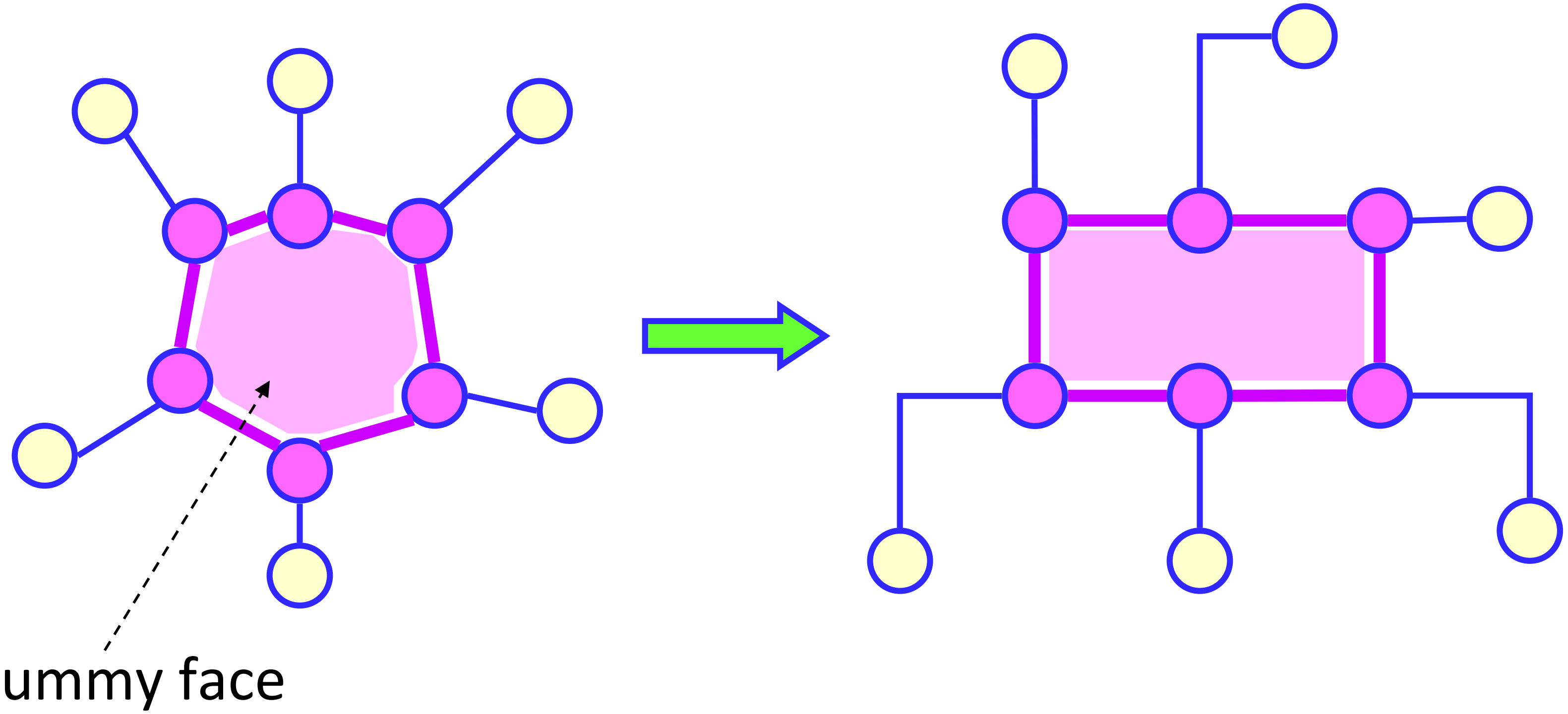
- After the planarization step, replace each high-degree vertex with a dummy face, having all vertices of degree 3





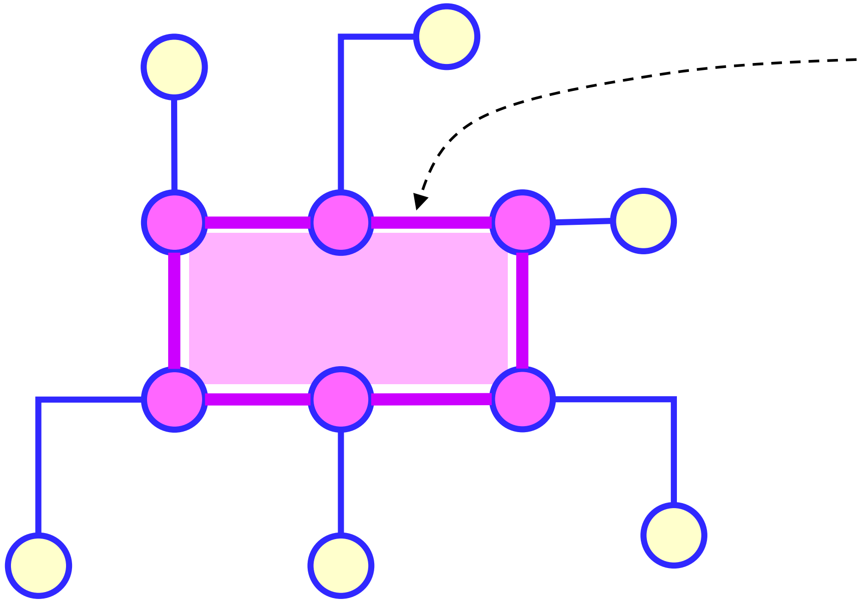
High-degree vertices: First strategy

- Apply the topology-shape-metrics approach with some constraints that guarantee that each dummy face is drawn as a rectangle



High-degree vertices: First strategy

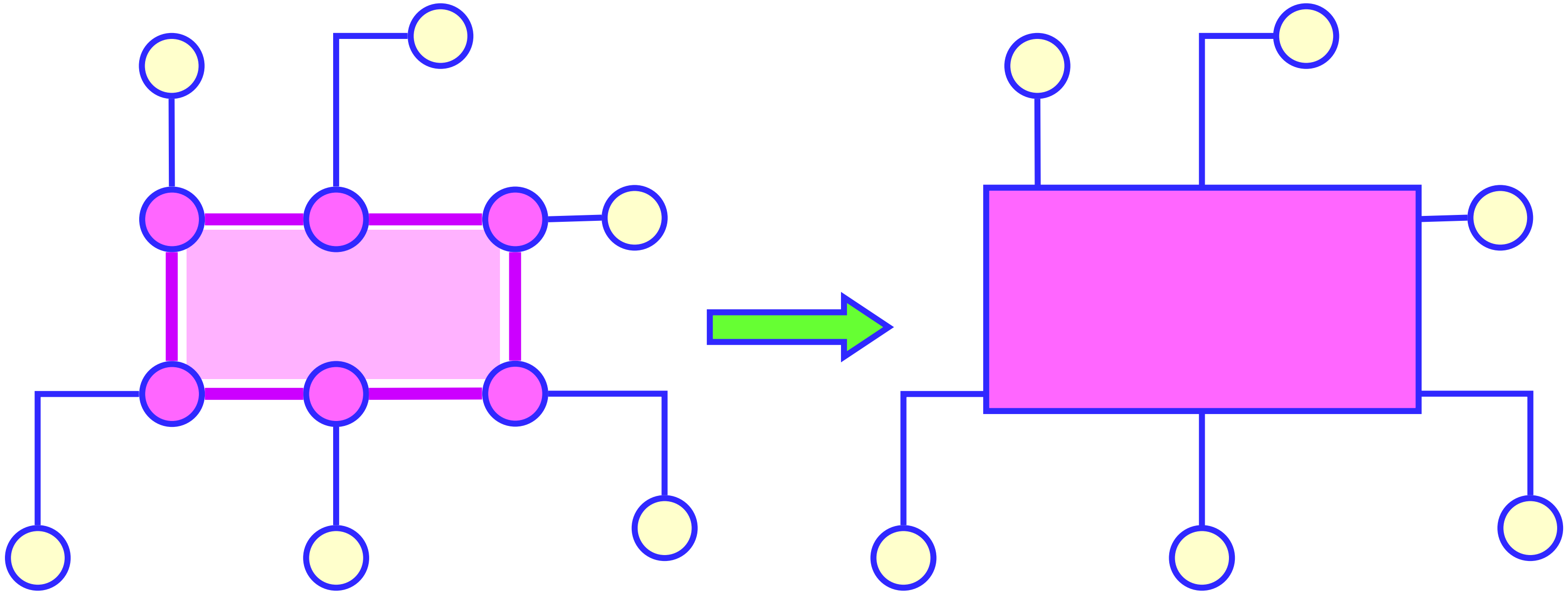
- constraints on the orthogonalization algorithm



- each edge of the dummy face boundary is forced to be straight
- this is done by deleting the face-to-face arcs incident to the dummy face node in the flow network

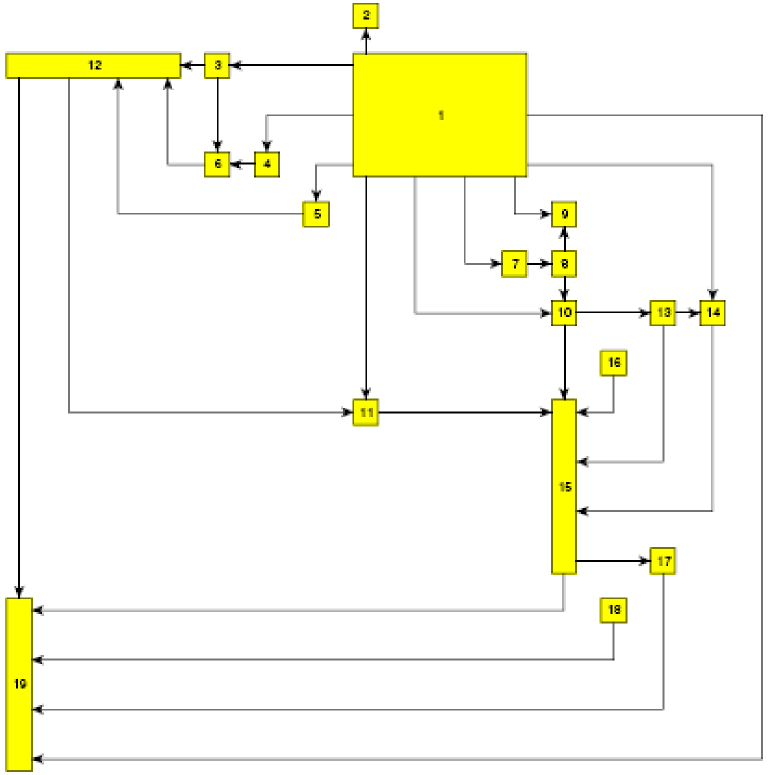
High-degree vertices: First strategy

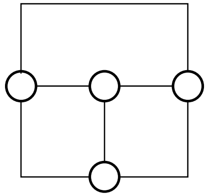
- In the final drawing dummy faces will be shown as boxes



Drawbacks of this strategy

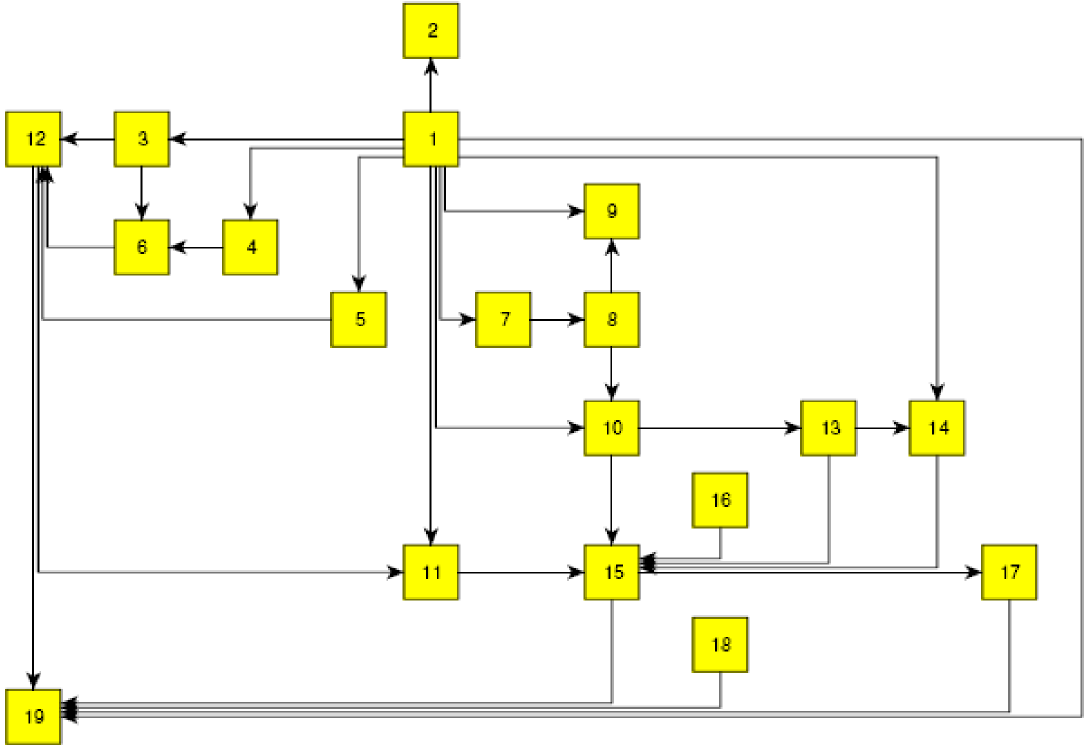
- No control on the dimensions of high-degree vertices
  - the corresponding dummy faces may be stretched a lot in the compaction phase
- Real-world applications may require all vertices of the same dimensions





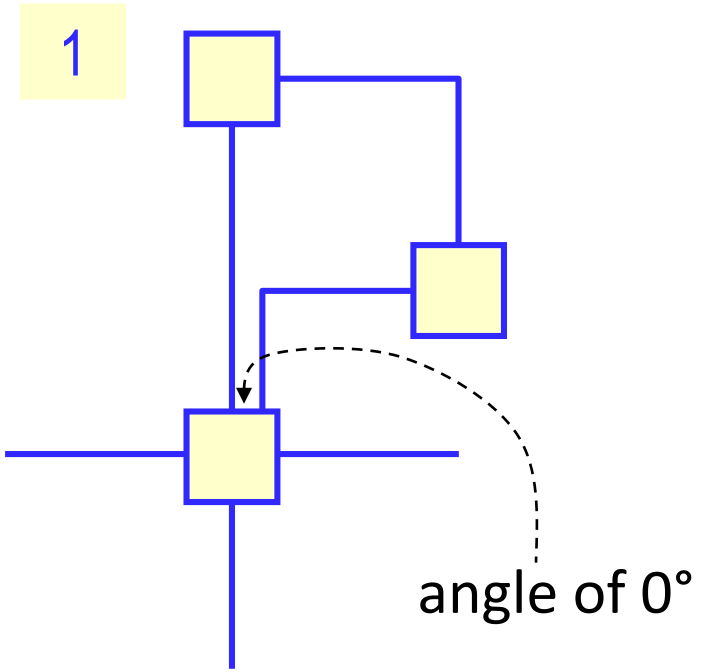
High-degree vertices: Second strategy

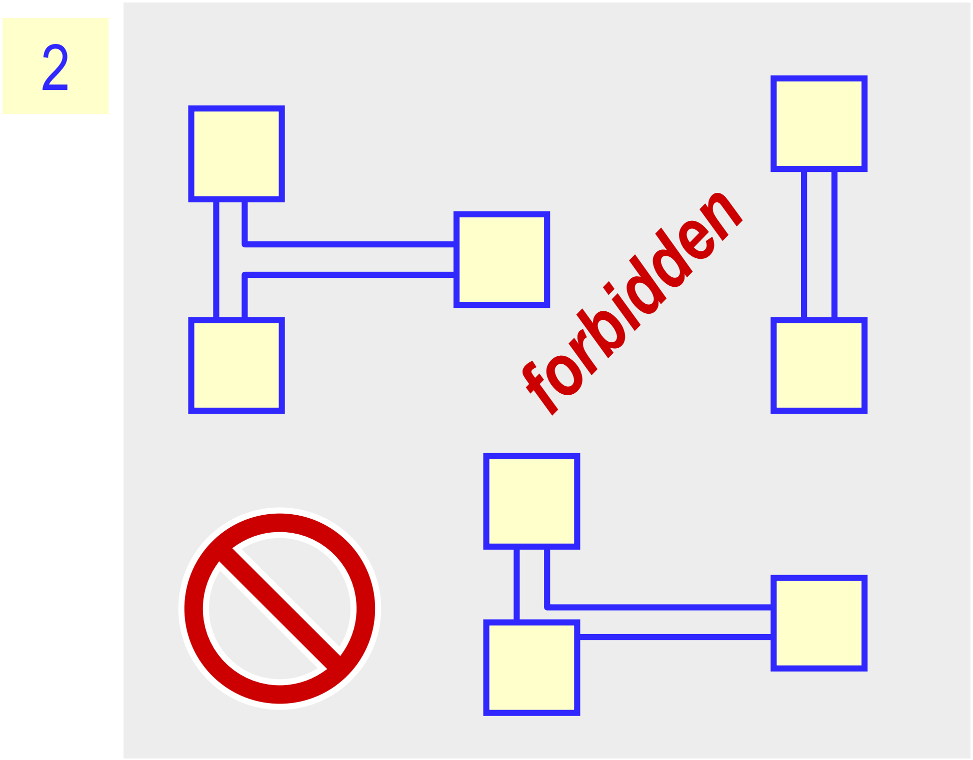
- Use a different model with all vertices of the same size (Kandinsky)
  - *Fößmeier and Kaufmann*: Drawing high degree graphs with low bend numbers, Graph Drawing (1995)



