Orthogonal Drawings of Graphs and Their Relatives

Part 3 – Relatives of Orthogonal Drawings

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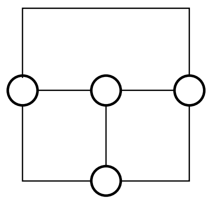
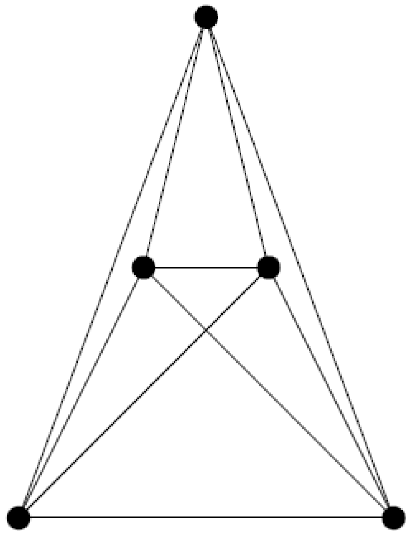


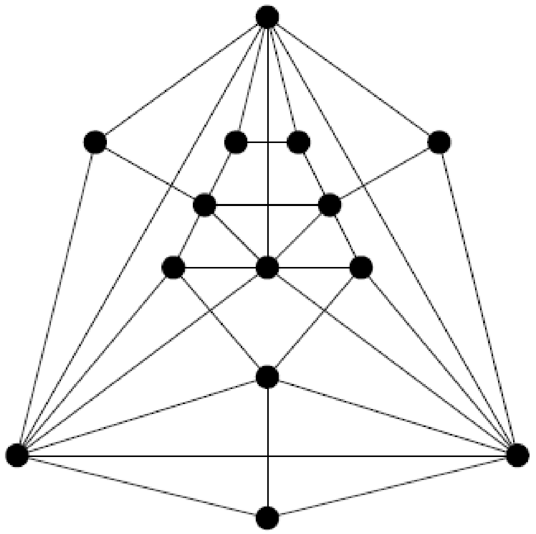
Right Angle Crossing Drawings

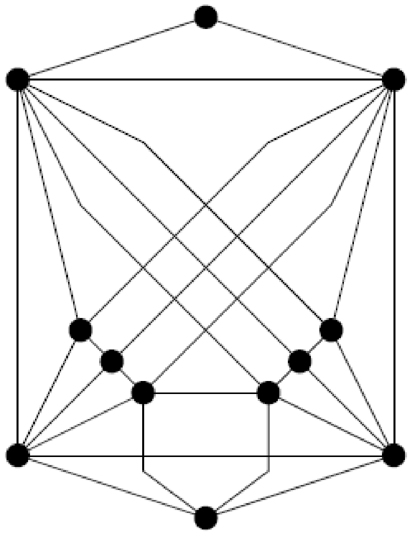
RAC drawings

Right Angle Crossing drawing (RAC drawing) :

- each vertex is drawn as a point in the plane
- each edge is drawn as a poly-line
- edges cross at right angles



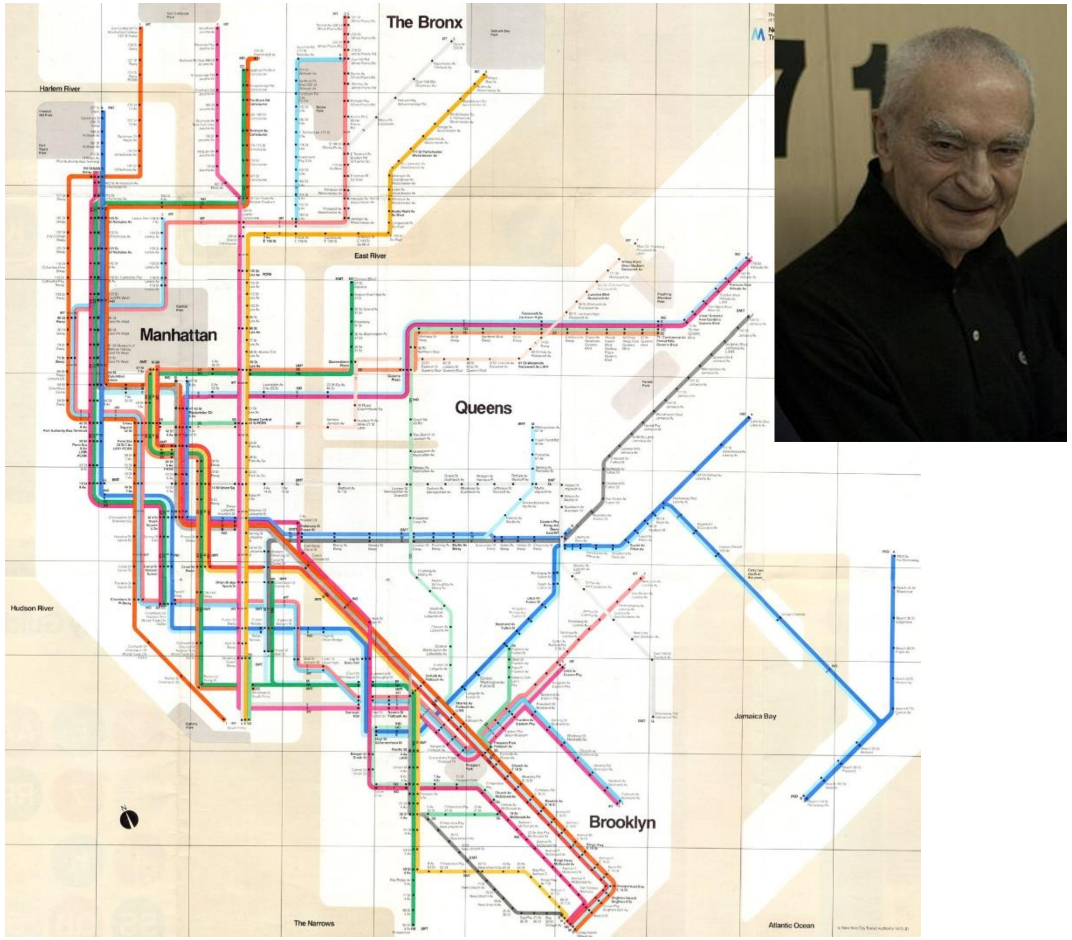




RAC drawings: Motivation

- Cognitive experiments suggest a positive correlation between large angle crossings and human understanding of graph layouts
  - W. Huang: Using eye tracking to investigate graph layout effects. APVIS (2007)
  - W. Huang, S.H. Hong, P. Eades: Effects of crossing angles. PacificVis (2008)
  - W. Huang, P. Eades, S.H. Hong: Larger crossing angles make graphs easier to read. JVLC 25(4) (2014)

RAC drawings: Witnesses



New York City subway map (1973)
Massimo Vignelli (1931-2014)

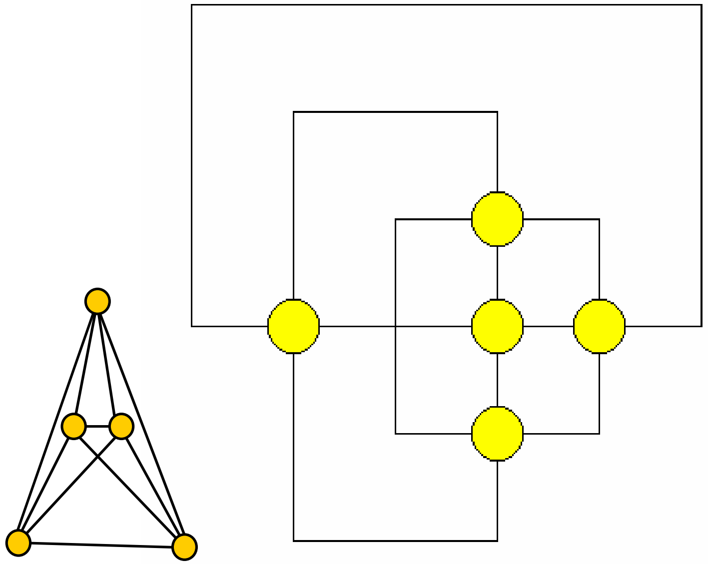
.. Reinterpreted by another artist

RAC drawings: Witnesses

Walking through Boston (2017) – 558 Washington St

Orthogonal and RAC drawings

Orthogonal drawings are "ancestors" and special cases of RAC drawings



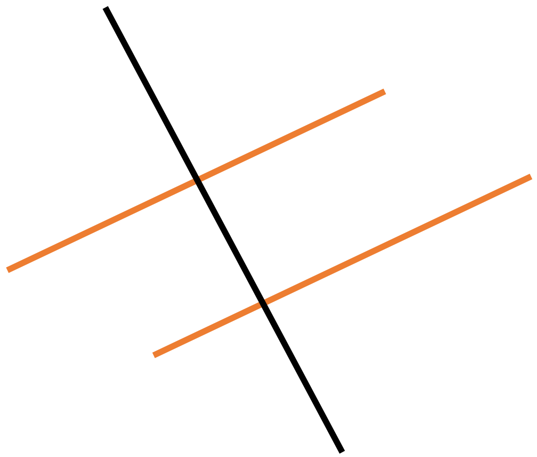
If vertices are represented as points, orthogonal drawings require vertex-degree at most 4, and may require higher curve complexity

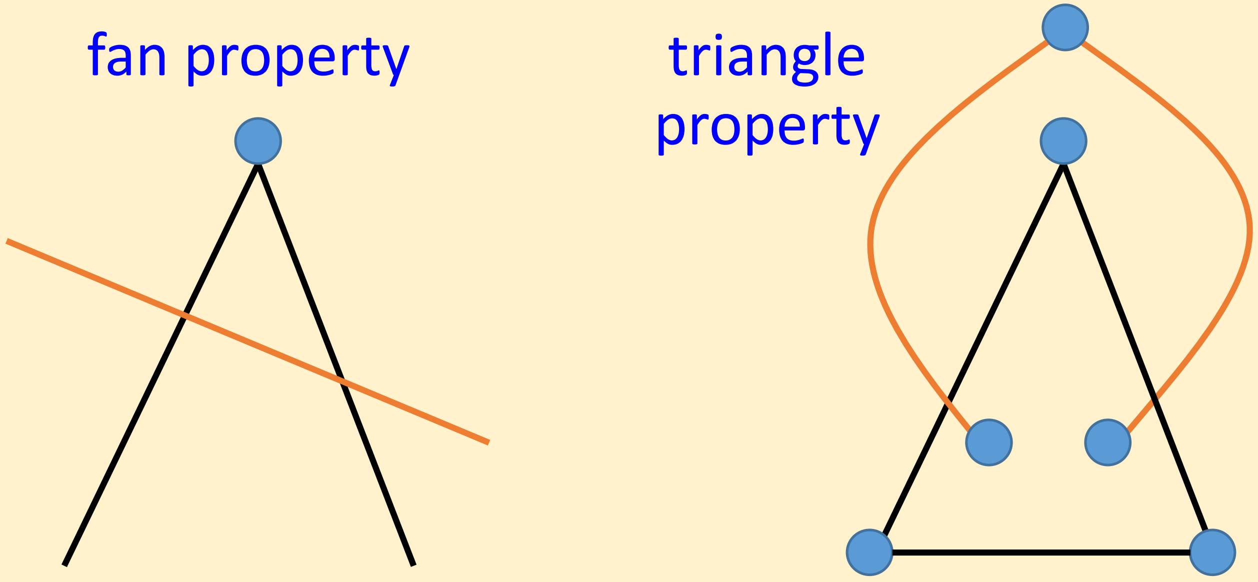
Research lines

- Edge density
  - What is the maximum number of edges in a RAC drawing?
- Drawing algorithms
  - What is the complexity of testing if a graph is RAC drawable?
  - Can we design algorithms that compute "readable" RAC drawings?
- Inclusion relationships
  - Are there interesting inclusion relationships between RAC drawable graphs and other classes of graphs that admit drawings with specific forbidden types of crossings?