High-degree vertices: Kandinsky

1. introduction of angles of 0°
2. each face has an area strictly greater than 0





High-degree vertices: Kandinsky

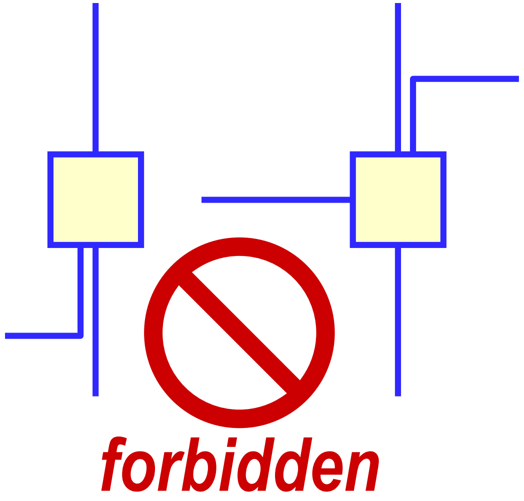
- Unfortunately, minimizing the number of bends in the Kandinsky model is NP-complete:
  - *T. Bläsius, G. Brückner, I. Rutter*: Complexity of Higher-Degree Orthogonal Graph Embedding in the Kandinsky Model. ESA (2014)

- But the problem is polynomial-time solvable with few additional restrictions (simple Kandinsky)
  - *P. Bertolazzi, G. Di Battista, W. Didimo*: Computing Orthogonal Drawings with the Minimum Number of Bends. IEEE Trans. Computers 49(8): 826-840 (2000)

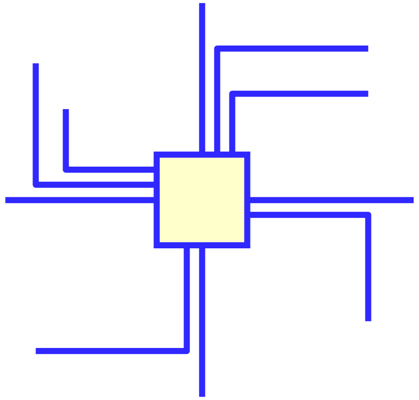
High-degree vertices: simple Kandinsky

1. there cannot be two edges incident to the same side of a vertex if there is at least one unused side of the vertex
2. If there are multiple edges incident to the same side of a vertex, all of them except the first (in clockwise order) must bend in the same direction (e.g. to the right)

1



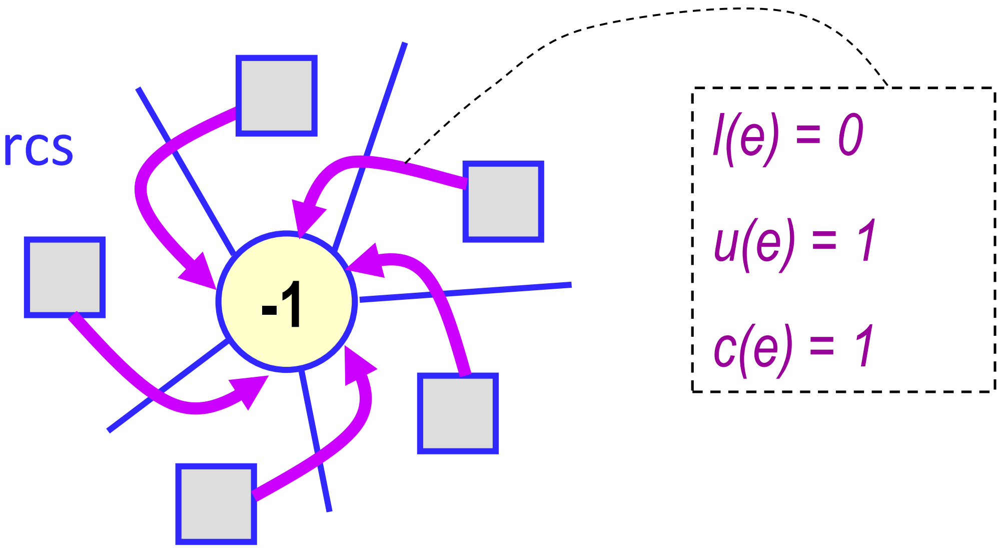
2



High-degree vertices: simple Kandinsky

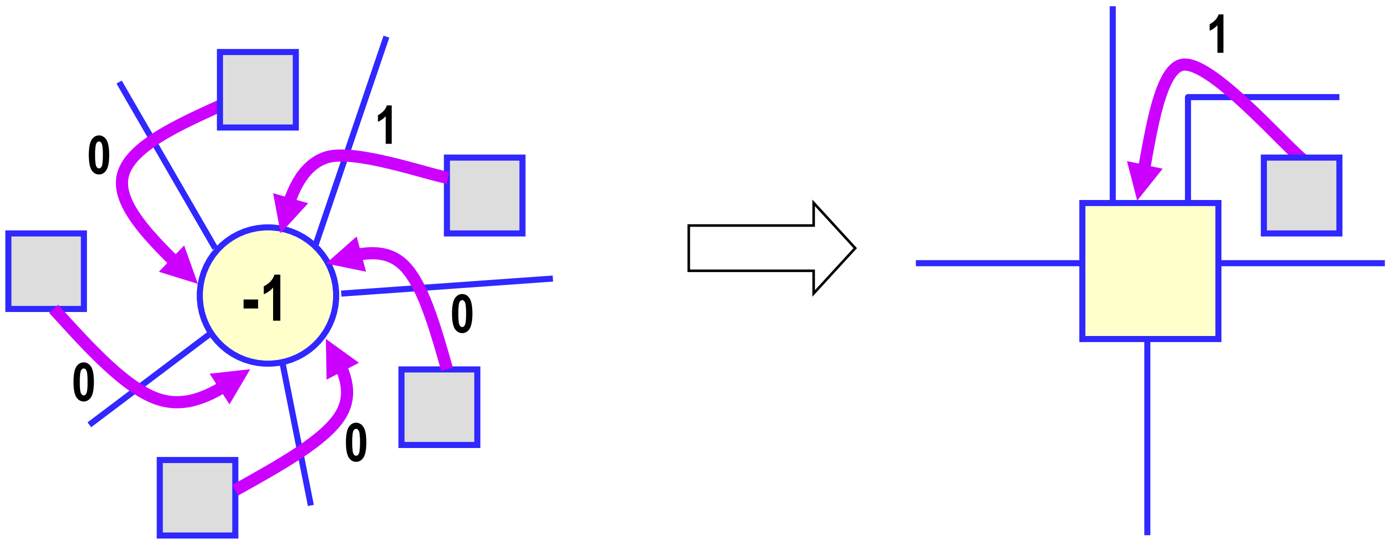
- To compute a bend-minimum orthogonal representation in the simple Kandinsky model extend Tamassia's flow network
  - each high-degree vertex v becomes a consumer instead of a producer; it consumes flow deg(v) – 4, received by its incident faces

face-to-vertex arcs



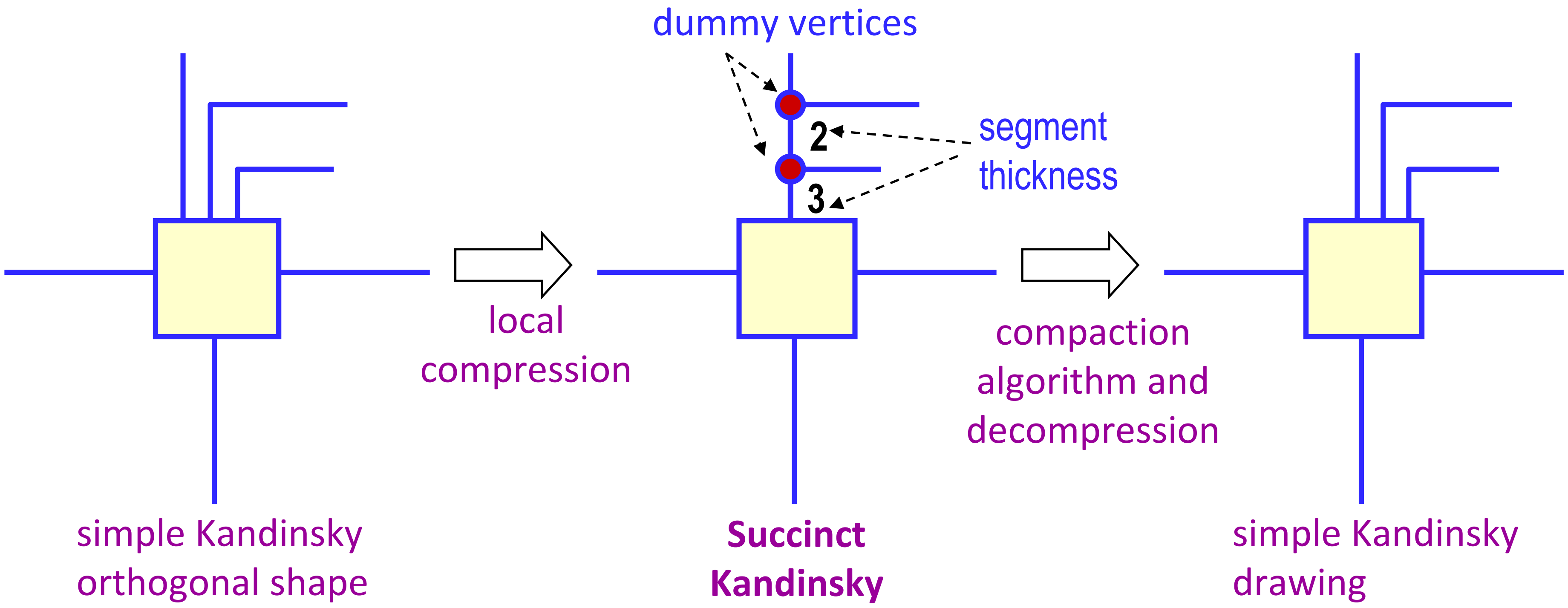
High-degree vertices: simple Kandinsky

- Interpretation of the flow on the new kind of arcs
  - one unit of flow on an arc (f,v) represents an angle of 0° and causes 1 bend



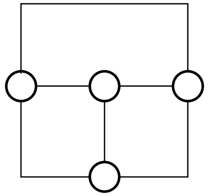
High-degree vertices: simple Kandinsky

- *Compaction* of simple Kandinsky
  - reduced to the compaction algorithm for classical orthogonal shapes



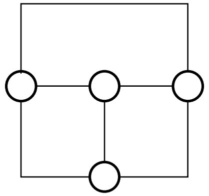
Handling constraints

- The topology-shape-metrics approach makes it possible to deal with several types of constraints in each phase:
  - topology constraints
  - shape constraints
  - metrics constraints



Topology constraints

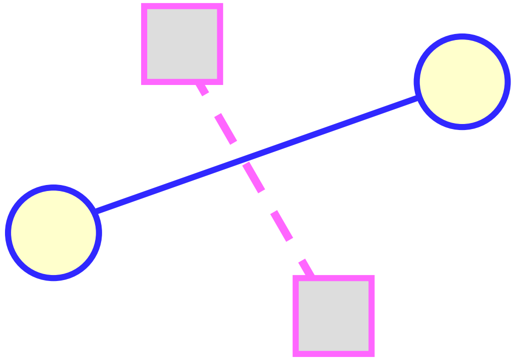
- Some topology constraints
  - edges that cannot cross (uncrossable edges)
  - subsets of vertices that must lie on the same face boundary
  - groups of edges that must be consecutive around a common end-vertex
- Handled in the planarization phase



Topology constraints

- Some topology constraints
  - edges that cannot cross (uncrossable edges)

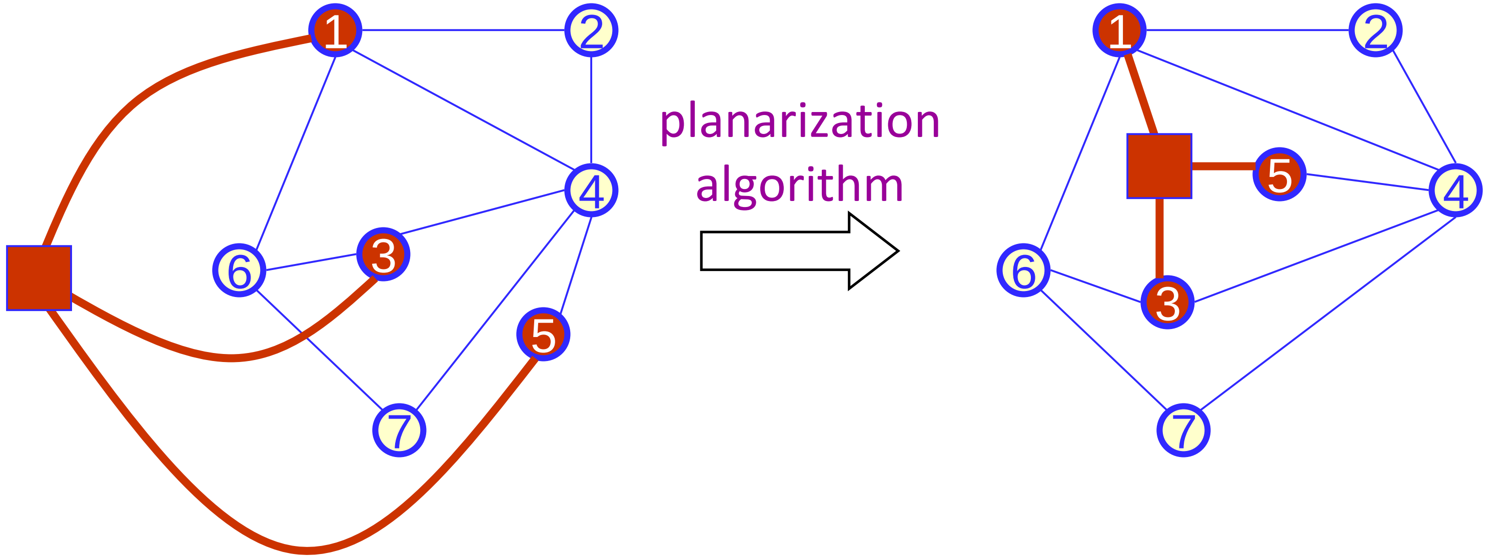
to make an edge *uncrossable*, the planarization algorithm is modified by removing the corresponding edge in the dual graph; *a shortest path in the dual cannot cross the primal edge*



Topology constraints

- Some topology constraints
  - subsets of vertices that must lie on the same face boundary

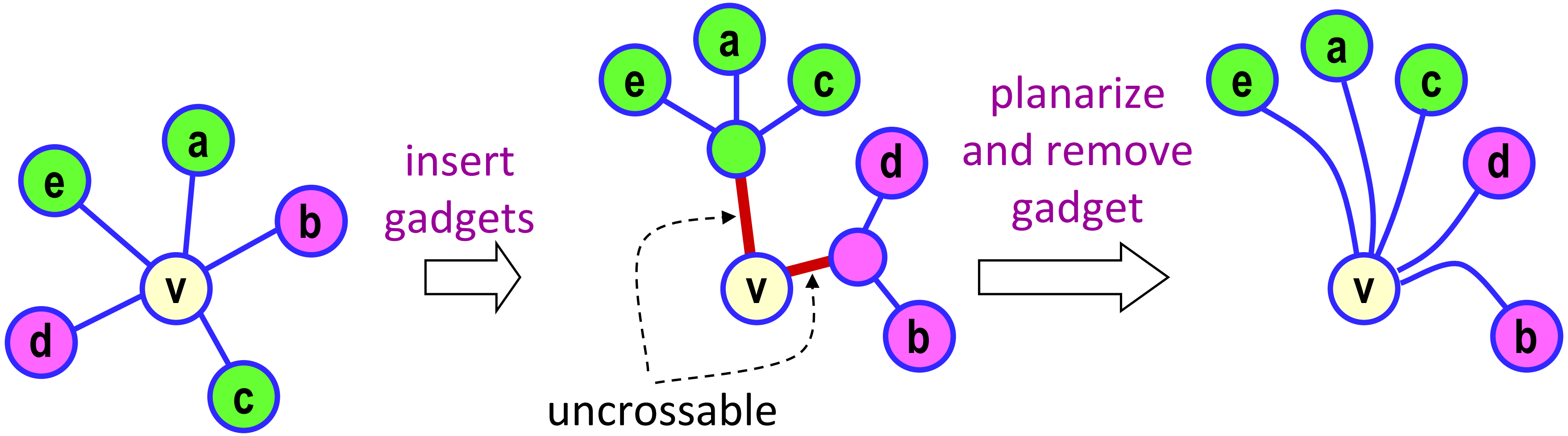
the planarization algorithm is applied after the insertion of a *"star-gadget" of uncrossable edges*

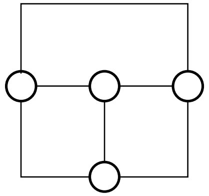


Topology constraints

- Some topology constraints
  - groups of edges that must be consecutive around a common end-vertex

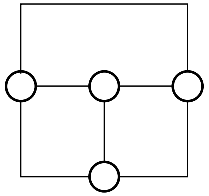
the planarization algorithm is applied after the insertion of a suitable *“star-gadget” for each group*





Topology constraints

- Other topology constraints
  - *C. Gutwenger, K. Klein, P. Mutzel*: Planarity Testing and Optimal Edge Insertion with Embedding Constraints. J. Graph Algorithms Appl. 12(1): 73-95 (2008)
  - *G. Liotta, I. Rutter, A. Tappini*: Graph Planarity Testing with Hierarchical Embedding Constraints. CoRR abs/1904.12596 (2019)

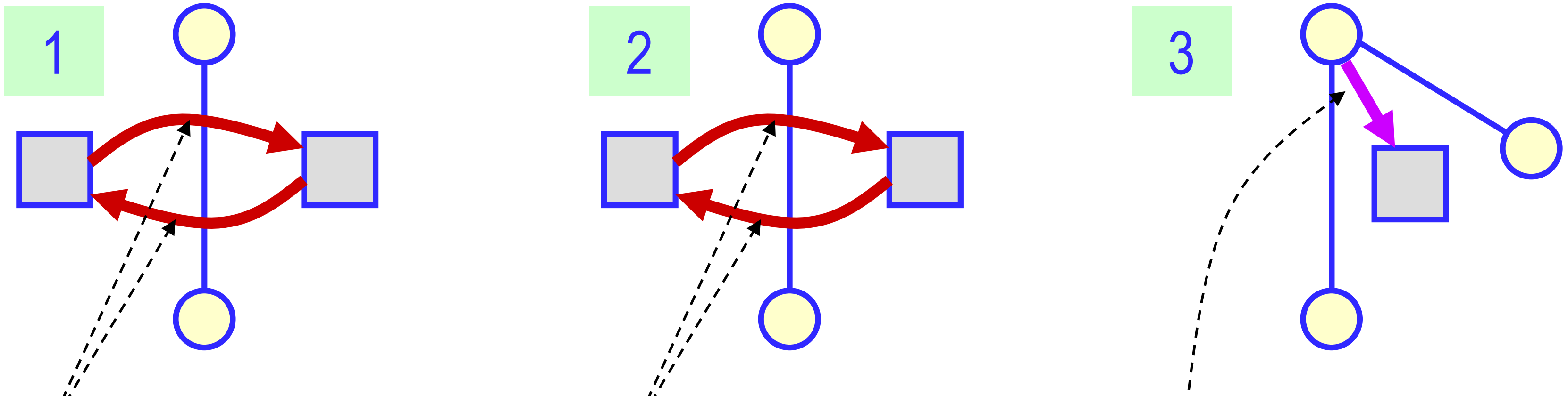


Shape constraints

- Some shape constraints
  - deciding the number of bends on an edge (e.g., no bend or any number or a specific number)
  - deciding the turn direction of an edge (left or right)
  - bounding or fixing the values of vertex angles
- Handled in the orthogonalization phase by suitably modifying capacities and/or costs of the arcs of the flow network
  - *R. Tamassia*: On Embedding a Graph in the Grid with the Minimum Number of Bends. SIAM J. Comput. 16(3): 421-444 (1987)

Shape constraints

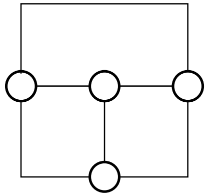
- Some shape constraints
  1. deciding the number of bends on an edge (e.g., no bend or any number or a specific number)
  2. deciding the turn direction of an edge (left or right)
  3. bounding or fixing the values of vertex angles



modify the cost of the face-to-face arcs or fix the flow

delete one of the two face-to-face arcs

change lower and upper capacities of the vertex-to-face arc



Metrics constraints

- Some metrics constraints
  - deciding vertex dimensions (width and the height of each single vertex)
  - deciding the attaching point of each edge

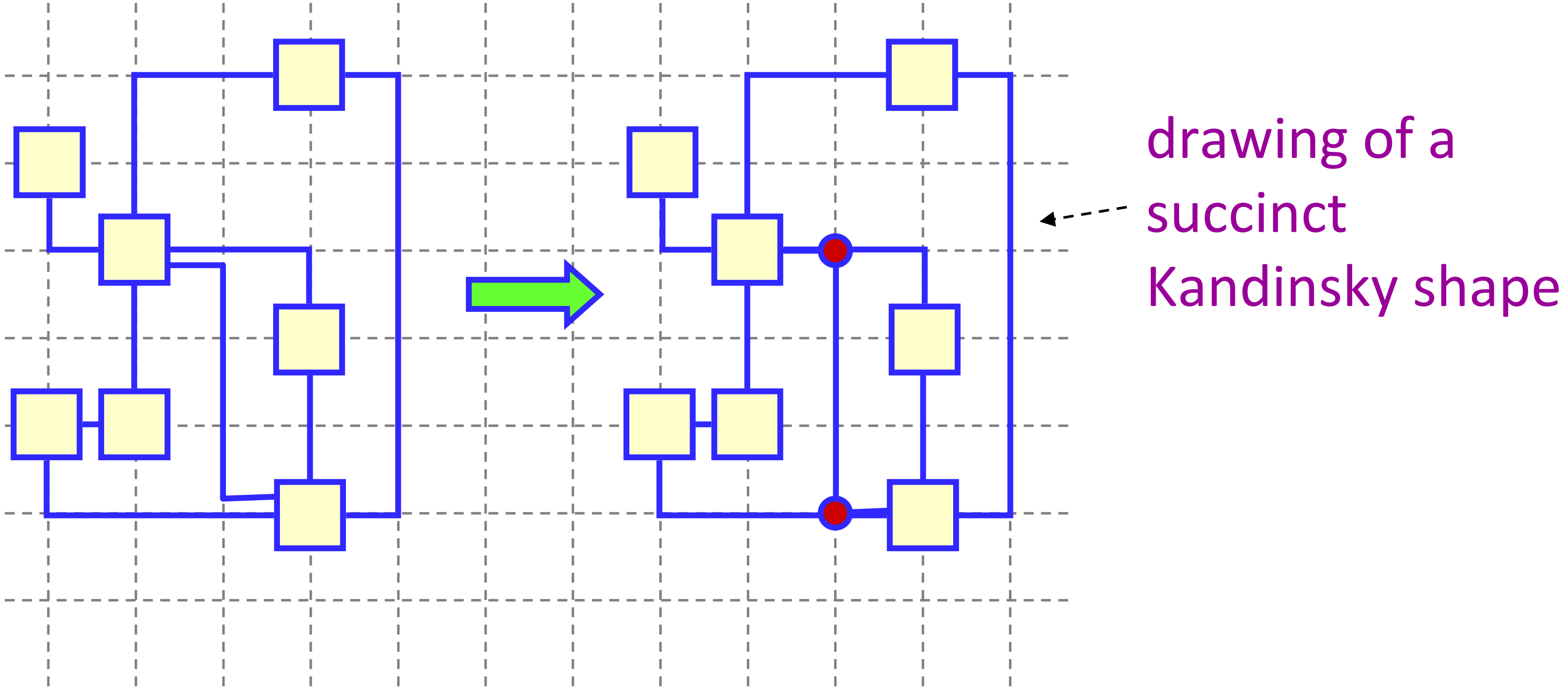
- Handled in the compaction phase

Metrics constraints

- Some metrics constraints
  - deciding vertex dimensions (width and height of each single vertex)
    - *G. Di Battista, W. Didimo, M. Patrignani, M. Pizzonia*: Orthogonal and quasi-upward drawings with vertices of arbitrary size. Graph Drawing (1999)
- **Idea**
  - start from a drawing of a succinct Kandinsky shape
  - expand vertices iteratively, by inserting extra rows and columns in the drawing, according to the desired vertex dimensions (expressed in terms of grid units)
  - compact the drawing again
  - uncompress edges to get the final drawing

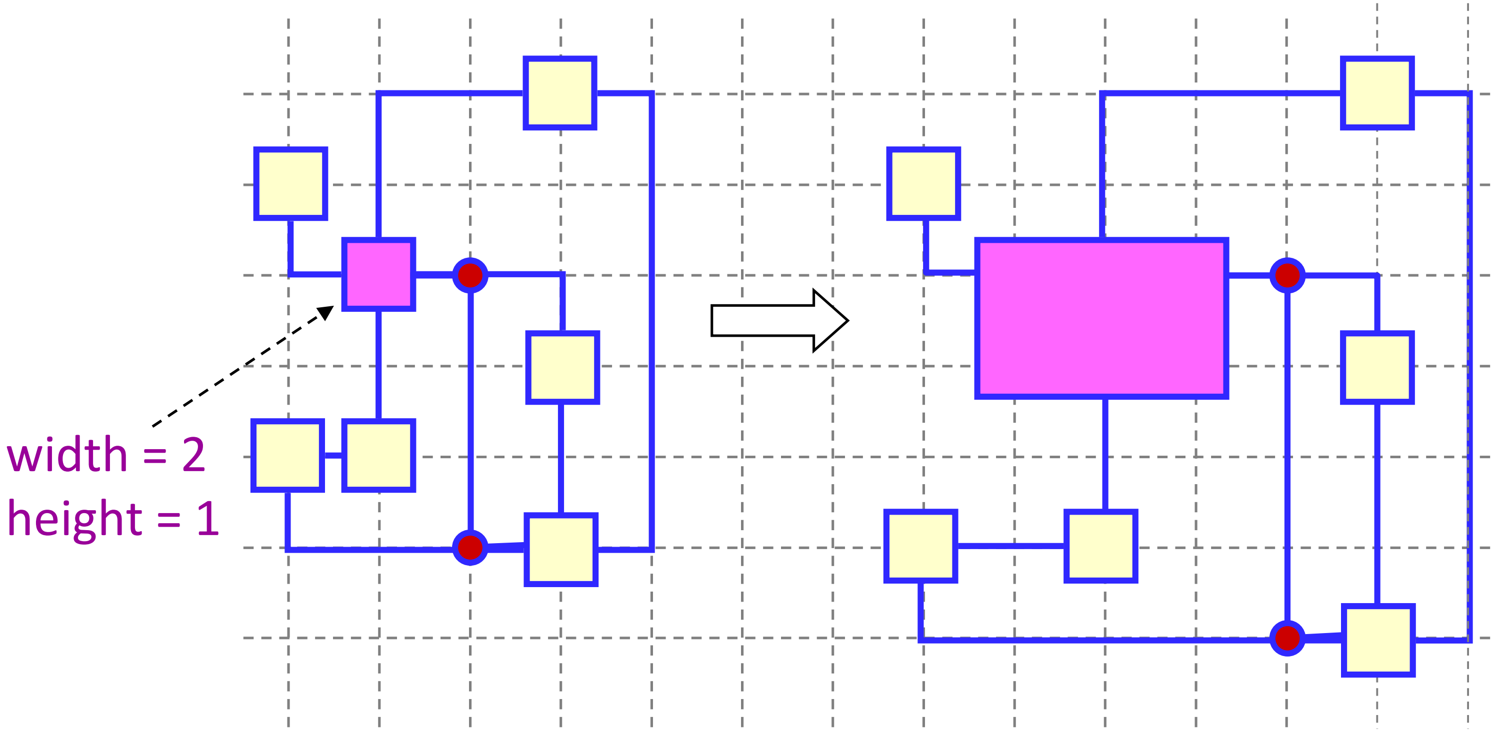
Metrics constraints

– start from a drawing of a succinct Kandinsky shape



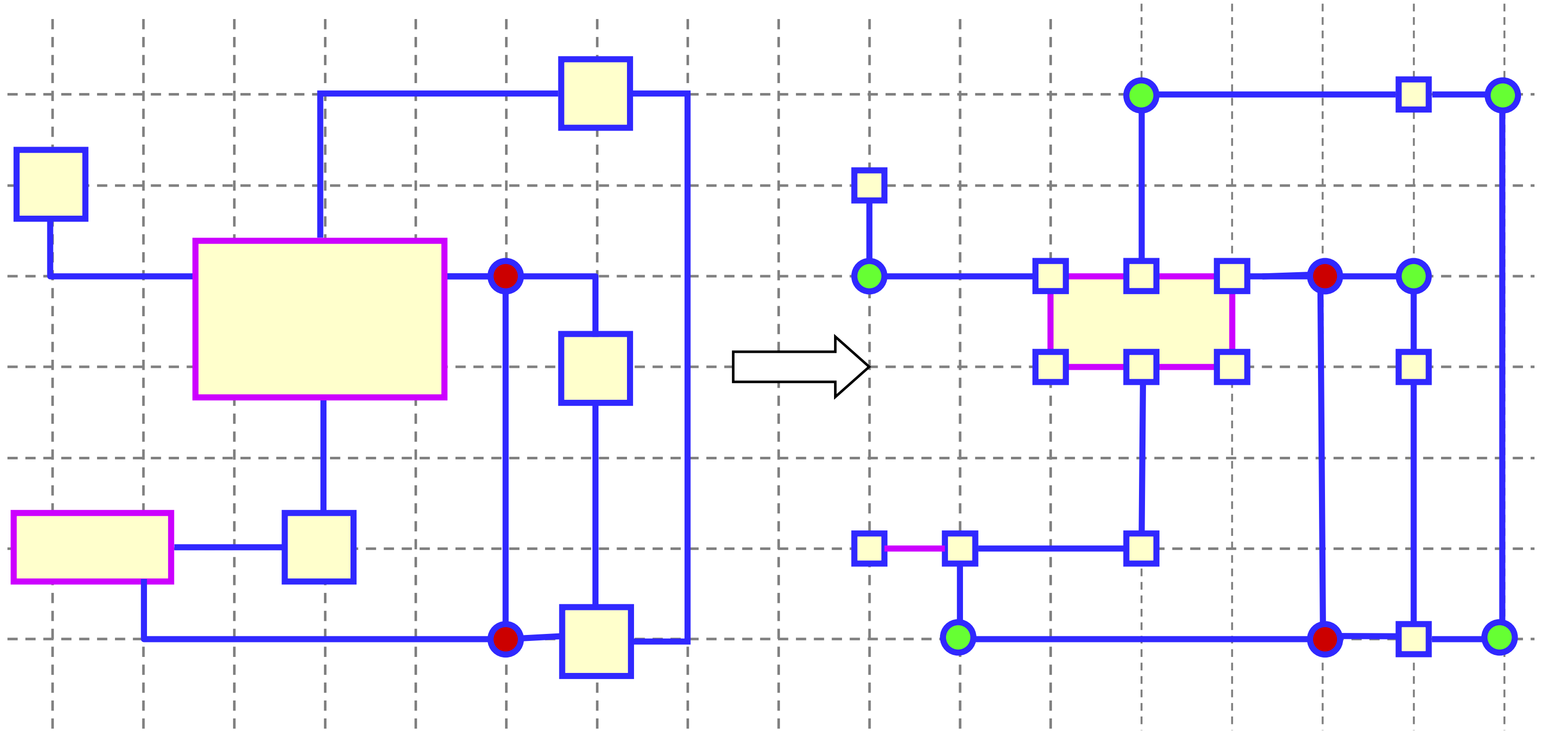
Metrics constraints

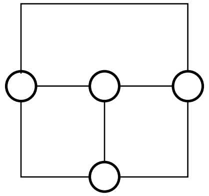
– expand vertices iteratively, by inserting extra rows and columns in the drawing, according to the desired vertex dimensions (expressed in terms of grid units)