Terminology and elementary properties

- k-bend RAC drawing: RAC drawing with at most k bends per edge
- straight-line RAC drawing ⇔ 0-bend RAC drawing

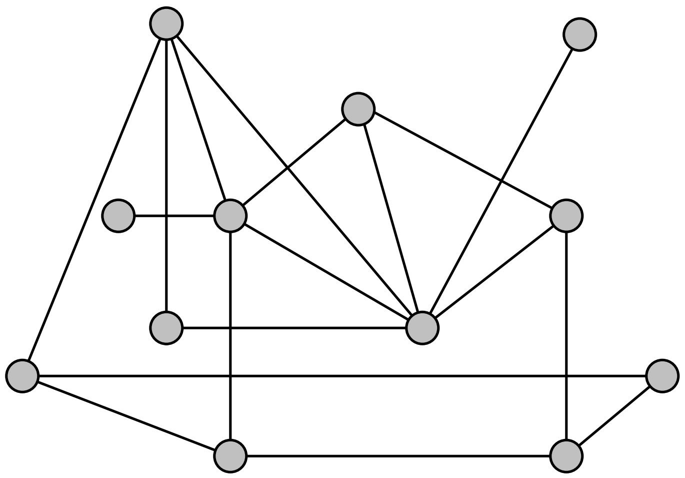




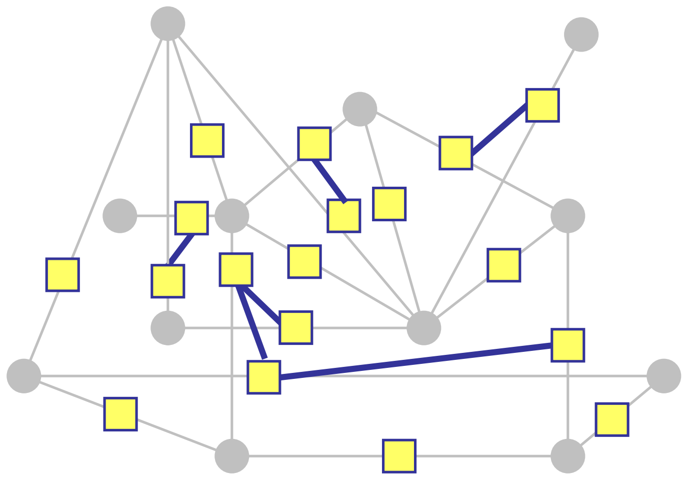
forbidden in a 0-bend RAC drawable *plane* graph

Crossing graph

CG-Lemma. The crossing graph of a straight-line RAC drawing is bipartite



a (RAC) drawing Γ

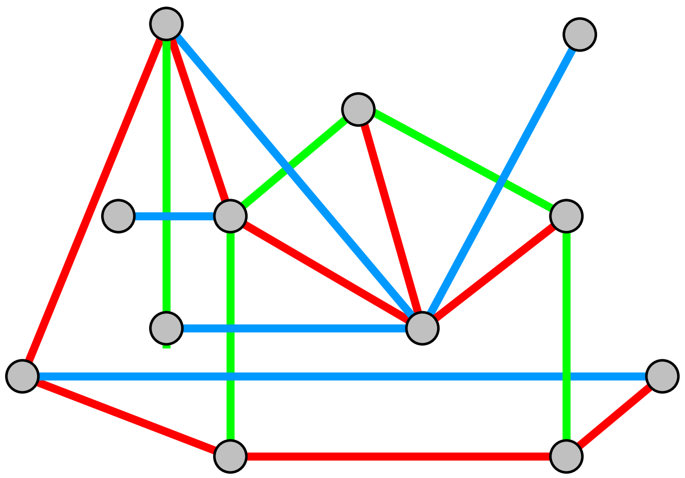


crossing graph of Γ

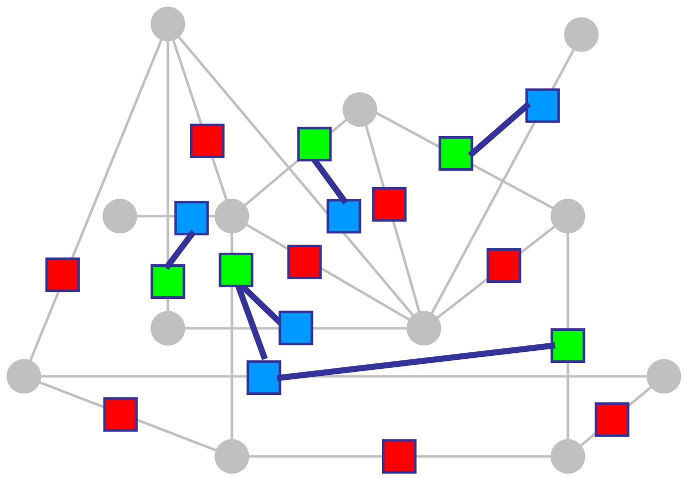
- a vertex for each edge
- an edge for each pair of crossing edges

Crossing graph

CG-Lemma. The crossing graph of a straight-line RAC drawing is bipartite



a (RAC) drawing Γ

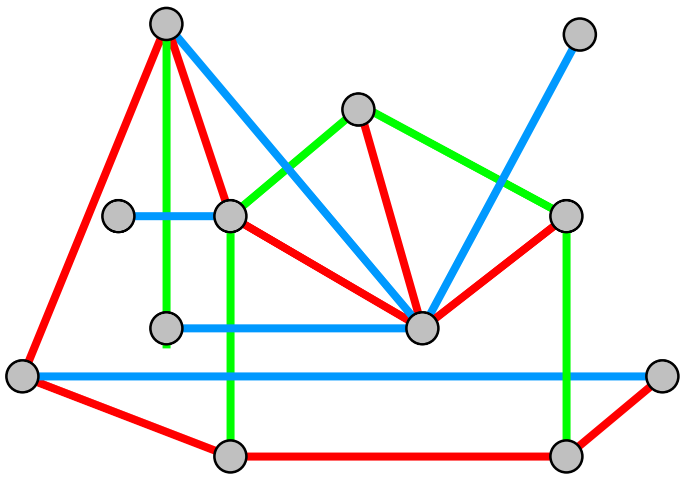


crossing graph of Γ

- red edges do not cross (they correspond to isolated vertices in the crossing graph)
- each green edge crosses with a blue edge
  - red-blue (embedded planar) graph = red + blue edges
  - red-green (embedded planar) graph = red + green edges

Crossing graph

CG-Lemma. The crossing graph of a straight-line RAC drawing is bipartite



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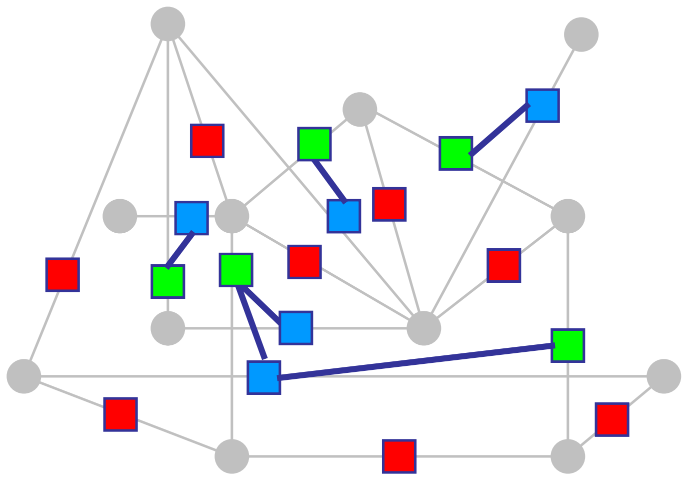
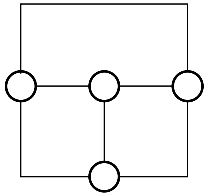


crossing graph of Γ

- Immediate consequence of CG-Lemma: $\mathbf{m \leq 6n - 12}$
  - n = number of vertices;
  - m = number of edges



Edge density

C_k = class of graphs that admit a k-bend RAC drawing

- **Theorem 0.** $G \in C_0 \Rightarrow m \leq 4n - 10$ (tight)
  - *W. Didimo, P. Eades, G. Liotta*: Drawing graphs with right angle crossings. Theor. Comput. Sci. 412(39) (2011)
- **Theorem 1.** $G \in C_1 \Rightarrow m \leq 6.5n - 13$
- **Theorem 2.** $G \in C_2 \Rightarrow m \leq 74.2n$
  - *K. Arikushi, R. Fulek, B. Keszegh, F. Moric, C. D. Tóth*: Graphs that admit right angle crossing drawings. Comput. Geom. 45(4) (2012)
- **Theorem 3.** $G \in C_3 \Rightarrow m = \text{any}$ (see later …)
  - *W. Didimo, P. Eades, G. Liotta*: Drawing graphs with right angle crossings. Theor. Comput. Sci. 412(39) (2011)

Edge density: 0-bend drawings

Theorem 0. A 0-bend RAC drawing with $n \geq 4$ vertices has at most $4n-10$ edges. Also, for any $k \geq 3$ there exists a straight-line RAC drawing with $n = 3k-5$ vertices and $4n-10$ edges

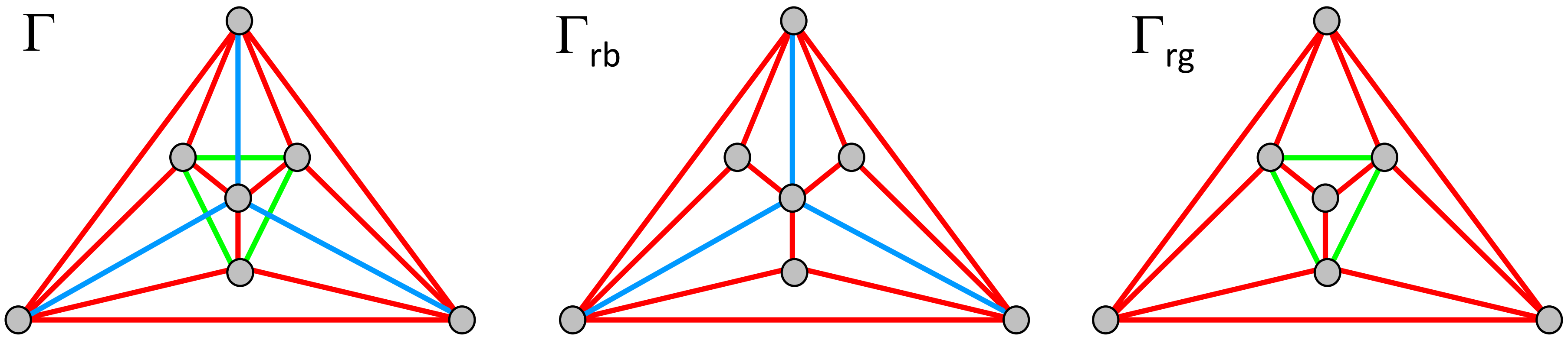
Proof ingredients.

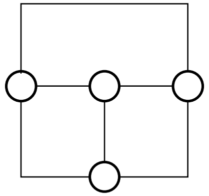
- part I (upper bound): an interesting property of the red-blue and the red-green graphs + several applications of Euler's formula
- part II (lower bound): constructive technique

Theorem 0 – A technical lemma

G is C_0-maximal if: (i) $G \in C_0$; (ii) G plus an edge $\notin C_0$

Face-Lemma. Let Γ be a 0-bend RAC drawing of a C_0-maximal graph G, and let Γ_{rb} and Γ_{rg} be a red-blue and a red-green subdrawing of Γ, respectively. Then Γ_{rb} and Γ_{rg} have only external red edges and every internal face has at least two red edges





Theorem 0 – Proof of part I

- G = C_0-maximal
- Γ = 0-bend RAC drawing of G with red-green-blue coloring

Notation:

- ω = number of edges of the external boundary of Γ
- m_r , m_b , m_g = number of red, blue, and green edges of Γ
- f_{rb} = number of faces of the red-blue graph Γ_{rb}

Assumption:

- $m_g \leq m_b$

Theorem 0 – Proof of part I

By the Face-Lemma, each internal face of Γ_{rb} has at least 2 red edges and the external face of Γ_{rb} has ω red edges; also each edge is shared by at most 2 distinct faces $\Rightarrow$

$$2m_r \geq 2(f_{rb} - 1) + \omega \quad \Leftrightarrow \quad m_r \geq f_{rb} - 1 + \omega/2$$

By Euler's formula for planar graphs $\Rightarrow$ $m_r + m_b \leq n + f_{rb} - 2$

$$m_b \leq n - 1 - \omega/2$$

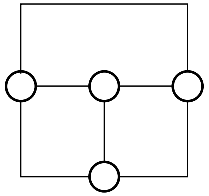
Theorem 0 – Proof of part I

$$m_b \leq n - 1 - \omega/2$$

Γ_{rg} has the same external face as Γ_{rb} and this face consists of ω edges $\Rightarrow$

$$m_r + m_g \leq 3n - 6 - (\omega - 3) \Leftrightarrow m_r + m_g \leq 3n - 3 - \omega$$

$$m \leq 4n - 4 - 3/2\ \omega$$



Theorem 0 – Proof of part I

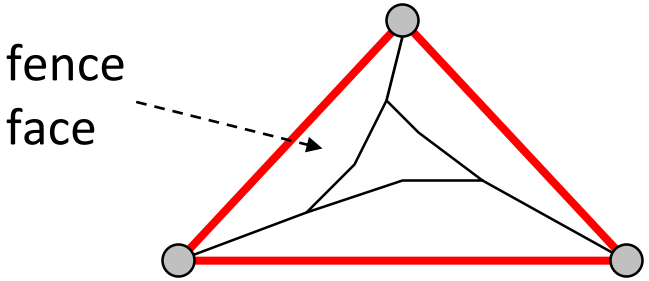
$m \leq 4n - 4 - 3/2\ \omega$

Two cases are possible:

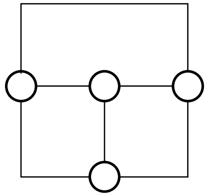
<u>Case 1</u>: $\omega \geq 4 \Rightarrow m \leq 4n - 10$ ✓

<u>Case 2</u>: $\omega = 3$

consider the internal faces of Γ_{rb} that share at least one edge with the external face (fence faces)



there are at least 1 and at most 3 fence faces



Theorem 0 – Proof of part I

Two sub-cases are possible if $\omega = 3$:

Sub-case 1: there is a fence face with at least 4 edges $\Rightarrow$

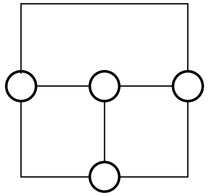
$$m_r + m_b \leq 3n - 6 - 1 \quad \Leftrightarrow \quad m_r + m_b \leq 3n - 7$$

By assumption we have $m_g \leq m_b$ $\Rightarrow$ $m_r + m_g \leq 3n - 7$

and we had $m_b \leq n - 1 - \omega/2$

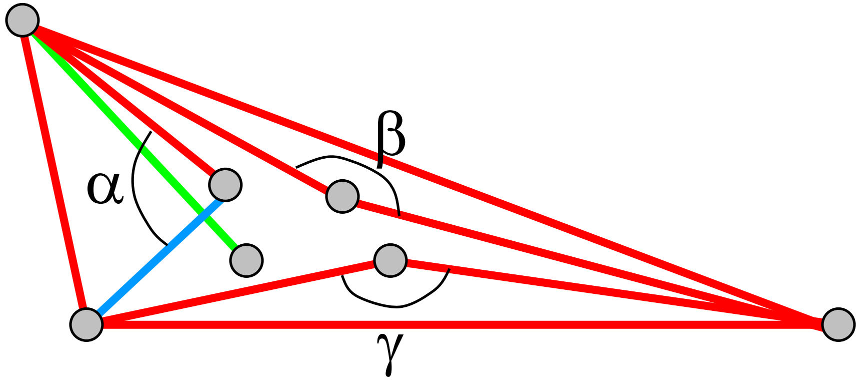
$\omega = 3$

$$m \leq 4n - 9.5 \quad \Rightarrow \quad m \leq 4n - 10 \quad \checkmark$$



Theorem 0 – Proof of part I

<u>Sub-case 2</u>: each fence face has 3 edges (in this case there are exactly three fence faces)

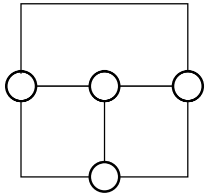


$\alpha + \beta + \gamma \geq 360°$

$\alpha < 90°$

$\Rightarrow \quad \beta \geq 90°$ and $\gamma \geq 90°$

$\Rightarrow \quad 2m_r \geq 2(f_{rb} - 3) + 3*3 \quad \Rightarrow \quad m_r \geq f_{rb} + 3/2$



Theorem 0 – Proof of part I

$m_r \geq f_{rb} + 3/2$

and we had $m_r + m_b \leq n + f_{rb} - 2$

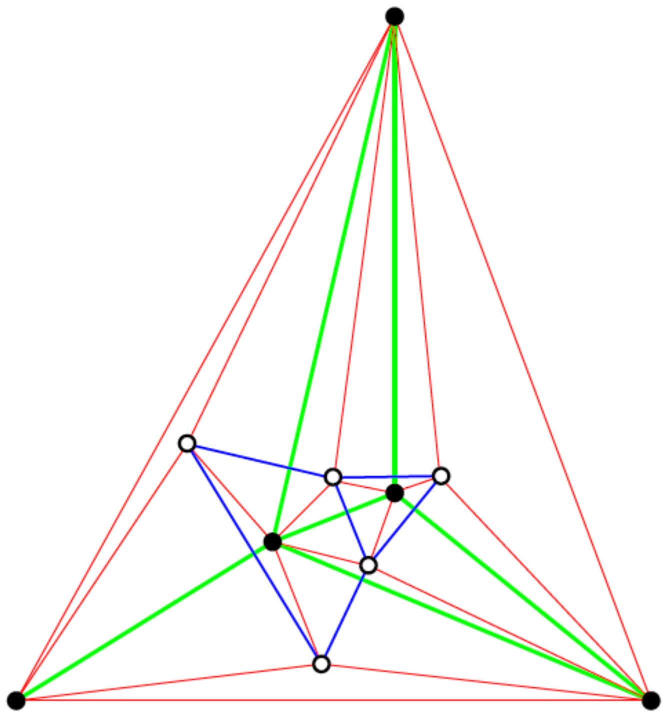
$m_b \leq n - 7/2$

and we had $m_r + m_g \leq 3n - 3 - \omega$

$\omega = 3$

$m \leq 4n - 9.5 \quad \Rightarrow \quad m \leq 4n - 10$ ✓

Theorem 0 – Proof of part II



Take the union of a maximal planar graph with **k** vertices (black) and its dual (white vertices), except the external face + three edges for each dual vertex

It has a 0-bend RAC drawing (consequence of a result by Brightwell and Scheinermann (1993))

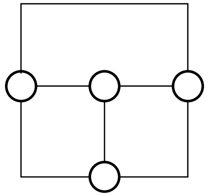
The dual graph has **2k-5** vertices (white vertices) and hence the total number of vertices is **$n = 3k - 5$**. The number of edges is **$m = (3k - 6) + 3(2k - 5) + (3k - 6 - 3) = 12k - 30 = 4n - 10$**

Edge density: Open problems

Type of RAC drawing	Maximum number of edges	Tightness
0-bend	$4n - 10$	✓
1-bend	$6.5n - 13$	×
2-bend	$74.2n$	×
3-bend	$n(n-1)/2$	✓

Problem ED1. Improve the upper bounds for 1- and 2-bend RAC drawings or prove that they are tight

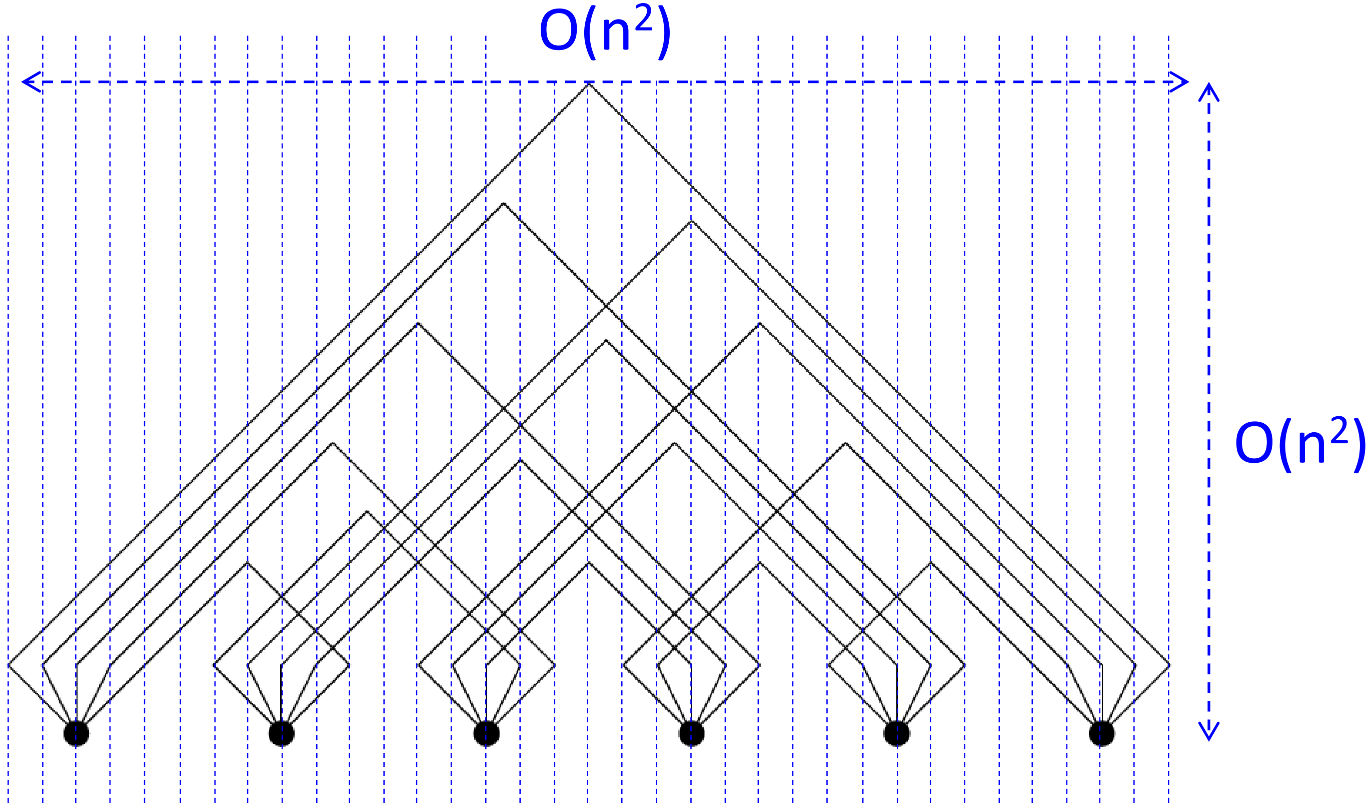
Problem ED2. What is the minimum number of edges that a C_k-maximal graph can have, for $k \in \{0, 1, 2\}$?