Metrics constraints

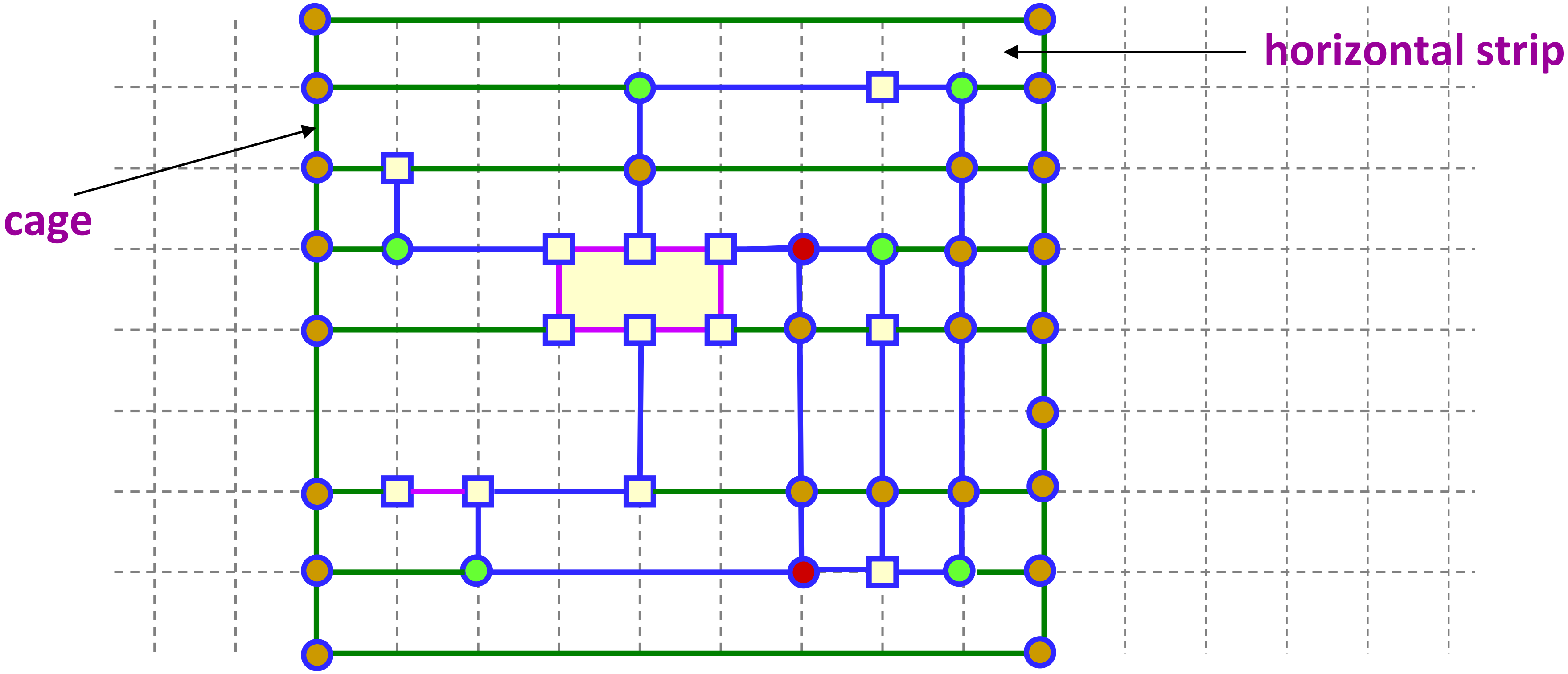
- compact the drawing again
  - replace each box with a “suitable” number of vertices of zero dimension (points)
  - replace each bend with a dummy vertex





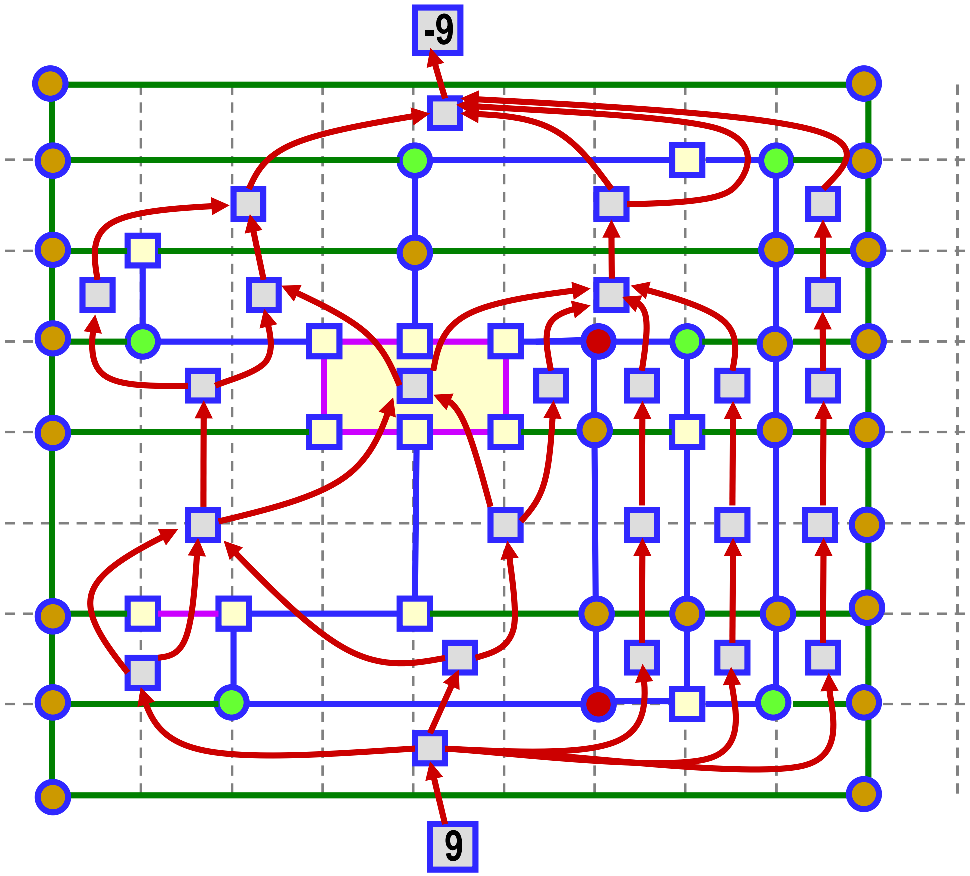
Metrics constraints

- compact the drawing again
  - create a dummy cage that includes the drawing and divide it into horizontal strips (extra dummy vertices and segments are created)

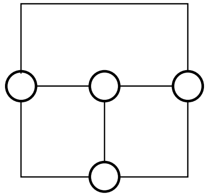


Metrics constraints

- compact the drawing again
  - compact horizontally by computing a min-cost-flow in a suitable network

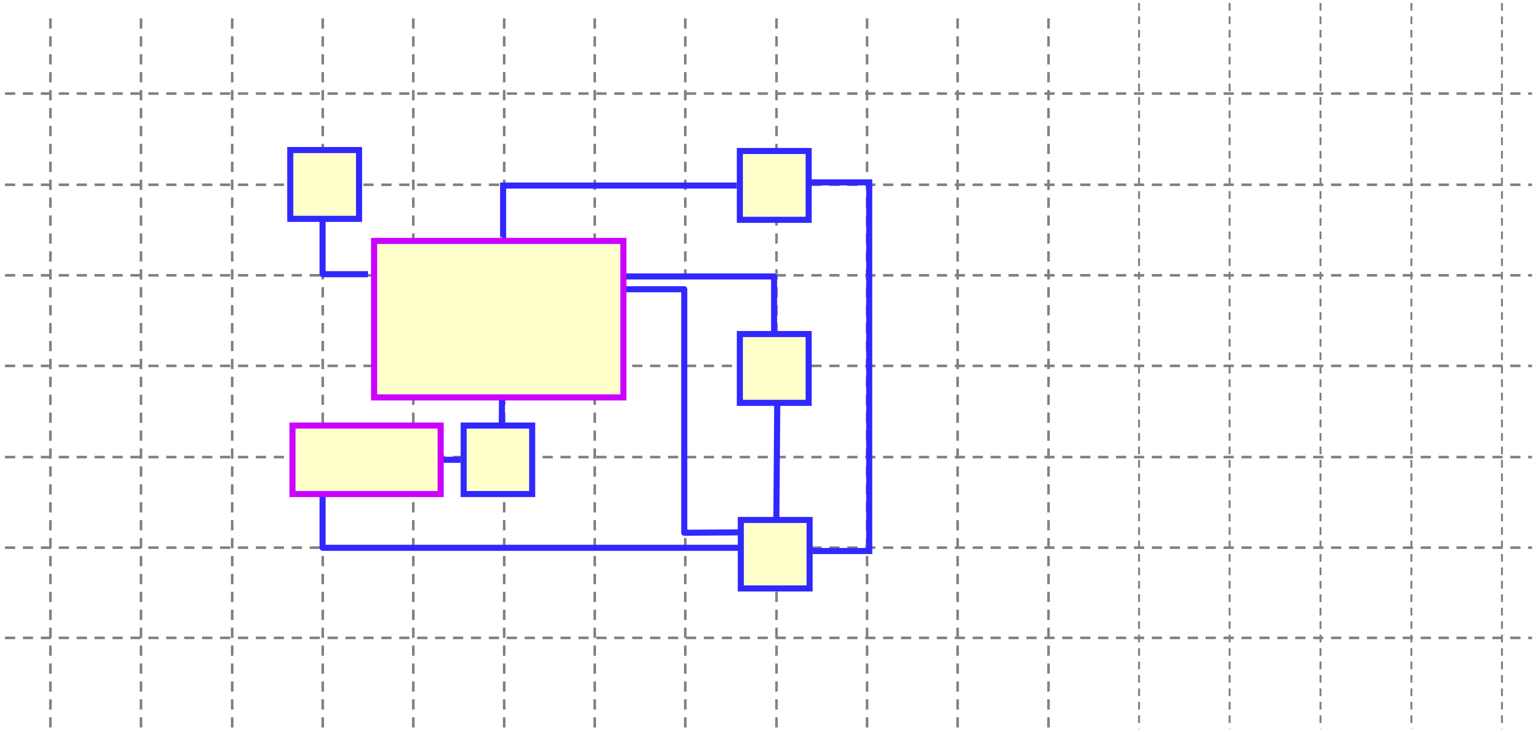


- flow represents edge lengths;
- produced flow = width of the cage
- arcs associated with box-vertex segments have fixed flow value (lower cap. = upper capacity)
- arcs associated with dummy segments have cost 0



Metrics constraints

- compact the drawing again
  - do the same to compact vertically – and repeat until no improvement happens
- decompress edges to get the final drawing



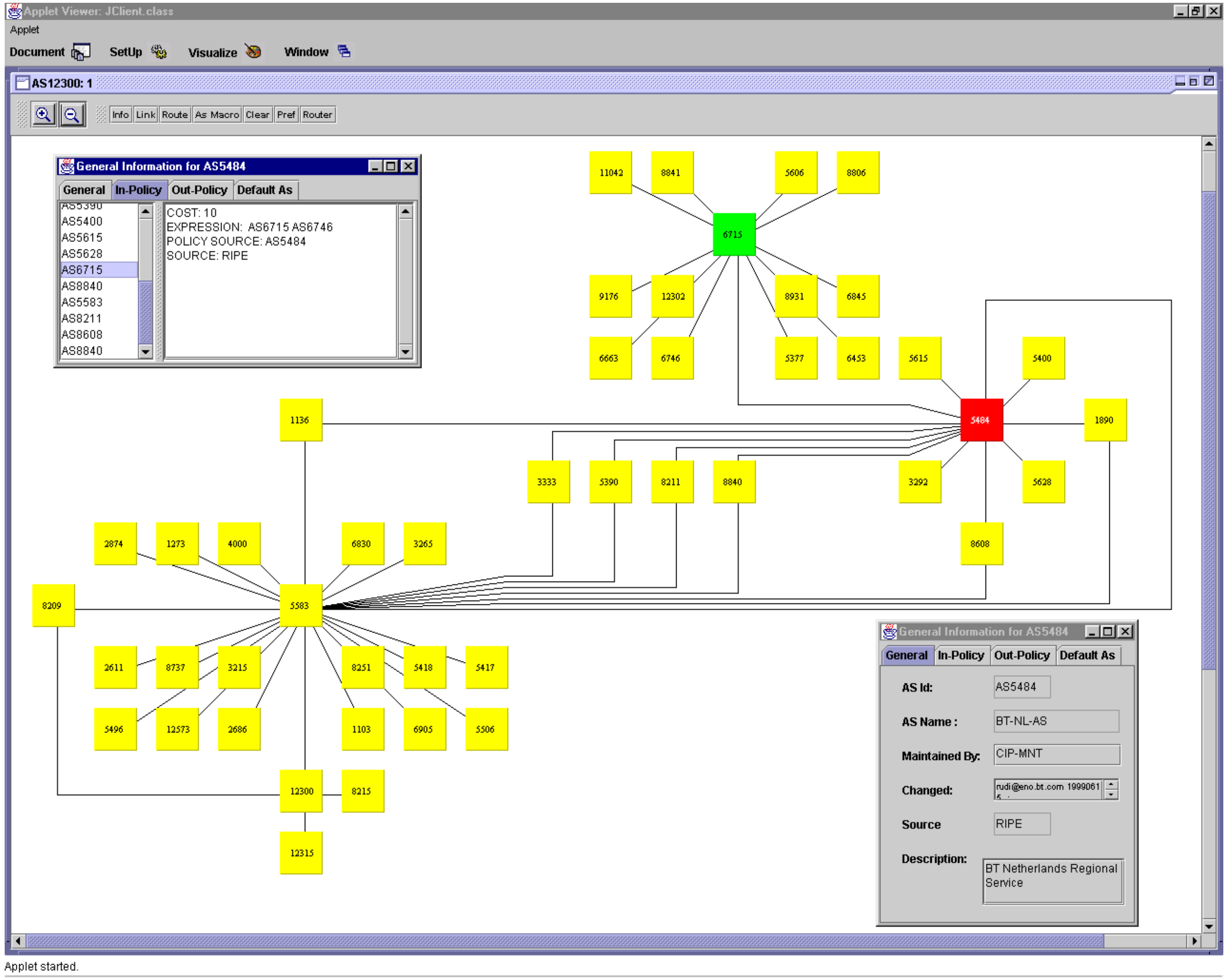
Further references

- *M. Eiglsperger, U. Fößmeier, M. Kaufmann*: Orthogonal graph drawing with constraints. SODA 2000: 3-11
- *M. Eiglsperger, M. Kaufmann*: Fast Compaction for Orthogonal Drawings with Vertices of Prescribed Size. Graph Drawing 2001: 124-138