Drawing algorithms: 3-bend RAC

- **Theorem 3**. Every graph G belongs to C_3. A 3-bend RAC drawing of G can be computed in O(n+m) time on an integer grid of size $O(n^2) \times O(n^2)$

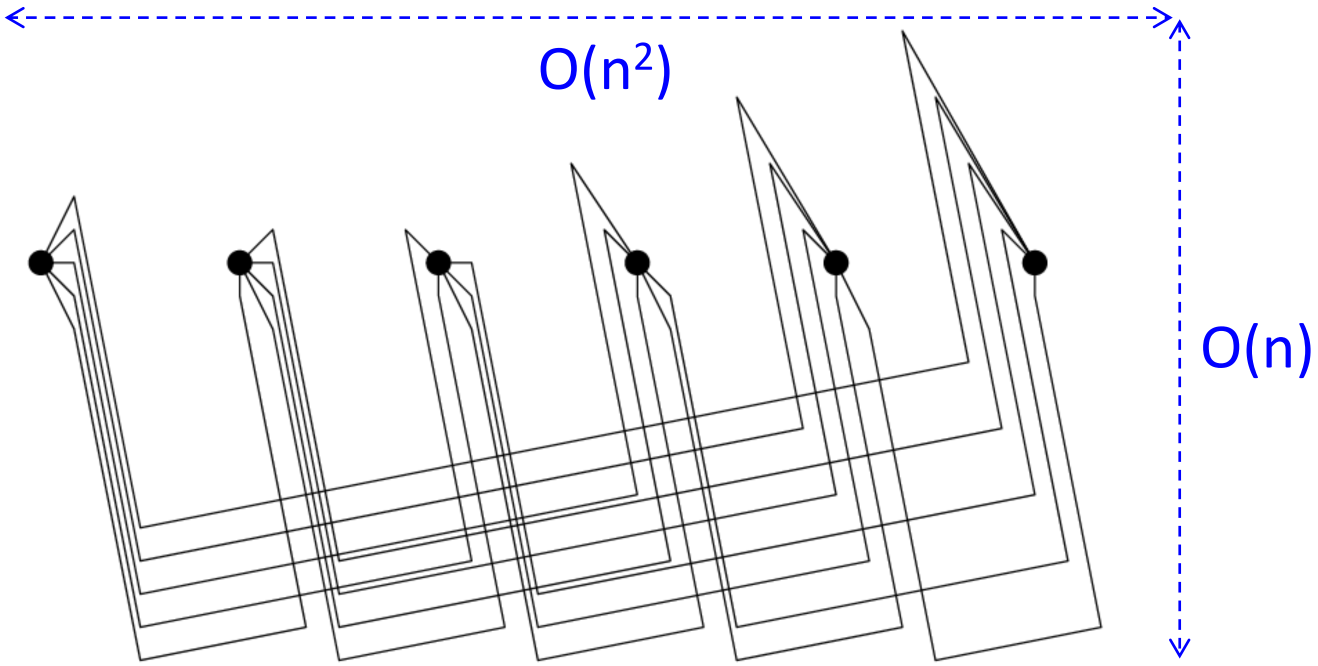
3-bend drawing of K_6



Drawing algorithms: 4-bend RAC

- **Theorem 4**. For every graph G, a 4-bend RAC drawing of G can be computed in O(n+m) time on an integer grid of size $O(n^2) \times O(n)$
  – E. Di Giacomo, W. Didimo, G. Liotta, H. Meijer: Area, Curve Complexity, and Crossing Resolution of Non-Planar Graph Drawings. Theory Comput. Syst. 49(3) (2011)

4-bend drawing of K_6



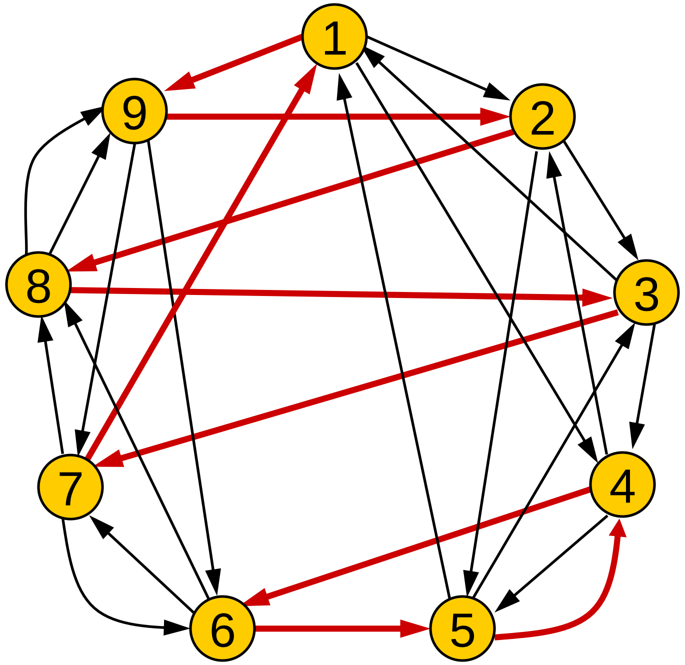
Drawing algorithms: 1-bend and 2-bend RAC

- **Δ-Theorem**. Every Δ-graph, with $\Delta \in \{3, 6\}$, admits a Δ/3-bend RAC drawing in $O(n^2)$ area, which can be computed in $O(n)$ time
  - *P. Angelini, L. Cittadini, G. Di Battista, W. Didimo, F. Frati, M. Kaufmann, A. Symvonis*: On the Perspectives Opened by Right Angle Crossing Drawings. J. Graph Algorithms Appl. 15(1) (2011)

- **Proof idea**
  - constructive technique based on the concept of cycle cover
  - we sketch the proof for Δ=6 ($\Rightarrow$ 2-bend RAC)

Drawing algorithms: 2-bend RAC

A cycle cover of a directed multi-graph is a spanning subgraph consisting of vertex-disjoint *directed* cycles

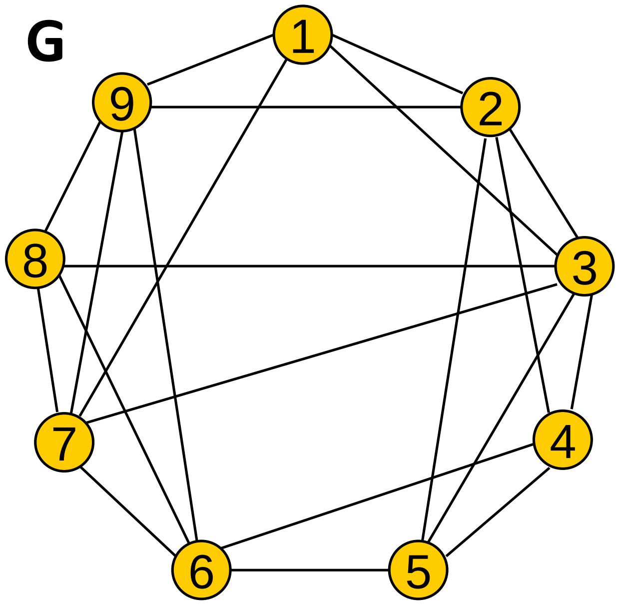


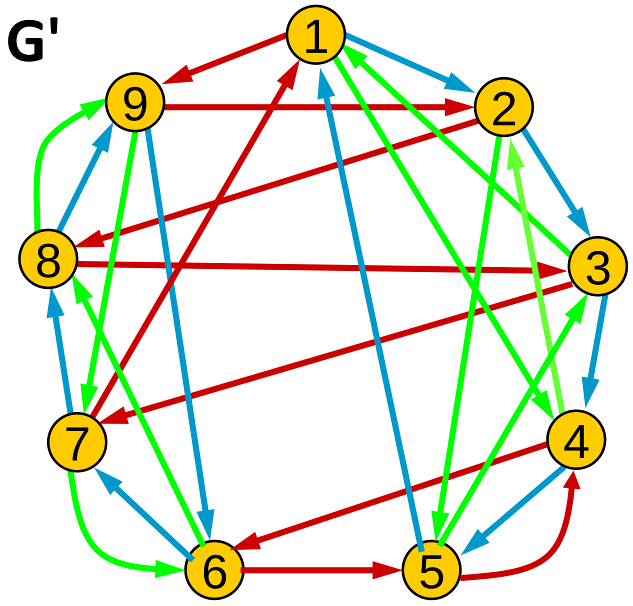
the two red cycles define a cycle cover

Drawing algorithms: 2-bend RAC

Lemma (Eades, Symvonis, Whitesides, 2000) [ESW'00]**.** For any Δ-graph G there exists a *directed* multi-graph G' with the same vertex set as G such that:

- each vertex of G' has in-degree and out-degree **d = ⌈Δ/2⌉**
- G is a subgraph of the underlying undirected graph of G’
- the edges of G' can be partitioned into **d** edge-disjoint cycle covers





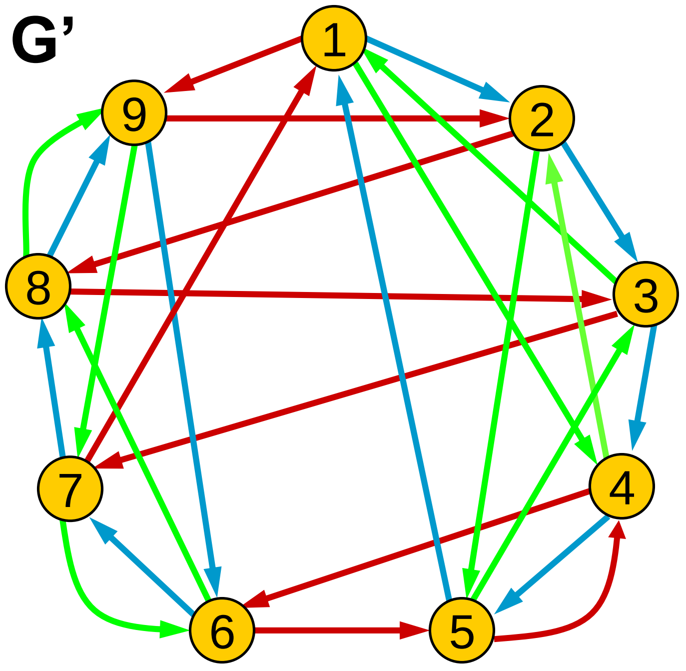
Remark: G' is a Δ-regular graph if Δ is even

Drawing algorithms: 2-bend RAC

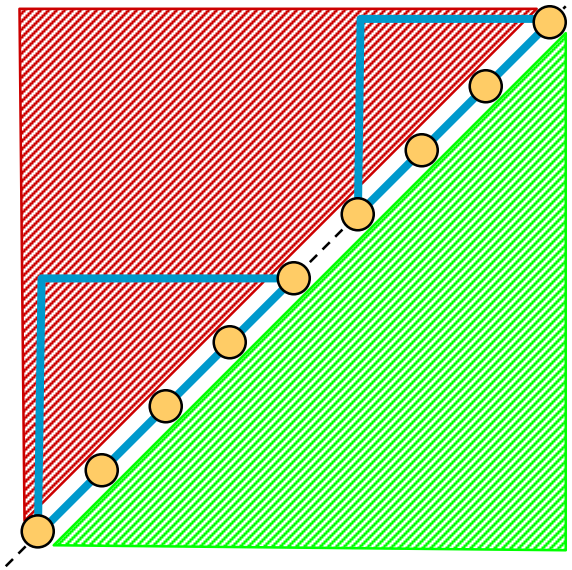
Constructive algorithm for a 6-graph G

1. compute a 6-regular multi-digraph G’ that contains G, and 3 edge-disjoint cycle covers of G', using [ESW'00]
2. use the cycle covers to construct a 2-bend RAC drawing of G’
3. remove dummy edges from G’

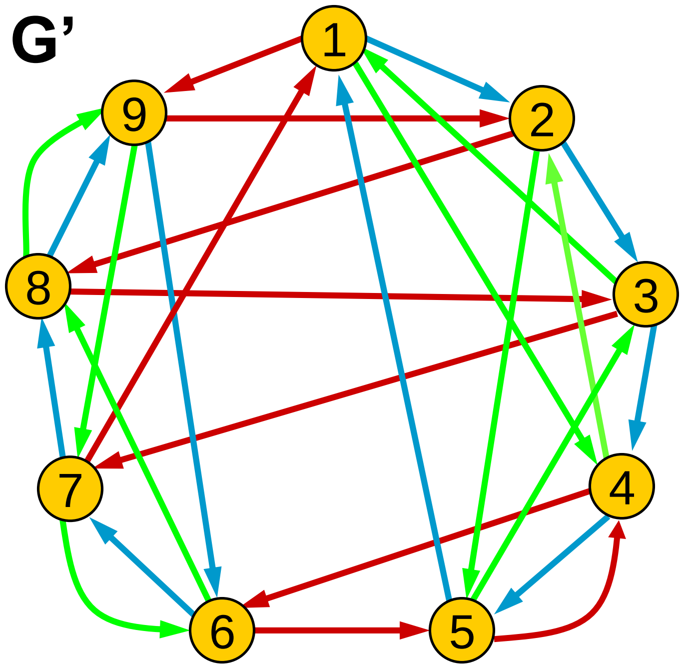
Drawing algorithms: 2-bend RAC



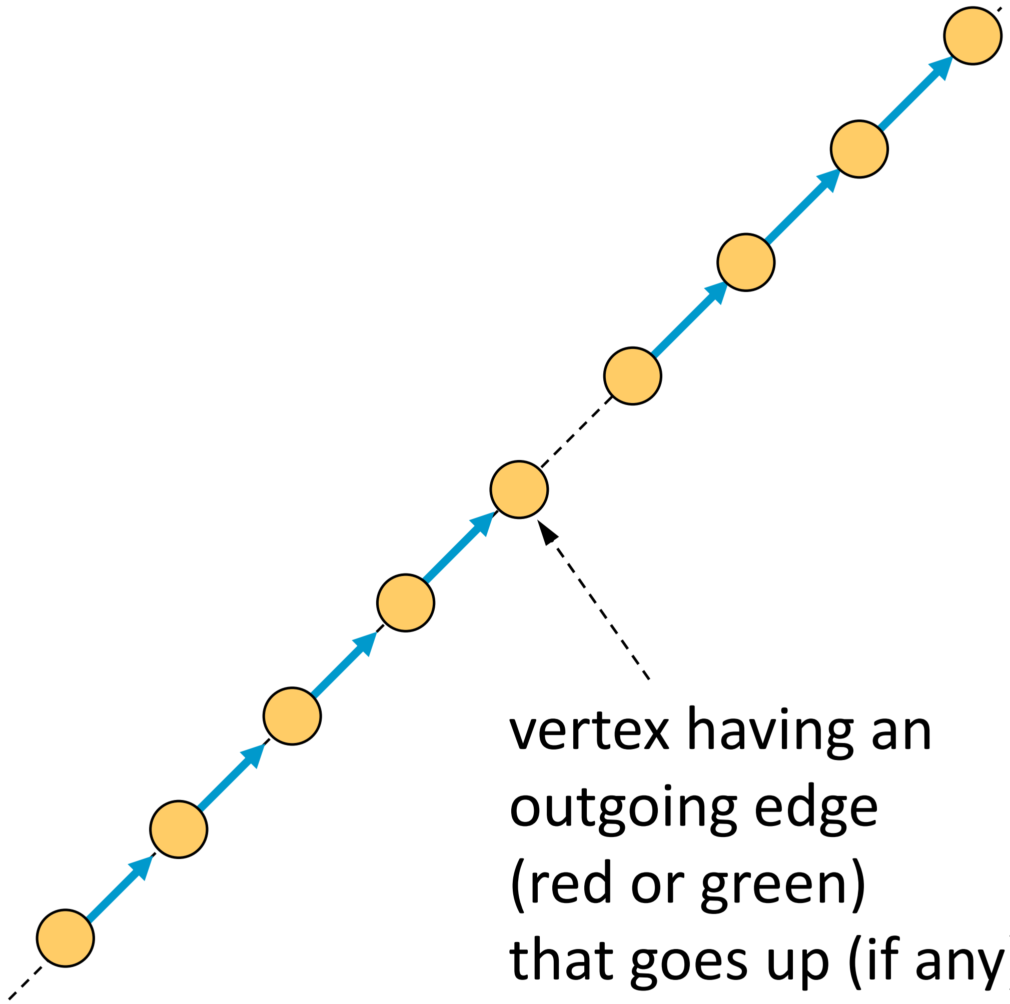
general idea



Drawing algorithms: 2-bend RAC

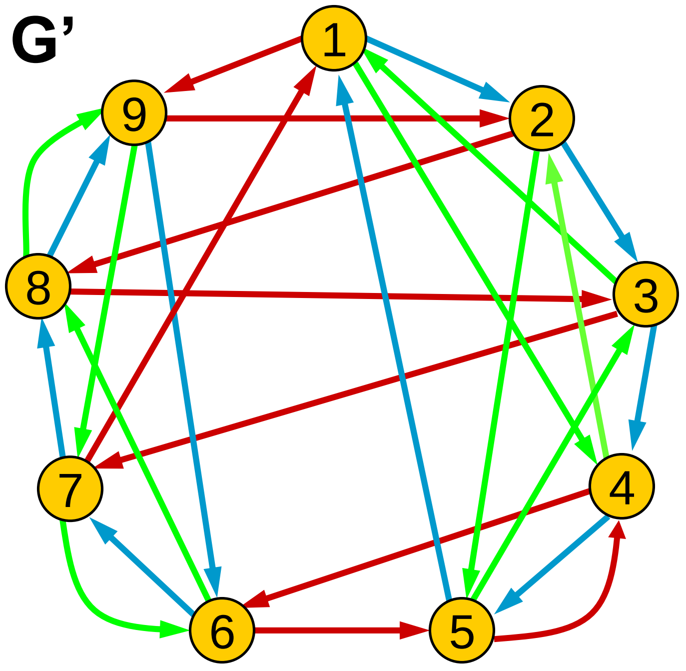


choosing the vertex order

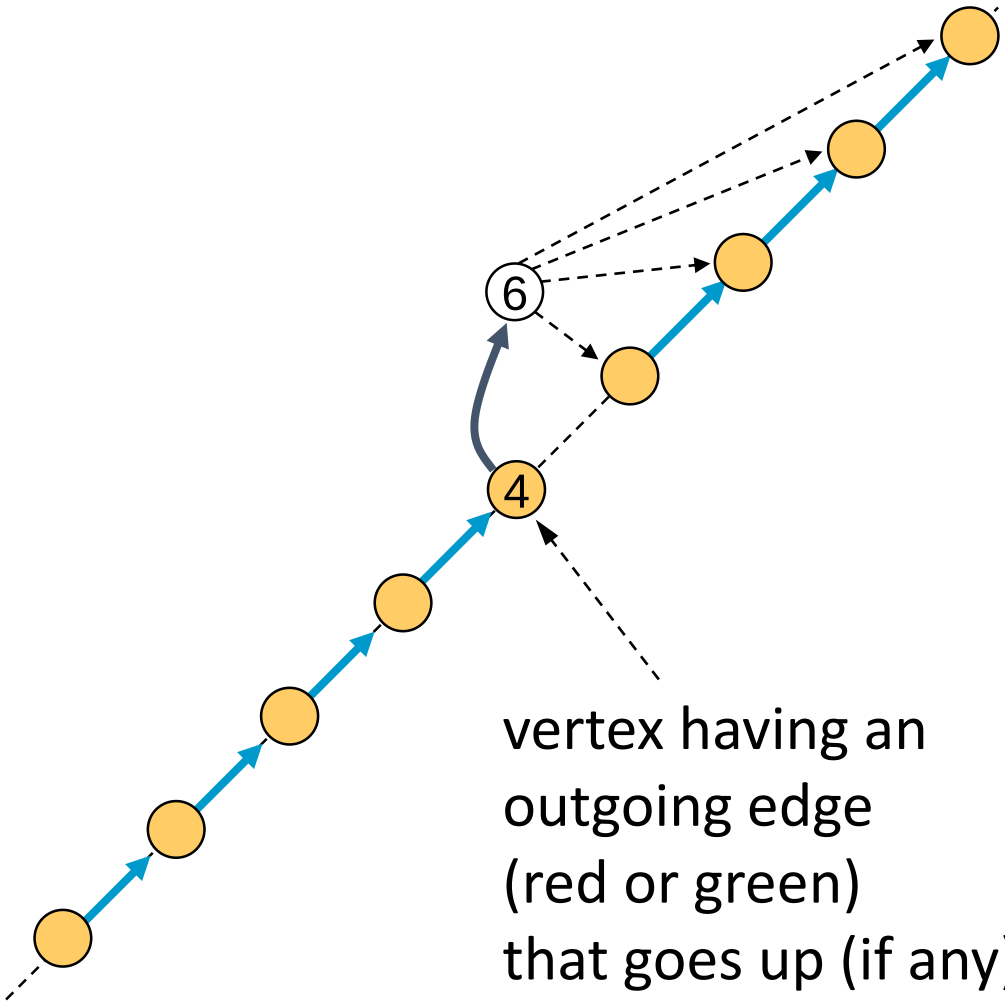


vertex having an
outgoing edge
(red or green)
that goes up (if any)

Drawing algorithms: 2-bend RAC

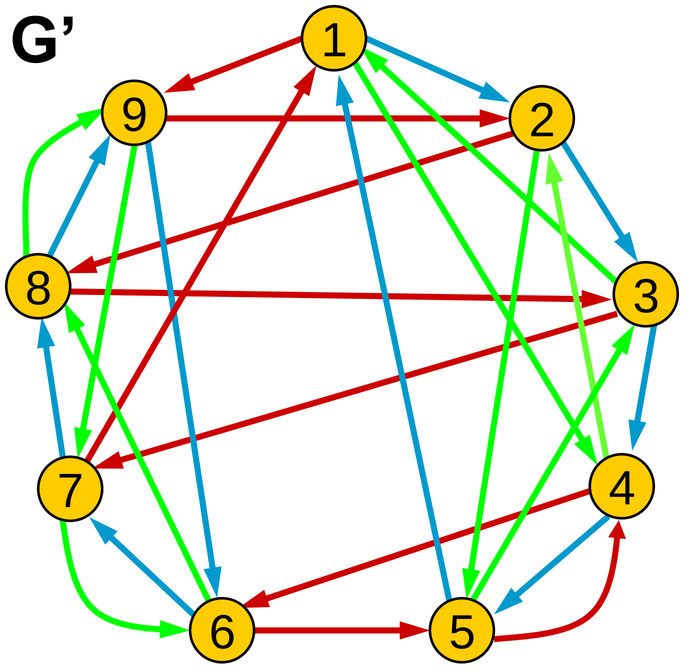


choosing the vertex order



vertex having an outgoing edge (red or green) that goes up (if any)