Implementations

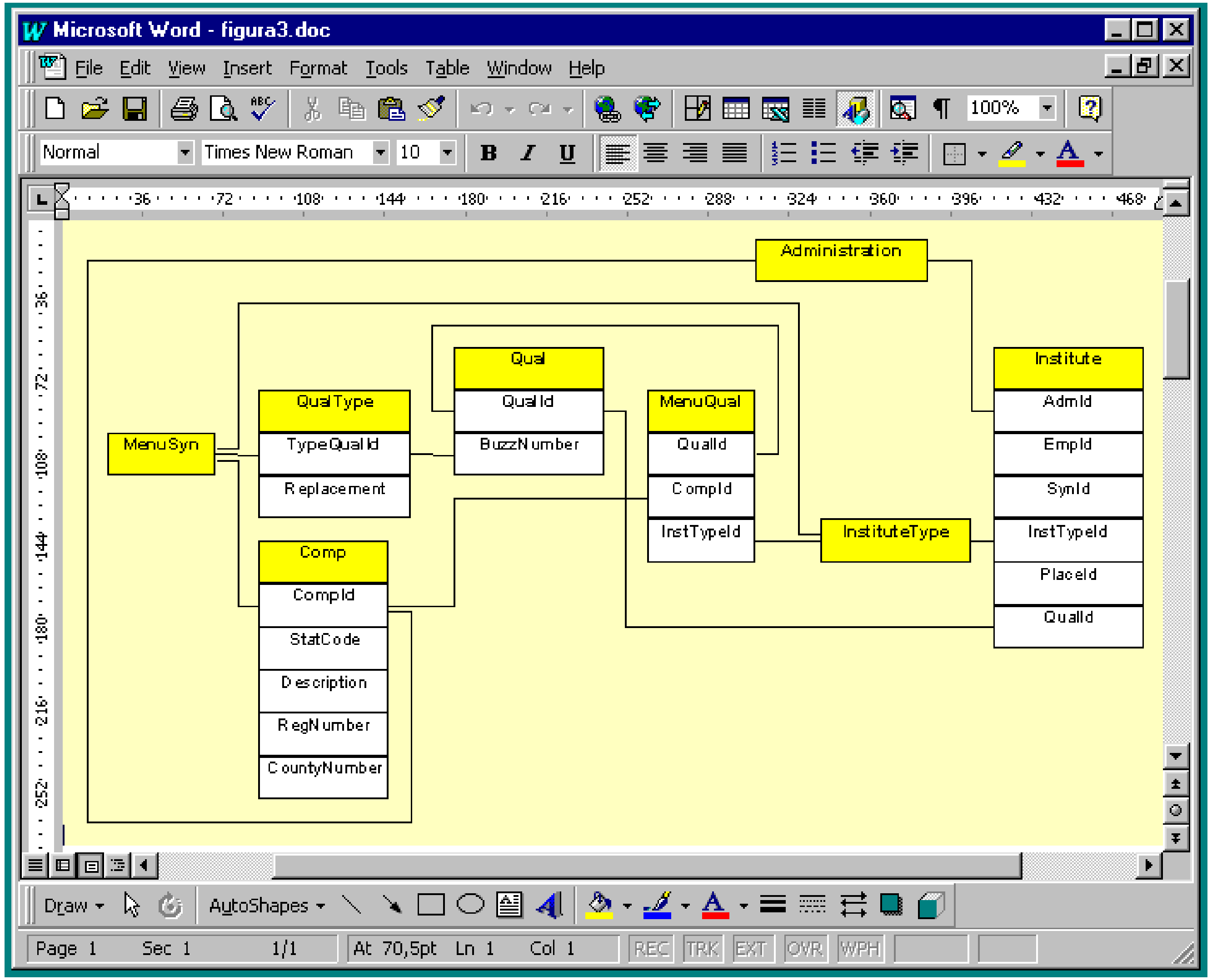
- Some graph drawing libraries that implement the topology-shape-metrics approach or other orthogonal drawing algorithms:
  - GDToolkit [*G. Di Battista, W. Didimo*: GDToolkit. Handbook of Graph Drawing and Visualization 2013: 571-597]
  - OGDF [*M. Chimani, C. Gutwenger, M. Jünger, G. W. Klau, K. Klein, P. Mutzel*: The Open Graph Drawing Framework (OGDF). Handbook of Graph Drawing and Visualization 2013: 543-569]
  - Tom Sawyer Software (www.tomsawyer.com/)
  - Yfiles [*R. Wiese, M. Eiglsperger, M. Kaufmann*: yFiles - Visualization and Automatic Layout of Graphs. Graph Drawing Software 2004: 173-191]

Applications: Hermes



A. Carmignani, G. Di Battista, W. Didimo, F. Matera, M. Pizzonia: Visualization of the High Level Structure of the Internet with HERMES. J. Graph Algorithms Appl. 6(3): 281-311 (2002)

Applications: DBDraw

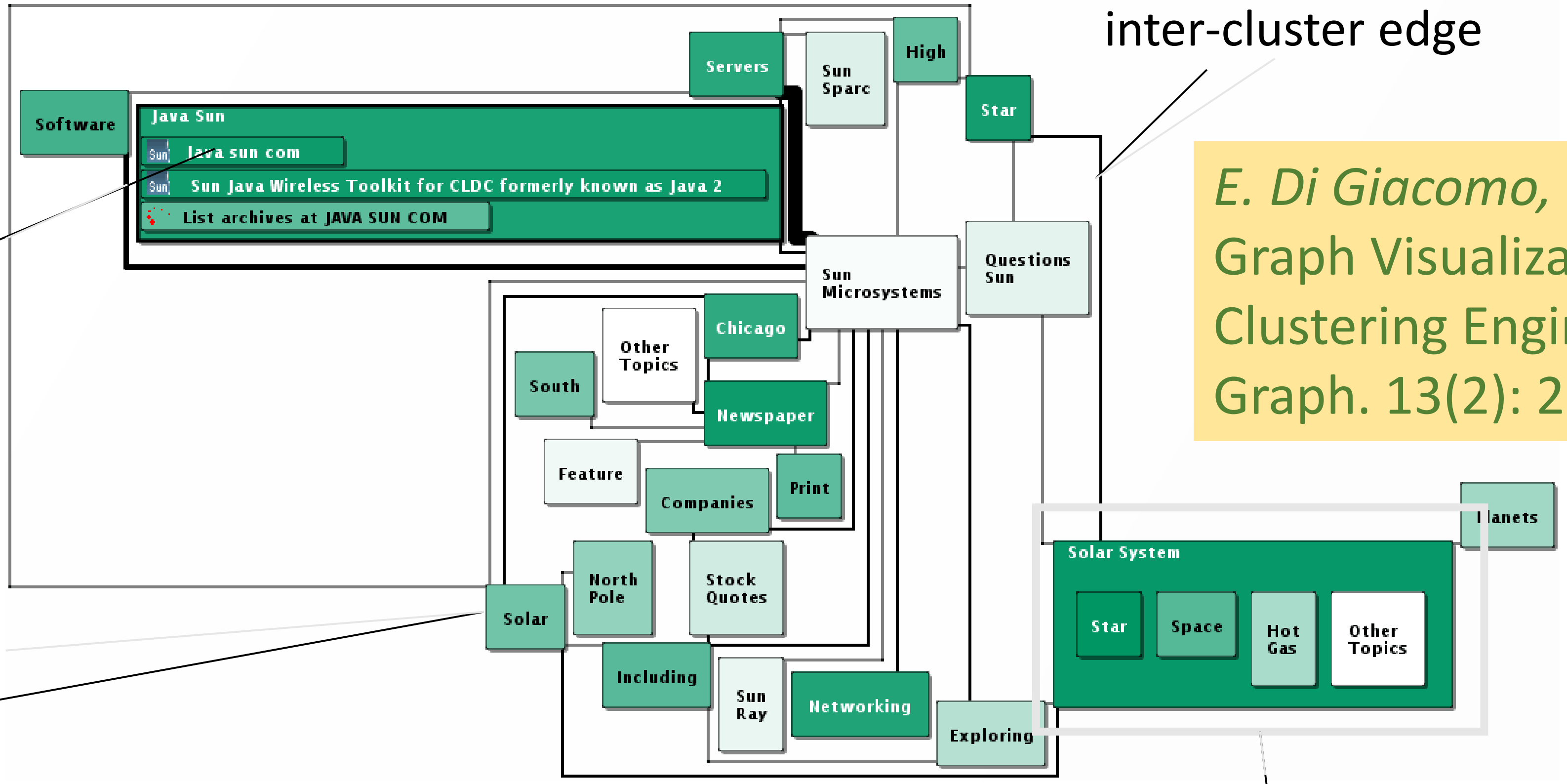


G. Di Battista, W. Didimo, M. Patrignani, M. Pizzonia: Drawing database schemas. Softw., Pract. Exper. 32(11): 1065-1098 (2002)

video

Applications: WhatsOnWeb (WOW)

inter-cluster edge



url

cluster

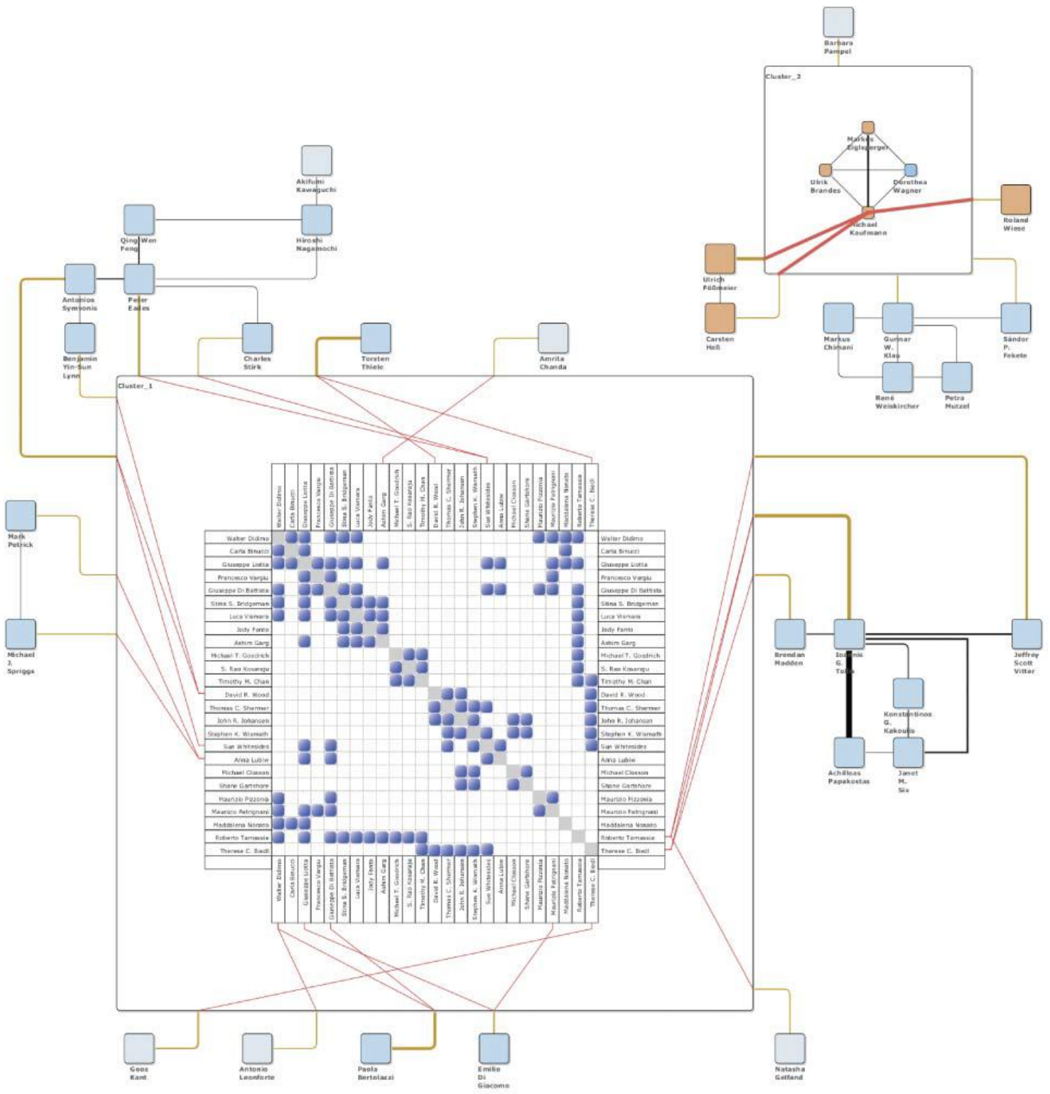
E. Di Giacomo, W. Didimo, L. Grilli, G. Liotta: Graph Visualization Techniques for Web Clustering Engines. IEEE Trans. Vis. Comput. Graph. 13(2): 294-304 (2007)

video

inclusion (parental edge)

most relevant least relevant

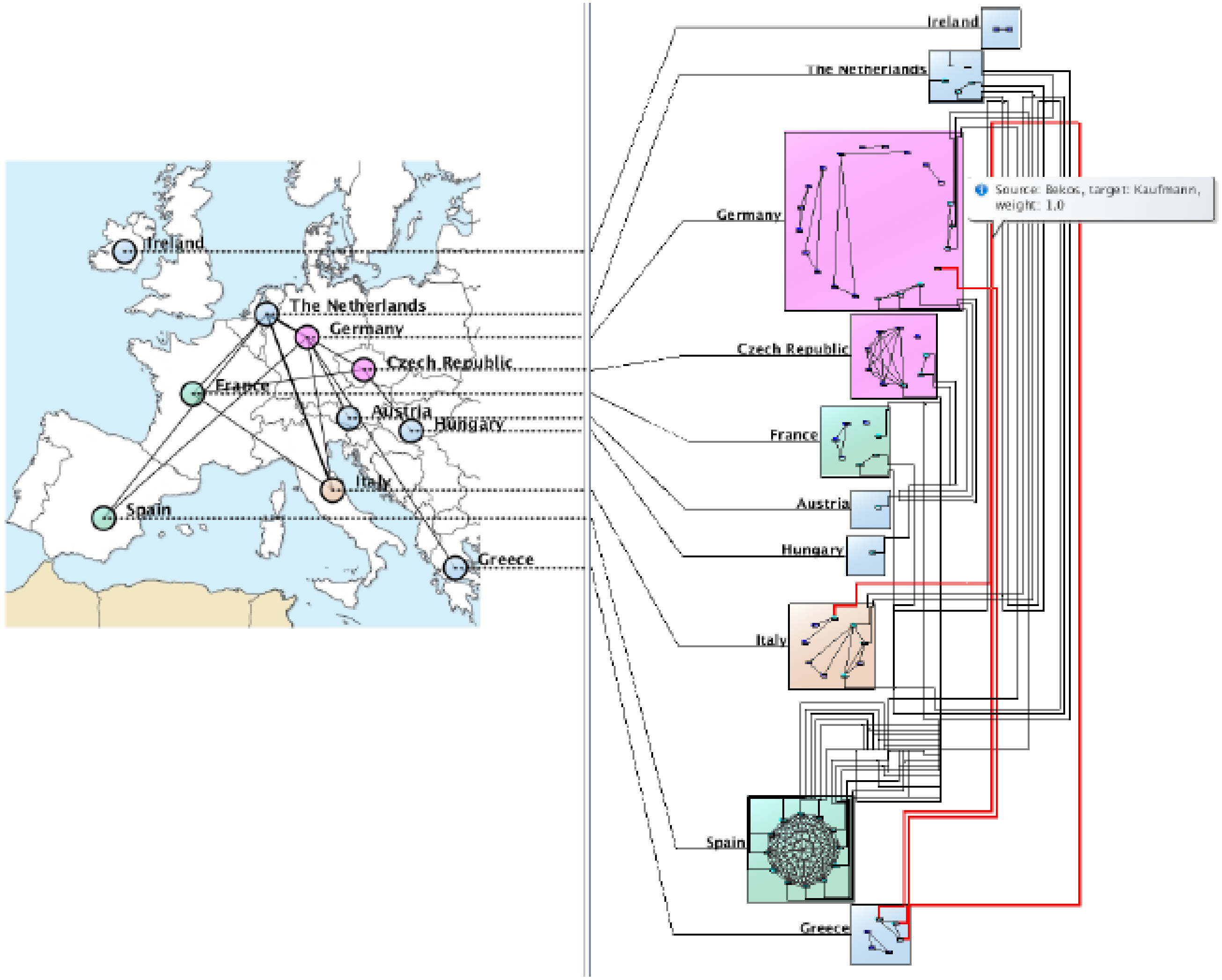
Applications: Hybrid visualizations



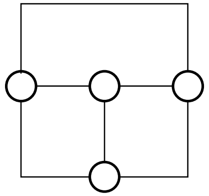
V. Batagelj, F. Brandenburg, W. Didimo, G. Liotta, P. Palladino, M. Patrignani:
Visual Analysis of Large Graphs Using (X,Y)-Clustering and Hybrid Visualizations. IEEE Trans. Vis. Comput. Graph. 17(11): 1587-1598 (2011)

video

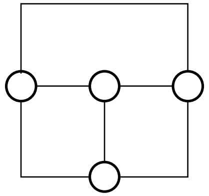
Applications: MatchOMan (MOM)



E. Di Giacomo, W. Didimo, G.Liotta, P. Palladino: Visual Analysis of One-To-Many Matched Graphs. J. Graph Algorithms Appl. 14(1): 97-119 (2010)