Drawing algorithms: 2-bend RAC



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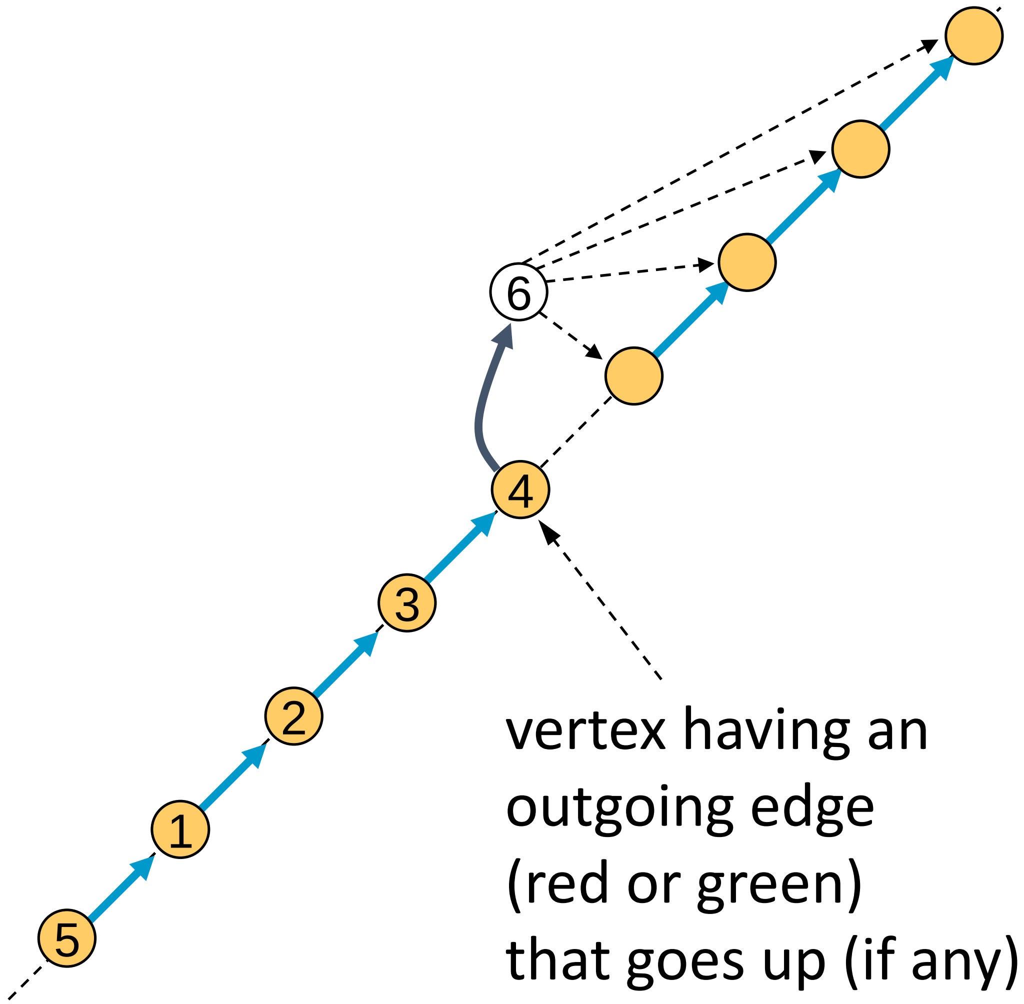
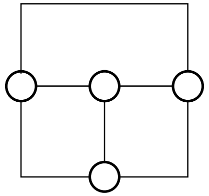
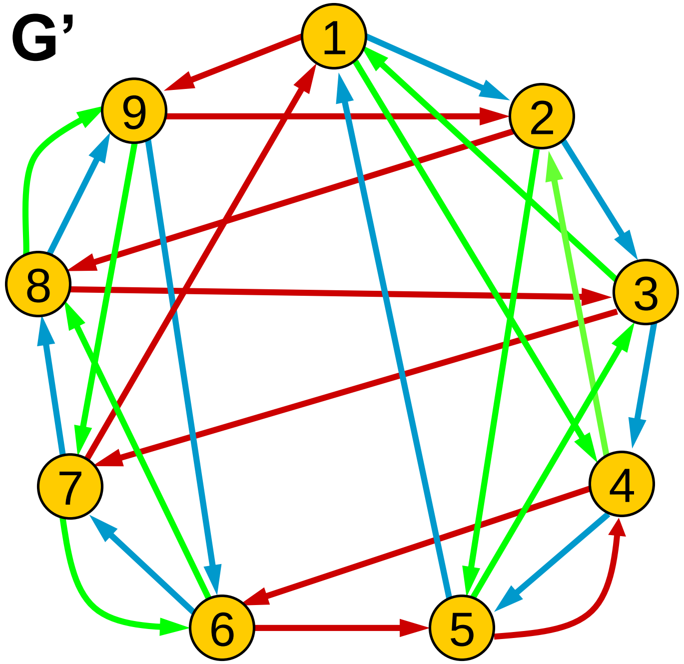


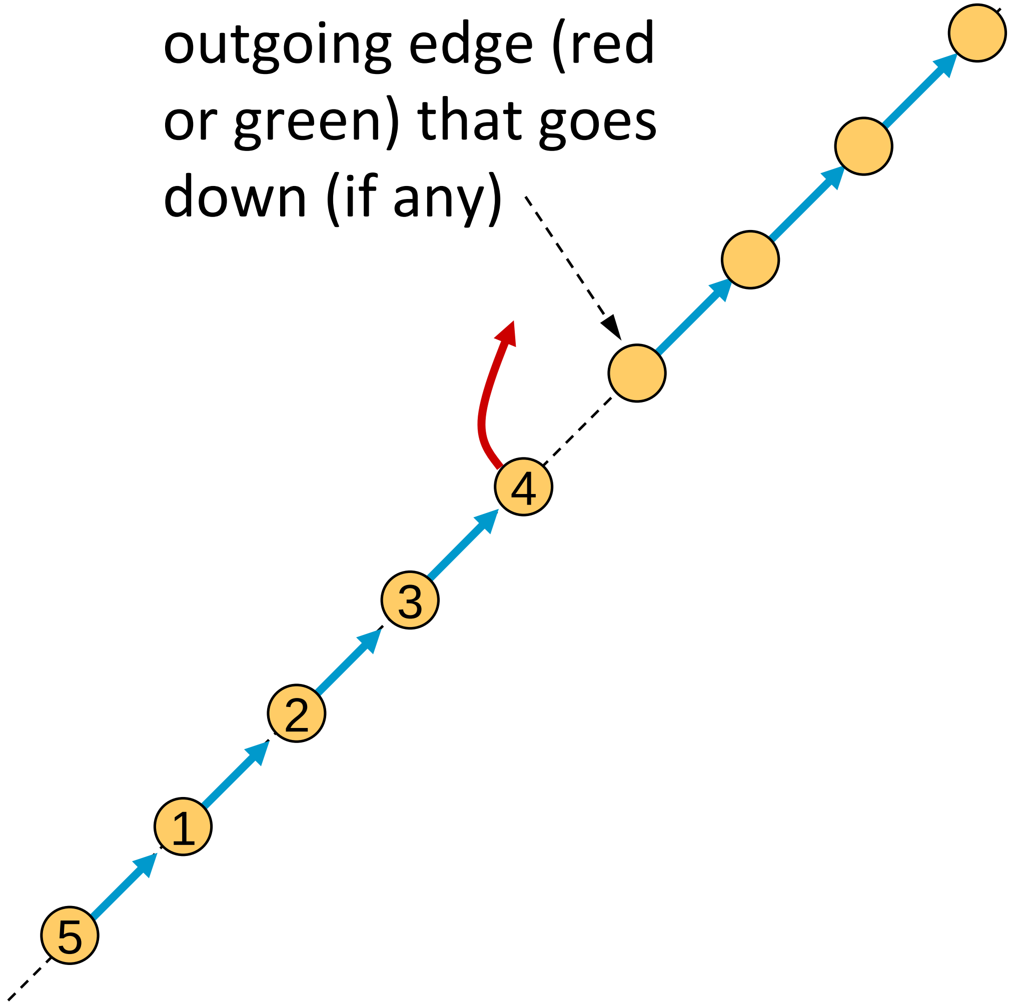


Drawing algorithms: 2-bend RAC

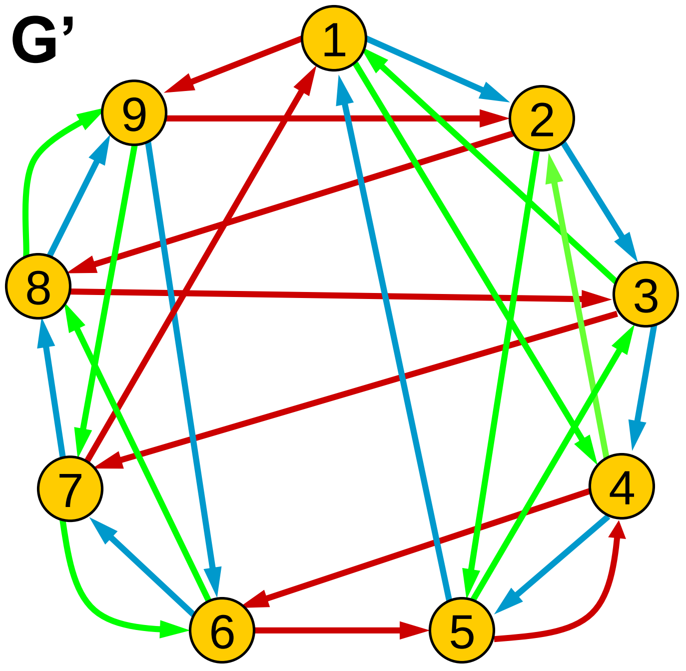


choosing the vertex order

vertex having an outgoing edge (red or green) that goes down (if any)



Drawing algorithms: 2-bend RAC



choosing the vertex order

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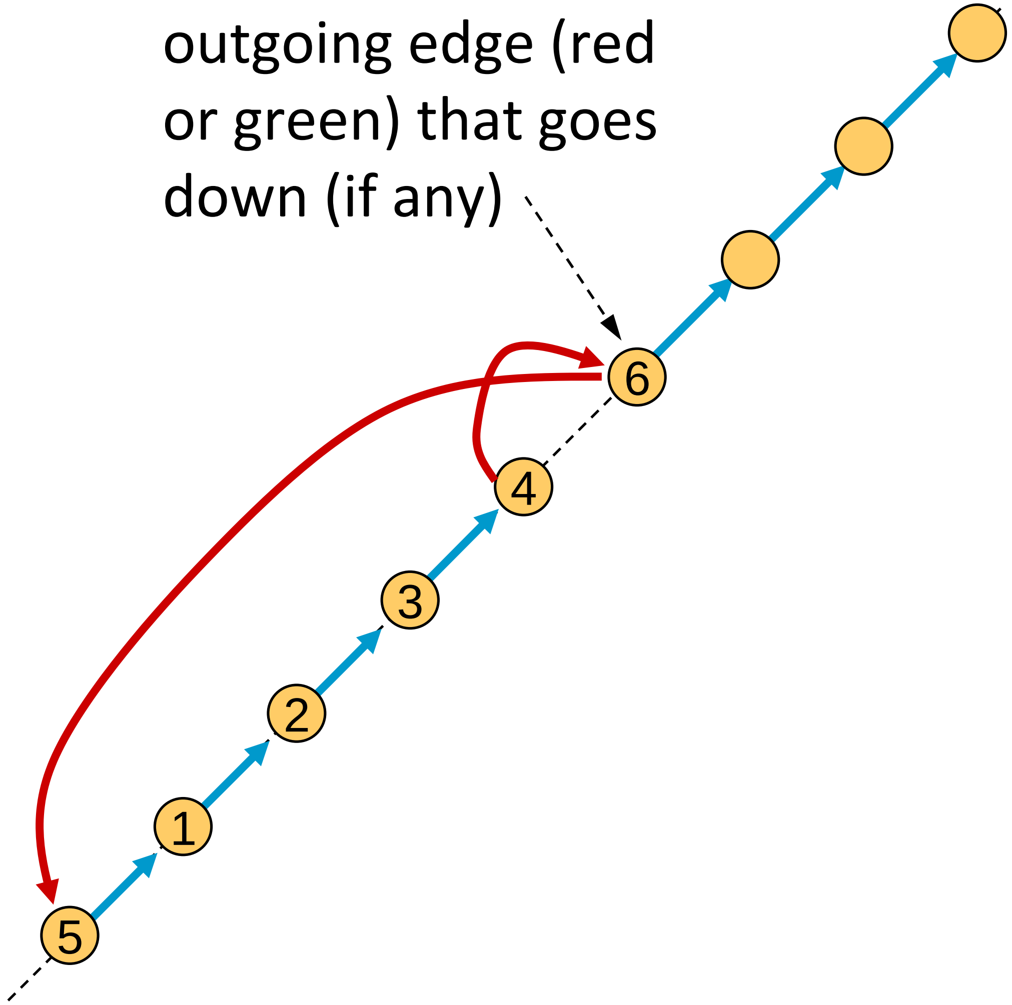
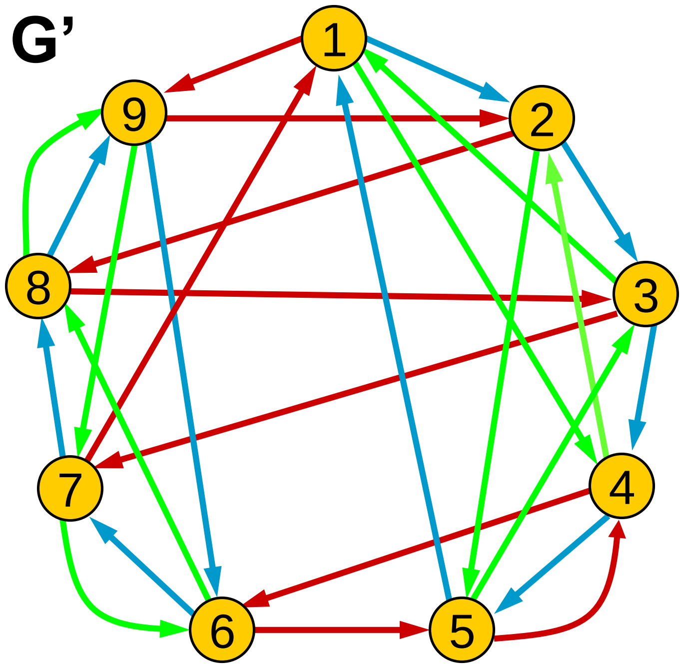


Drawing algorithms: 2-bend RAC



choosing the vertex order

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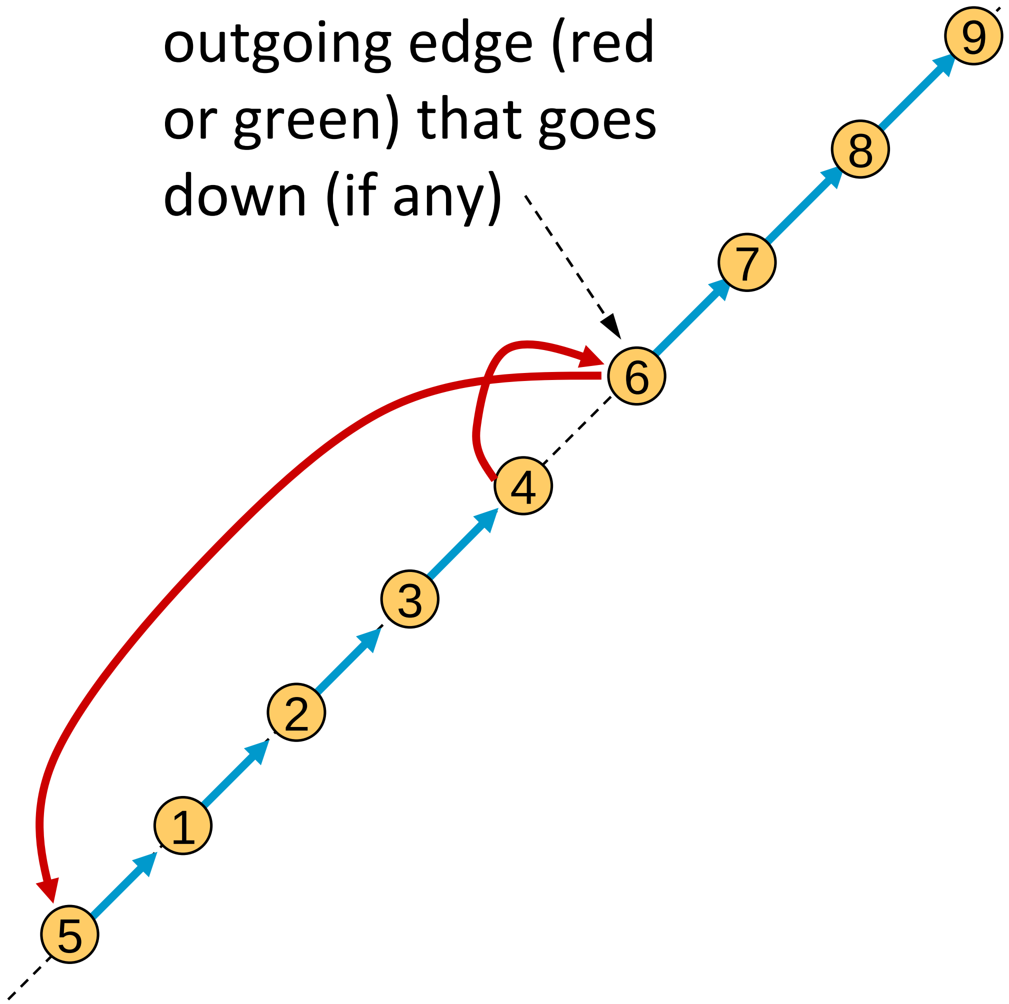
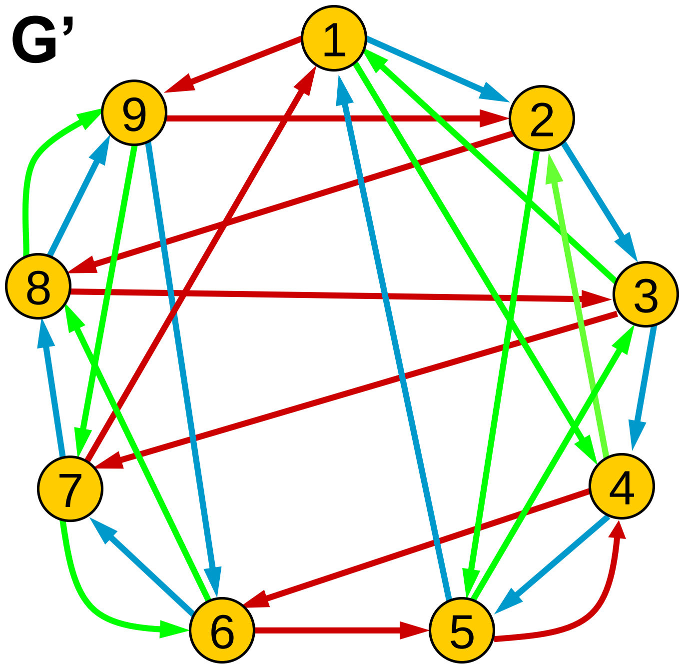
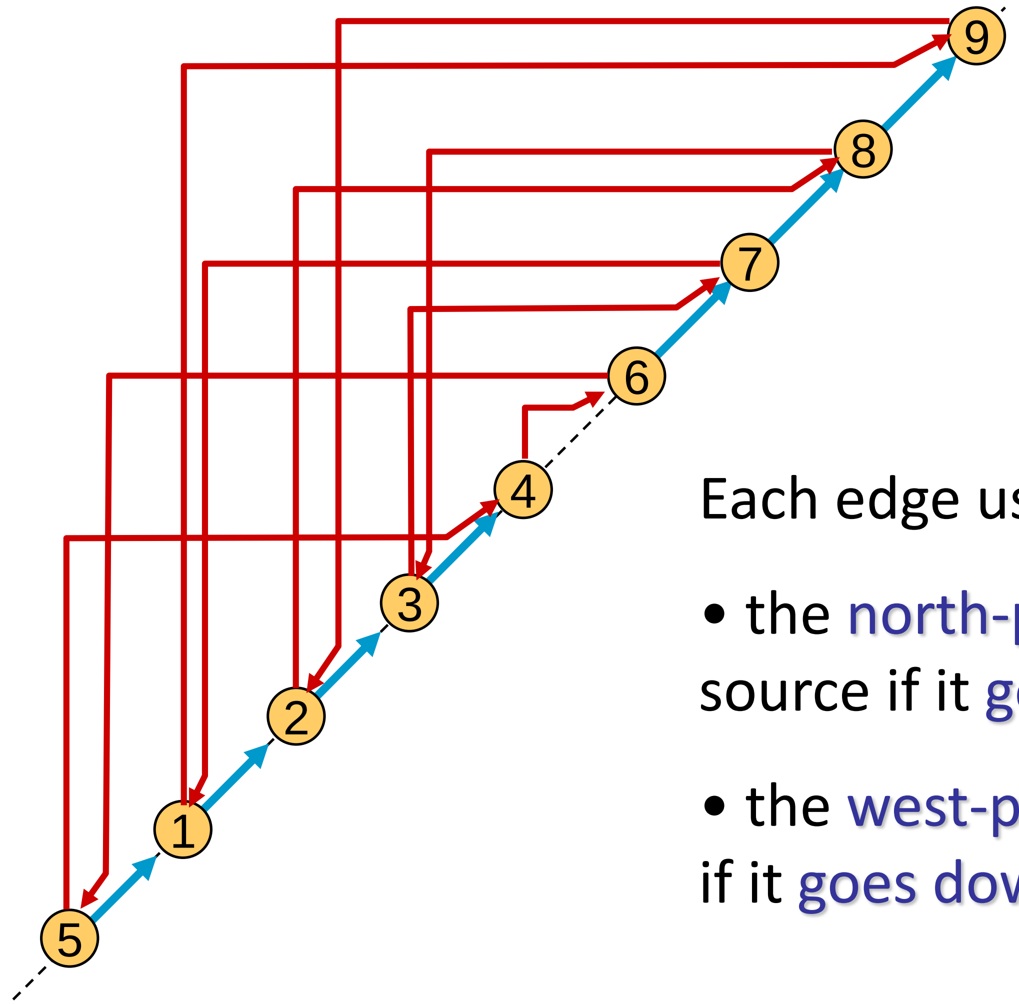