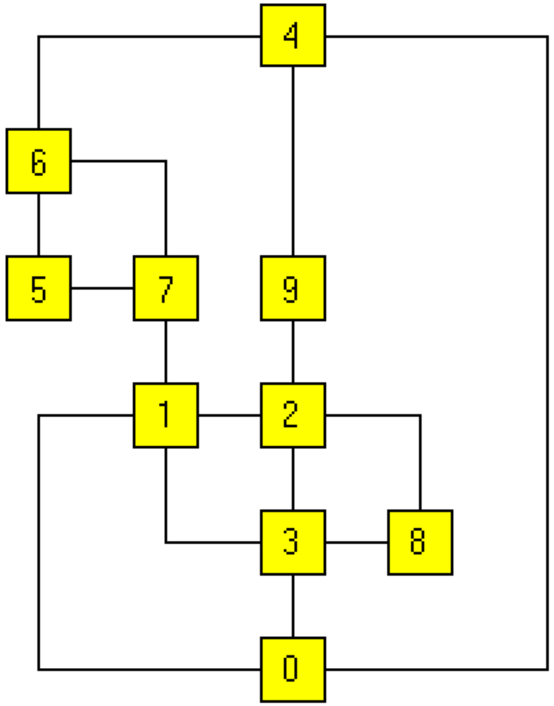


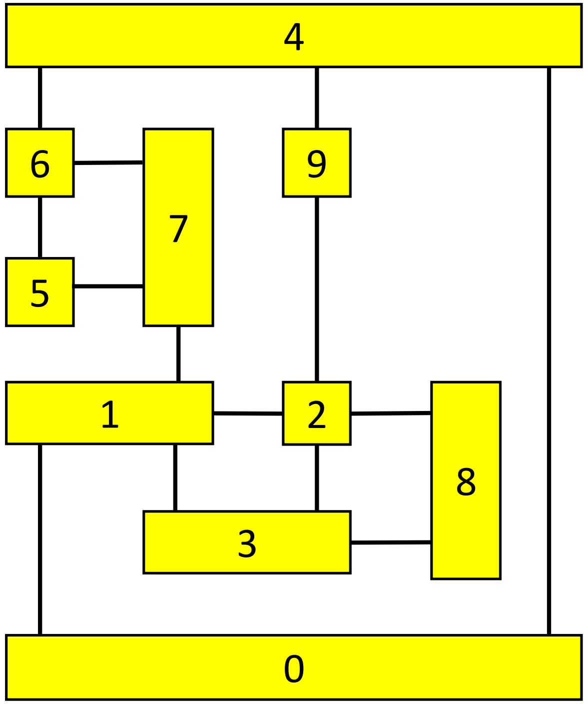
Part 1.3
Ortho-polygon Drawings



From edge complexity to vertex complexity

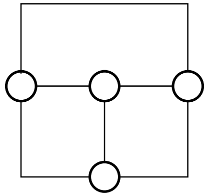
- If vertices are drawn as polygons, one may save edge bends



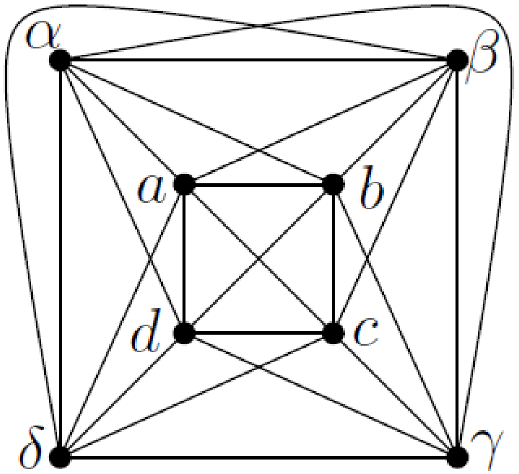


rectangle visibility representation

A. M. Dean and J. P. Hutchinson. Rectangle-visibility representations of bipartite graphs. Discrete Appl. Math., 75(1):9–25, (1997)

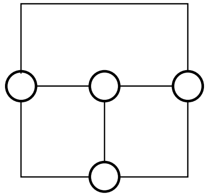


From edge complexity to vertex complexity



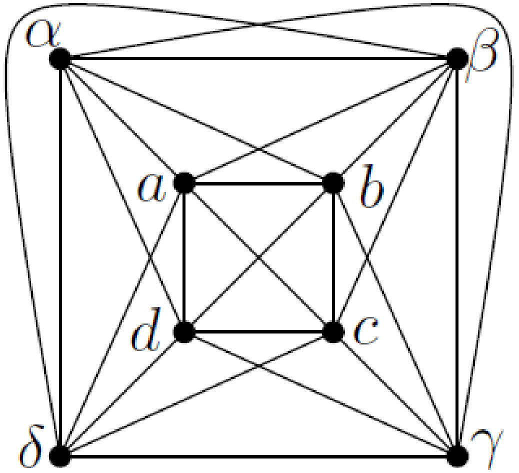
1-plane graph that does not admit a rectangle visibility representation

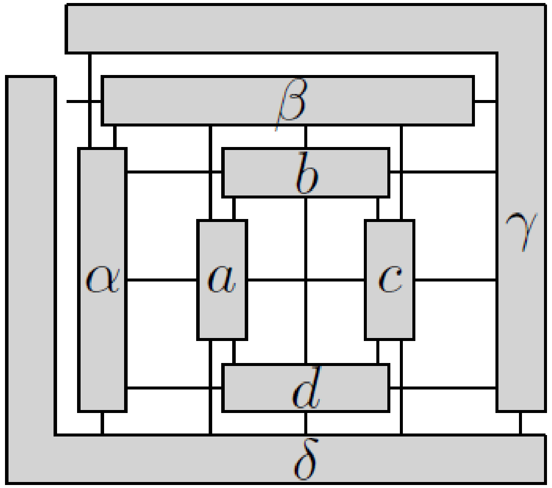
- It can be tested in polynomial time if an embedded graph admits a rectangle visibility representation
  – *T. C. Biedl, G. Liotta, F. Montecchiani*: Embedding-Preserving Rectangle Visibility Representations of Nonplanar Graphs. Discrete & Computational Geometry 60(2): 345-380 (2018)



Ortho-polygon drawings

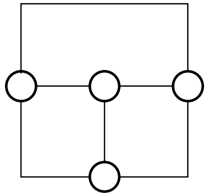
- Generalization of rectangle visibility representations – a vertex can be an ortho-polygon with both convex and reflex corners





ortho-polygon drawing with vertex-complexity 1

vertex-complexity = maximum number of reflex corners in a vertex

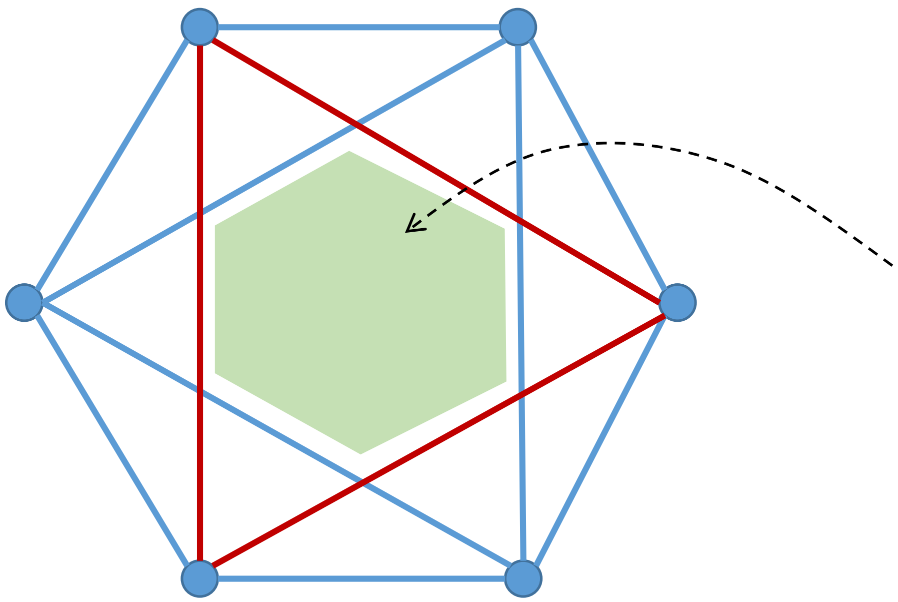


Ortho-polygon drawings: Existence

- Not all embedded graphs admit an ortho-polygon drawing
- Necessity:
  - the embedded graph is biplanar, i.e., the edge set can be partitioned into two planar subsets (e.g., vertical and horizontal in the drawing)

Ortho-polygon drawings: Existence

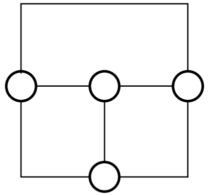
- Biplanarity is not sufficient



this face is not realizable, because it should have more than 4 convex corners and no reflex corners

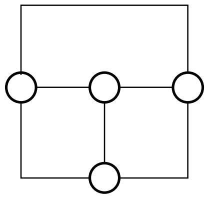
Ortho-polygon drawings: Existence

- Not all embedded graphs admit an ortho-polygon drawing
- Necessity:
  - the embedded graph is biplanar, i.e., the edge set can be partitioned into two planar subsets (e.g., vertical and horizontal in the drawing)
  - each face with only crossing-vertices has degree four



Ortho-polygon drawings: Existence

- **Questions:**
  - can we test whether an embedded graph admits an ortho-polygon drawing?
  - can we compute (if any) an ortho-polygon drawing with minimum vertex complexity? (i.e., minimum number of reflex corners per vertex)
  - if yes, can we also minimize the total number of reflex corners within the minimum vertex complexity?

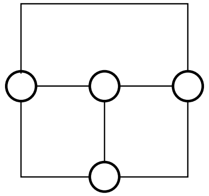


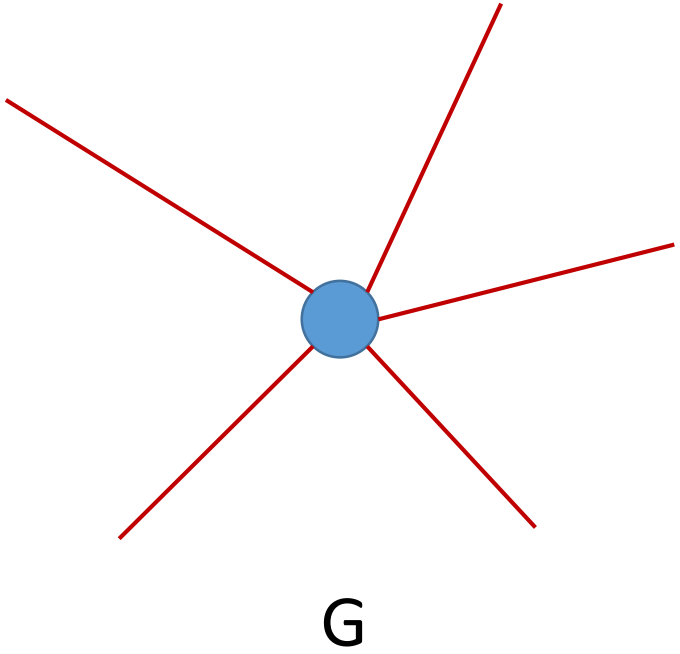
Ortho-polygon drawings: Existence

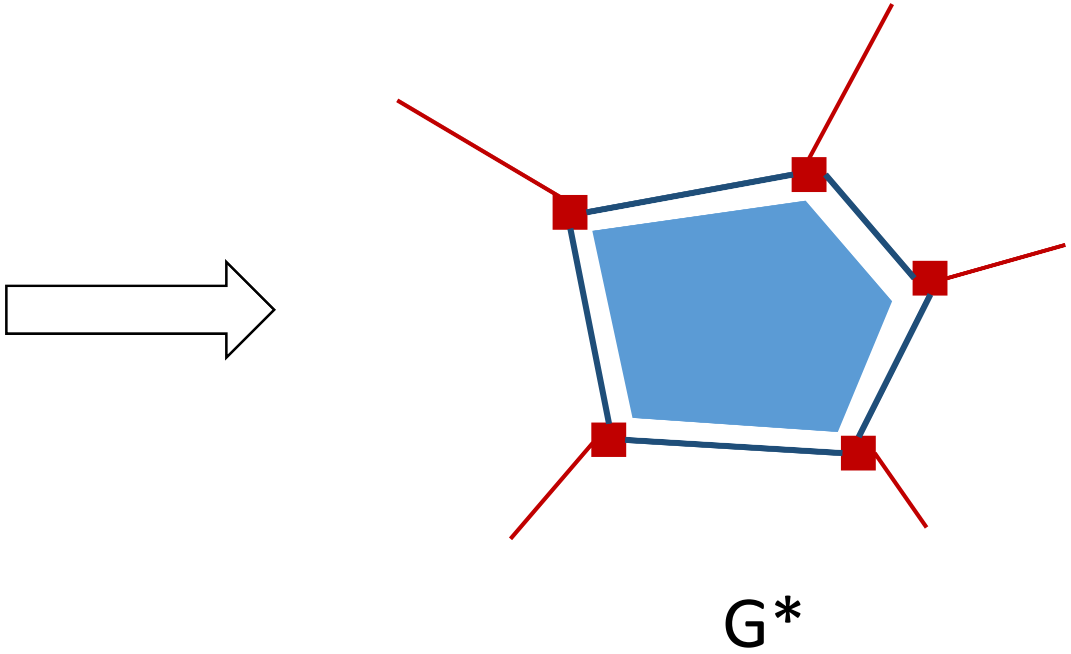
- **Questions:**
  - can we test whether an embedded graph admits an ortho-polygon drawing?
  - can we compute (if any) an ortho-polygon drawing with minimum vertex complexity? (i.e., minimum number of reflex corners per vertex)
  - if yes, can we also minimize the total number of reflex corners within the minimum vertex complexity?

- **Answer**: yes, by using a variant of Tamassia's flow network we can solve everything in polynomial time
  - *E. Di Giacomo, W. Didimo, W. S. Evans, G. Liotta, H. Meijer, F. Montecchiani, S. K. Wismath*: Ortho-polygon Visibility Representations of Embedded Graphs. Algorithmica 80(8): 2345-2383 (2018)

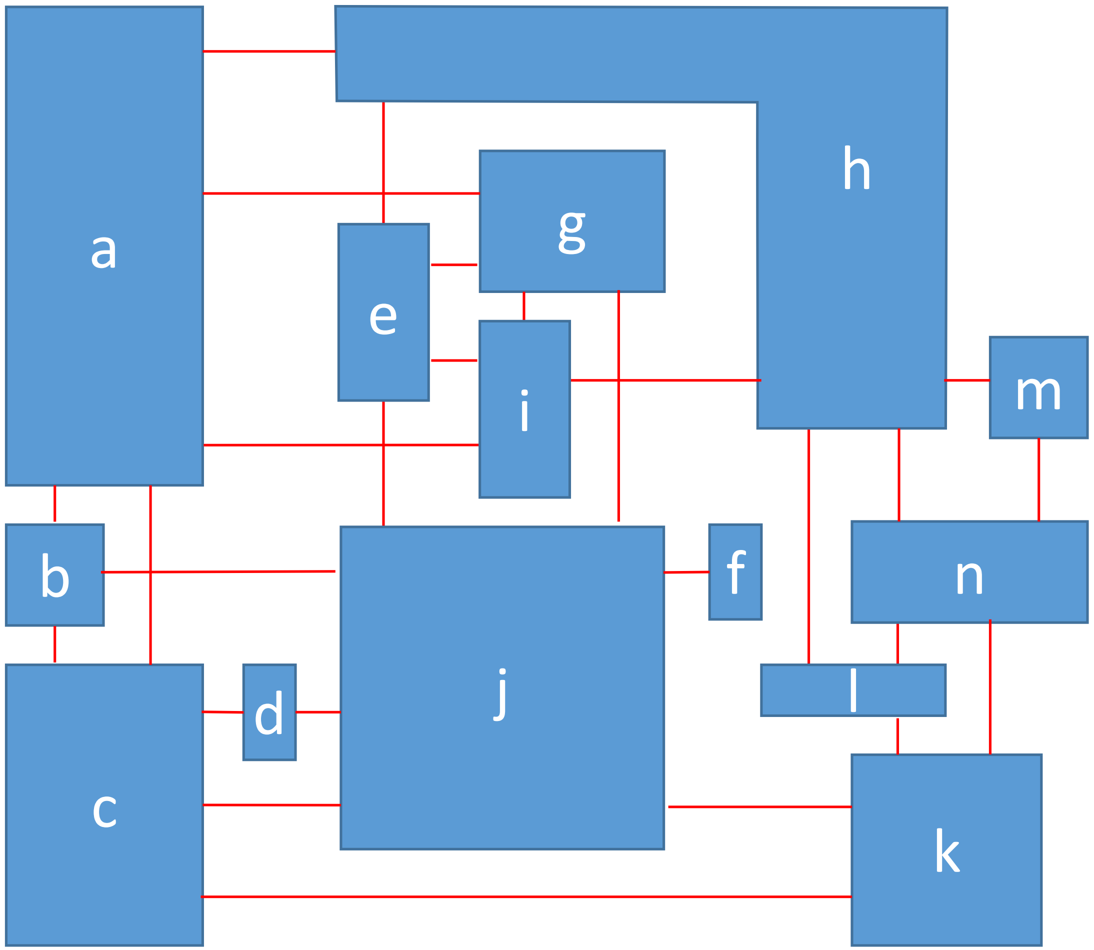
Ortho-polygon drawings: Expansion graph



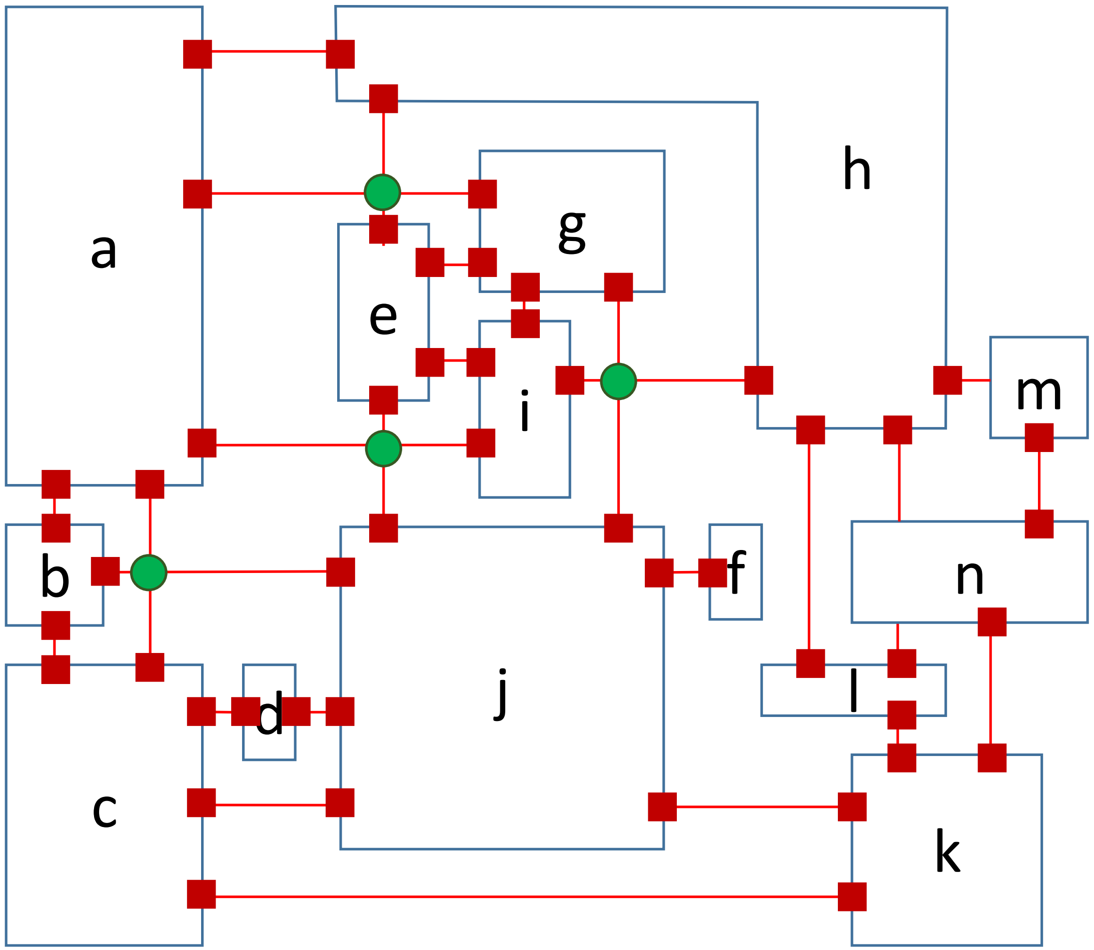




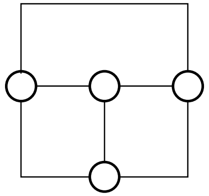
Ortho-polygon drawings: Characterization



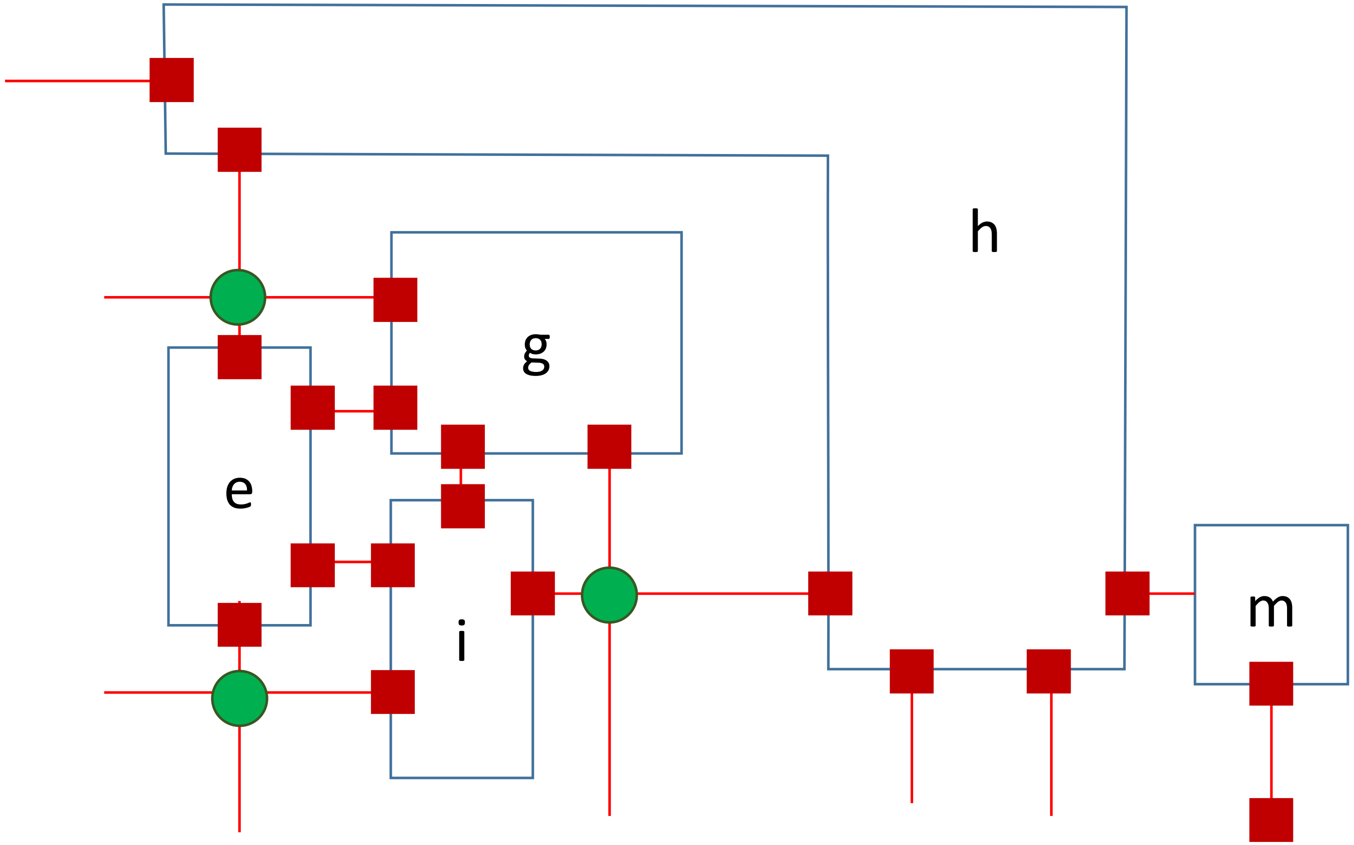
ortho-polygon drawing of G



orthogonal drawing of G*

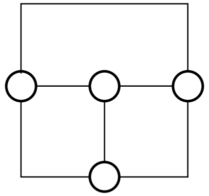


Ortho-polygon drawings: Characterization

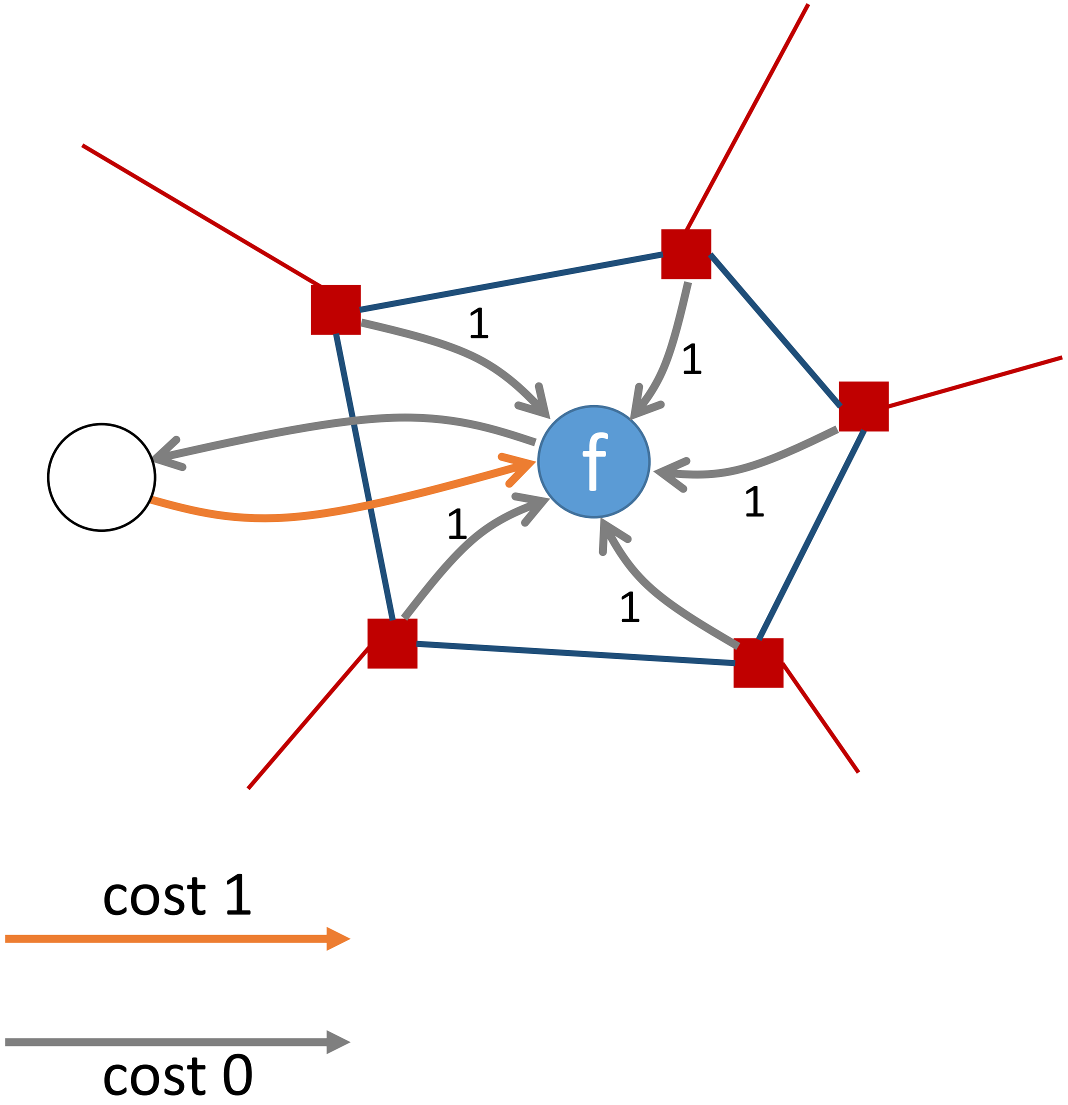


orthogonal drawing of G*

- P1. each **red** vertex has a 180° angle inside its node-face
- P2. each **real edge** has no bend

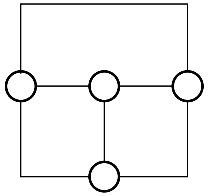


Ortho-polygon drawings: Flow network

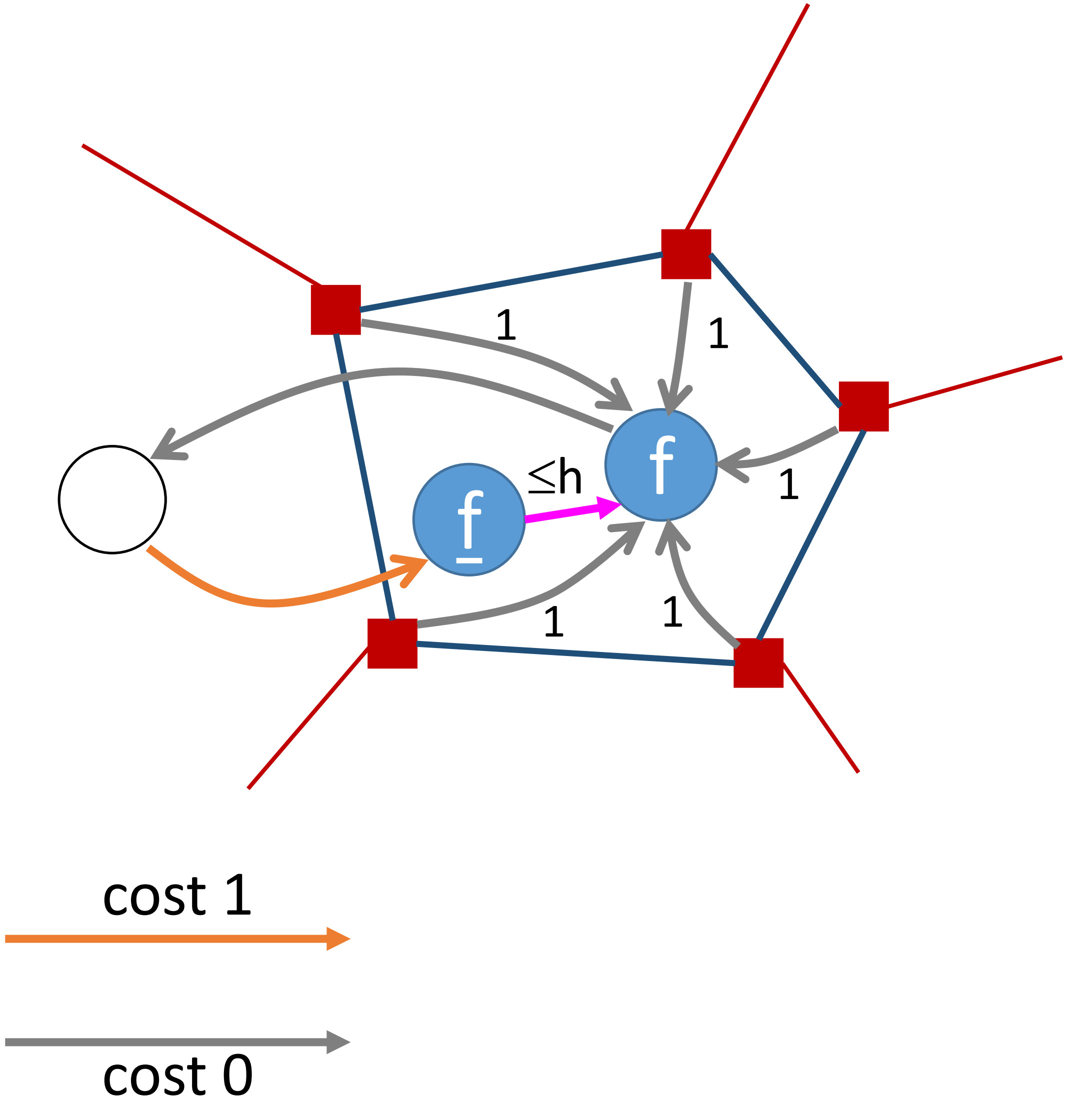


- P1. each **red** vertex has a 180° angle inside its node-face
- P2. each **real edge** has no bend

test and computation of an ortho-polygon drawing, with minimum number of reflex corners in total

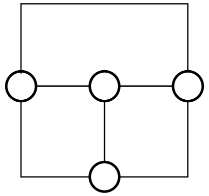


Ortho-polygon drawings: Flow network

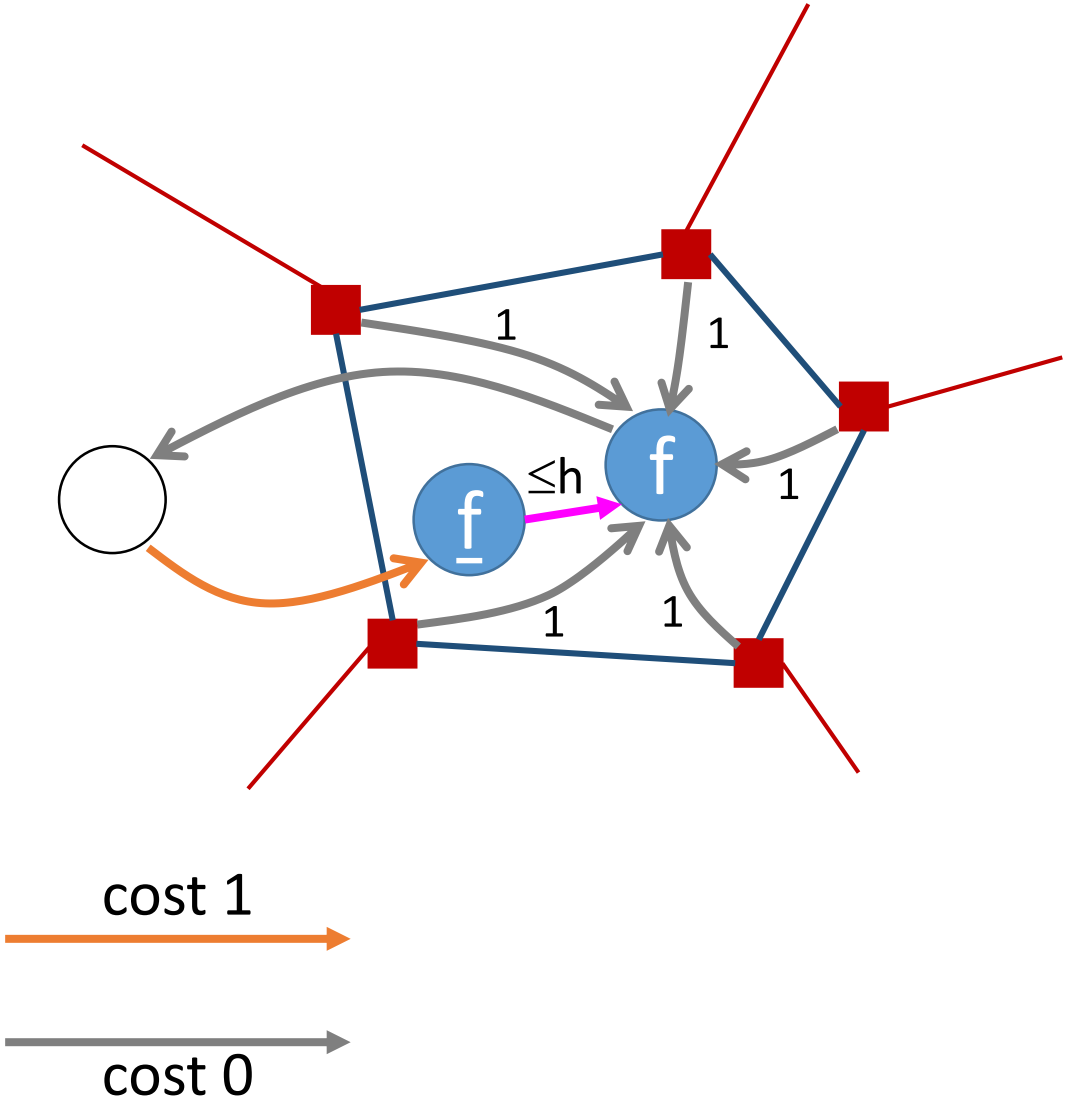


- P1. each **red** vertex has a 180° angle inside its node-face
- P2. each **real edge** has no bend

test and computation of an ortho-polygon drawing, with minimum number of reflex corners in total and at most h reflex corners per face



Ortho-polygon drawings: Flow network



each of the four units of flow corresponding to a convex corner in f will traverse a node-face at most once

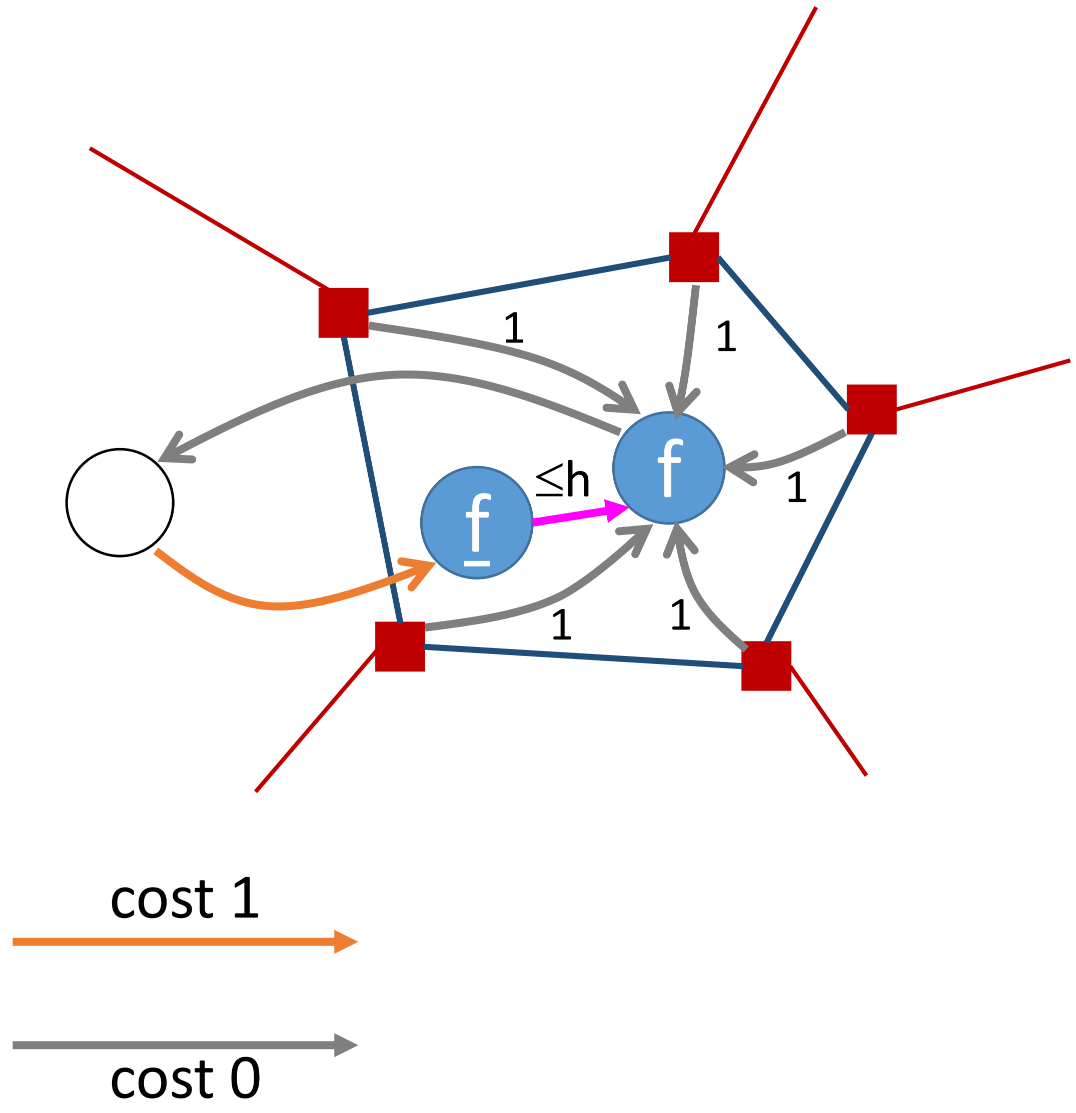
⇩

$h \leq 4n$

⇩

apply a binary search within [0,4n] for the determining the best value for h

Ortho-polygon drawings: Flow network



Computational complexity

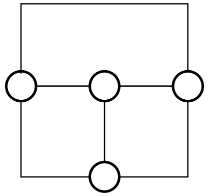
- flow network size = O(n)
- flow value = O(n)
- flow cost χ = $O(n^2)$

⇩

Min-cost flow algorithm time for fixed h:
$O(\chi^{3/4}\, n \log^{1/2} n) = O(n^{5/2} \log^{1/2} n)$

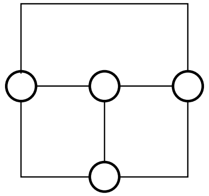
⇩

Min-cost flow algorithm time × binary-search time (O(log n)):
$O(n^{5/2} \log^{3/2} n)$



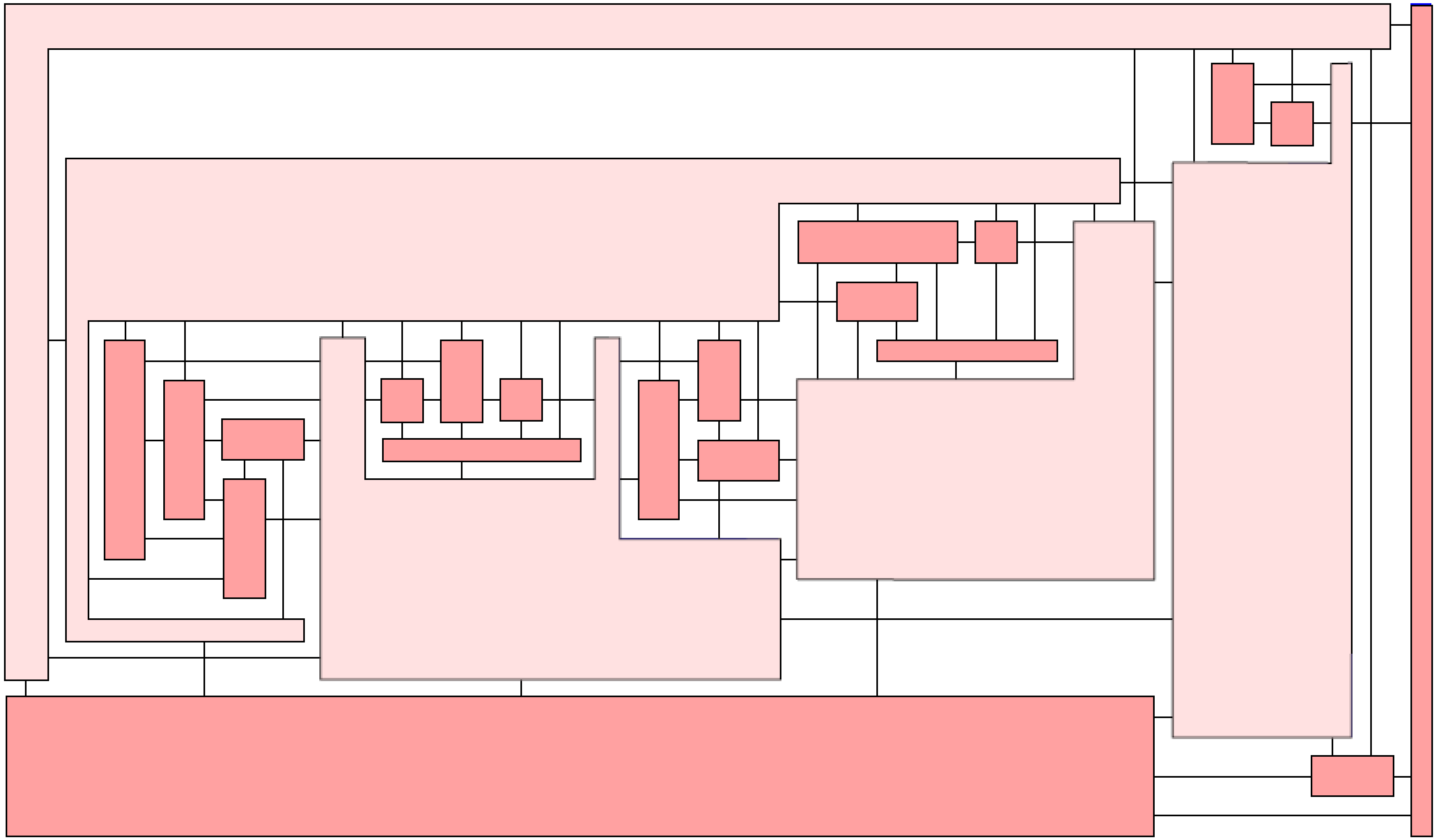
Ortho-polygon drawings: 1-plane graphs

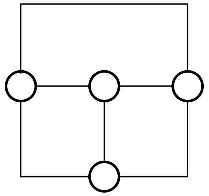
- **Remarks:**
  - every 1-plane graph admits an ortho-polygon drawing:
    - 2-connected 1-plane graphs may require vertex complexity $\Omega(n)$
    - 3-connected 1-plane graphs may require vertex complexity 2
    - 3-connected 1-plane graphs always admit an ortho-polygon drawing with vertex complexity at most 5 [*G. Liotta, F. Montecchiani, A. Tappini*: Ortho-Polygon Visibility Representations of 3-Connected 1-Plane Graphs. Graph Drawing 2018: 524-537]



Ortho-polygon drawings: Example

2-connected 1-plane graph with vertex complexity 3





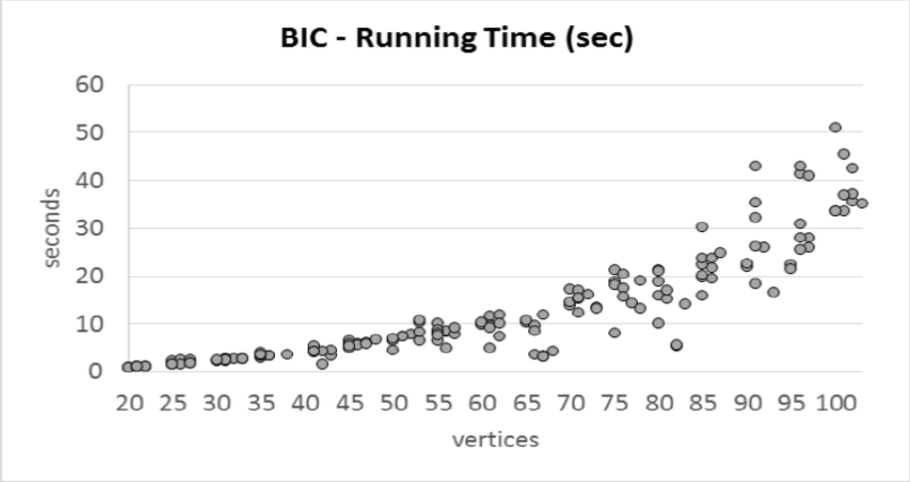
Ortho-polygon drawings: Open problems

- **Problem 1.** Reduce the time-complexity of computing ortho-polygon drawings of minimum vertex complexity on general graphs

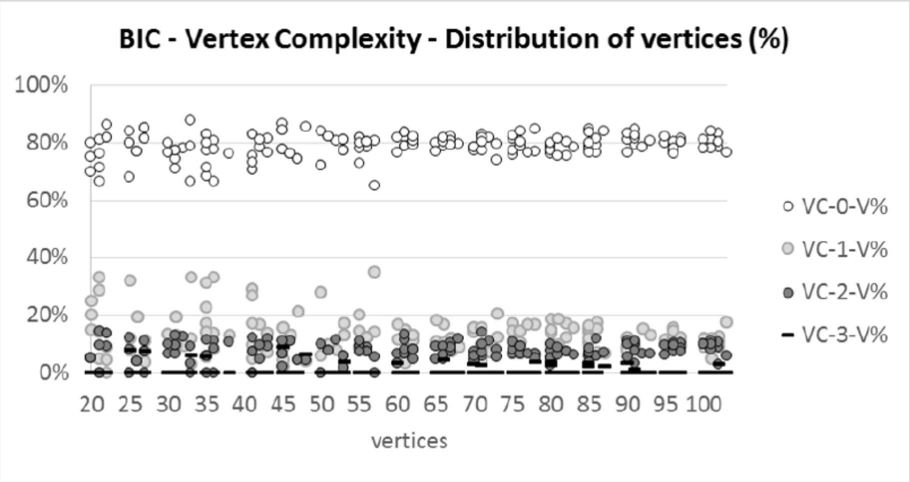
- **Problem 2.** Reduce the theoretical gap between upper bound (5) and lower bound (2) on the vertex complexity of ortho-polygon drawings of 3-connected 1-planar graphs

Ortho-polygon drawings: Experiments

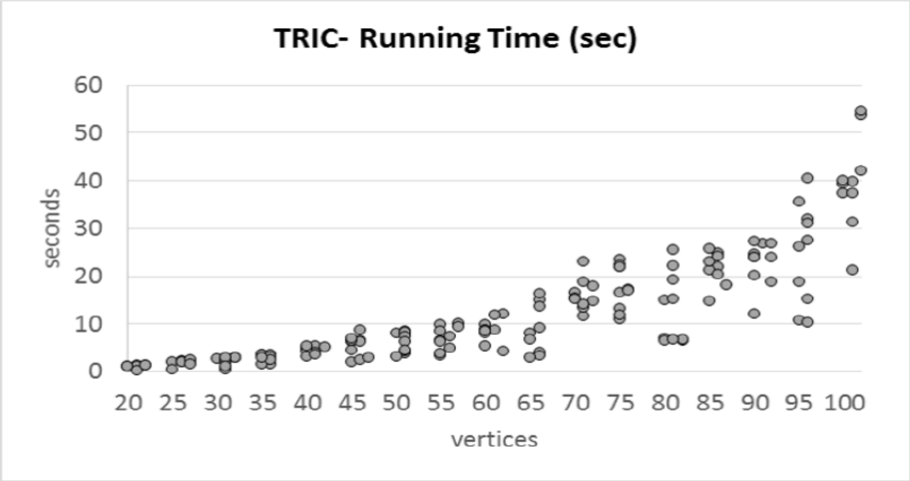
(a) Running time.



(b) % of vertices with complexity i (VC-i-V%).



(c) Running time.



(d) % of vertices with complexity i (VC-i-V%).