Drawing algorithms: 2-bend RAC



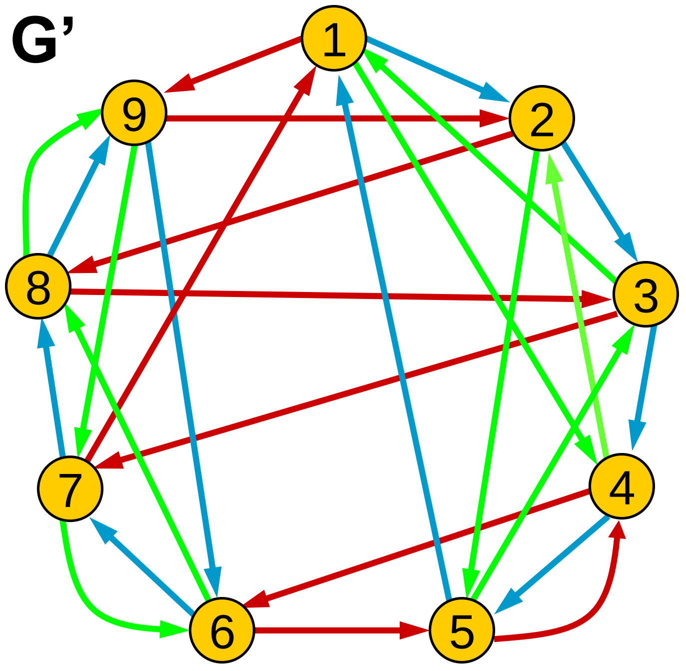
drawing the red cycles



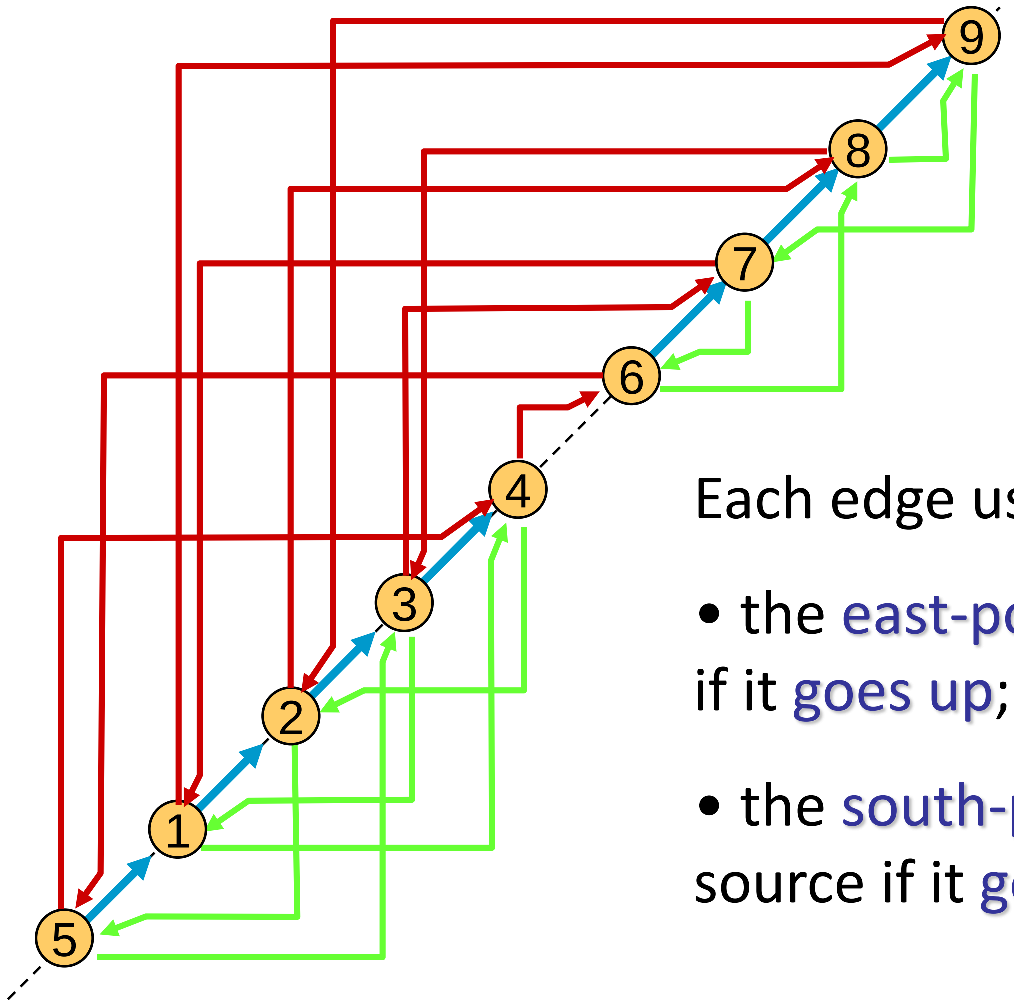
Each edge uses:

- the north-port of its source if it goes up;
- the west-port of its source if it goes down

Drawing algorithms: 2-bend RAC

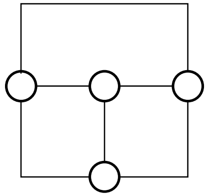


drawing the green cycles

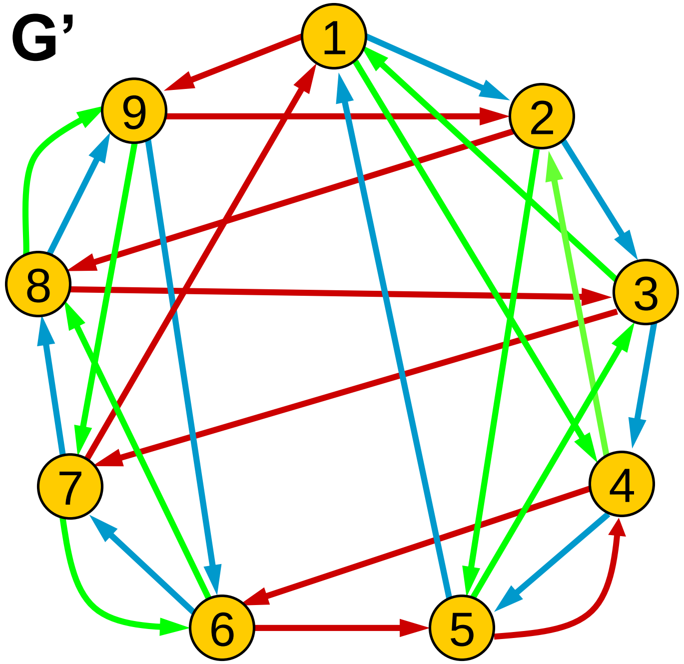


Each edge uses:

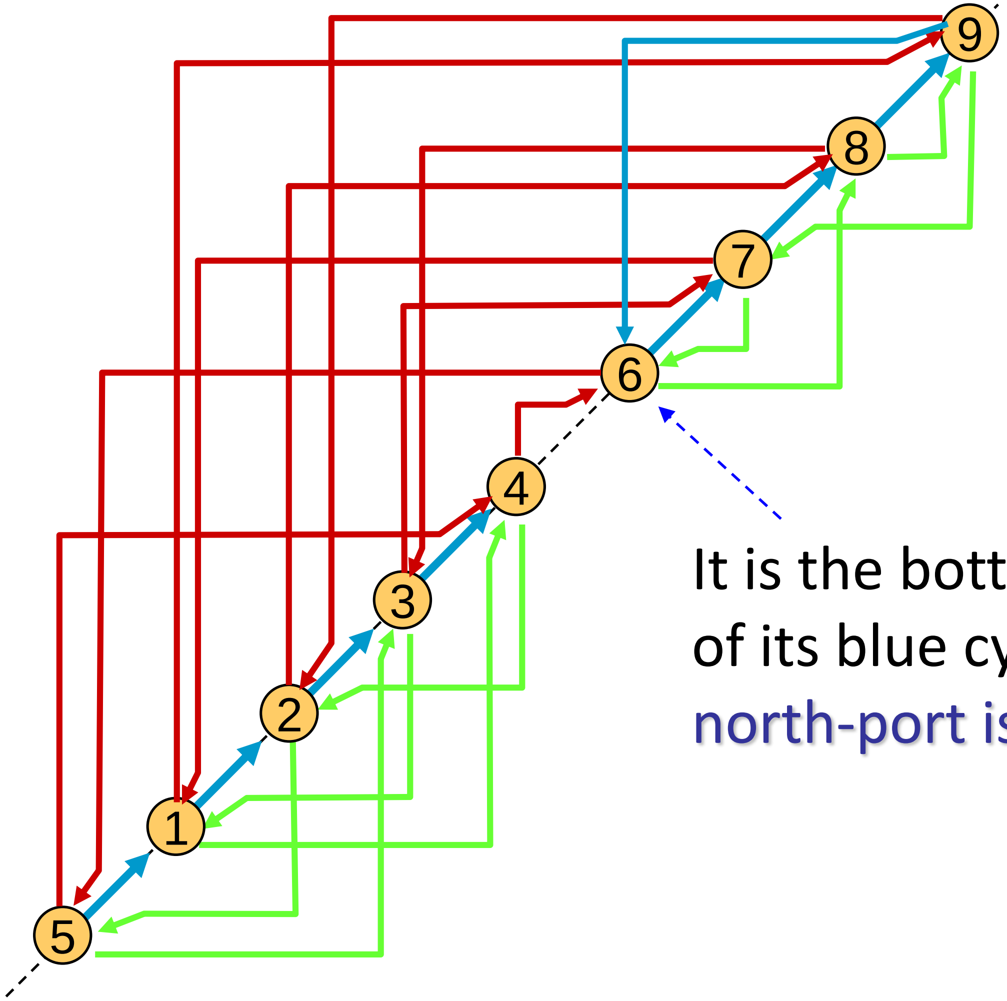
- the east-port of its source if it goes up;
- the south-port of its source if it goes down



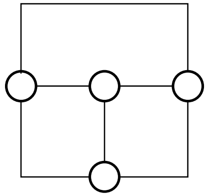
Drawing algorithms: 2-bend RAC



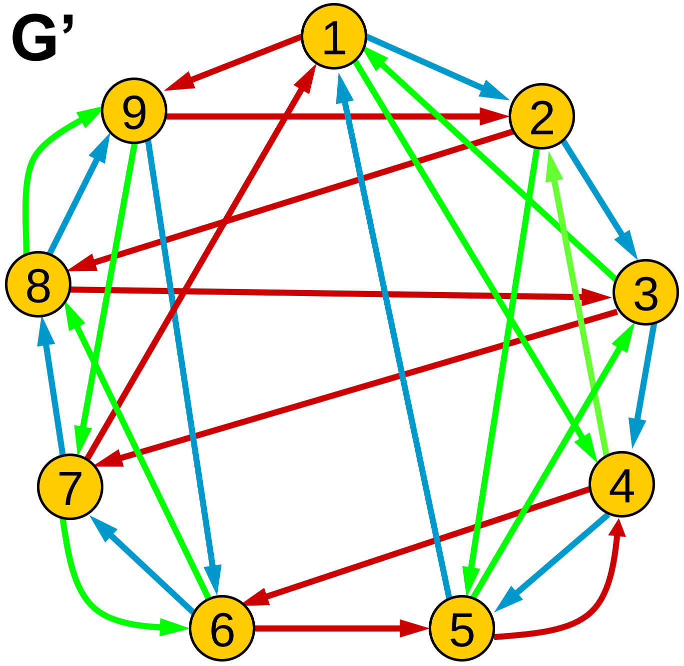
adding the edges that close the blue cycles



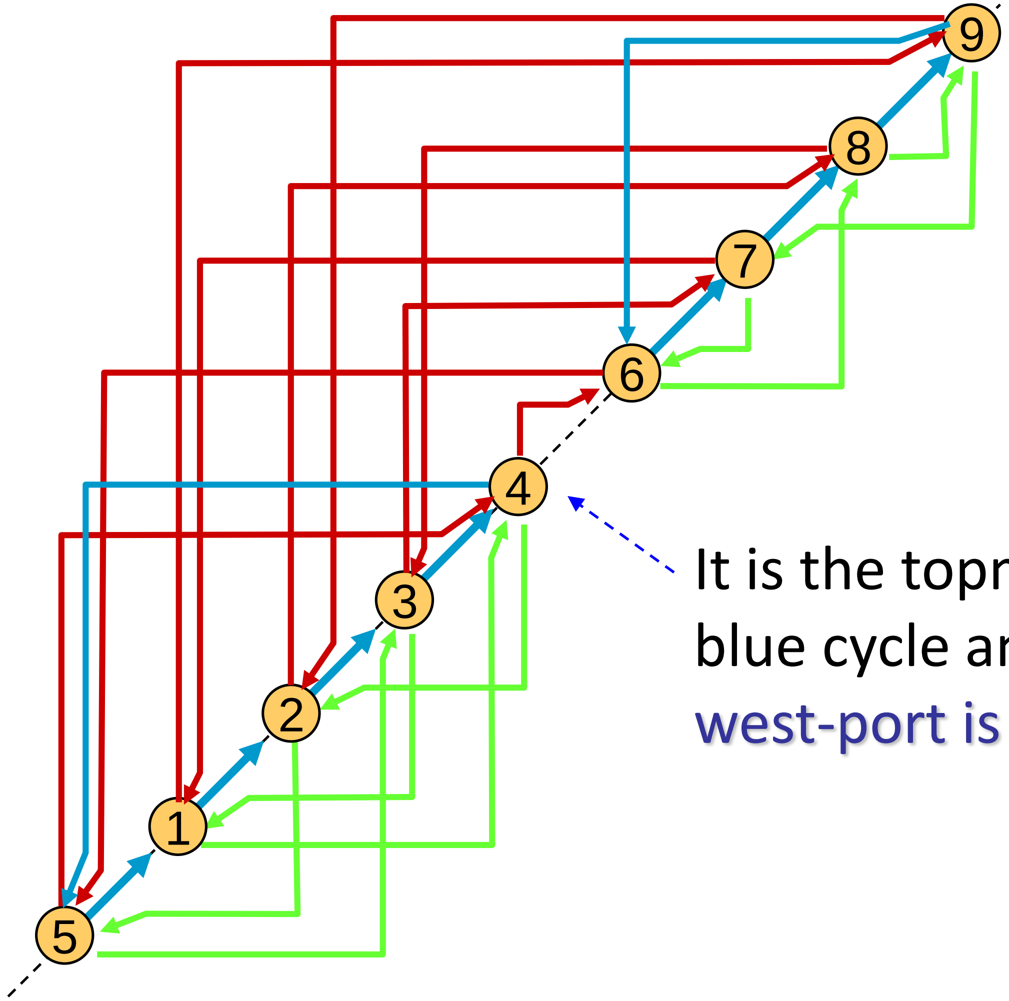
It is the bottommost vertex of its blue cycle and its north-port is free



Drawing algorithms: 2-bend RAC



adding the edges that close the blue cycles



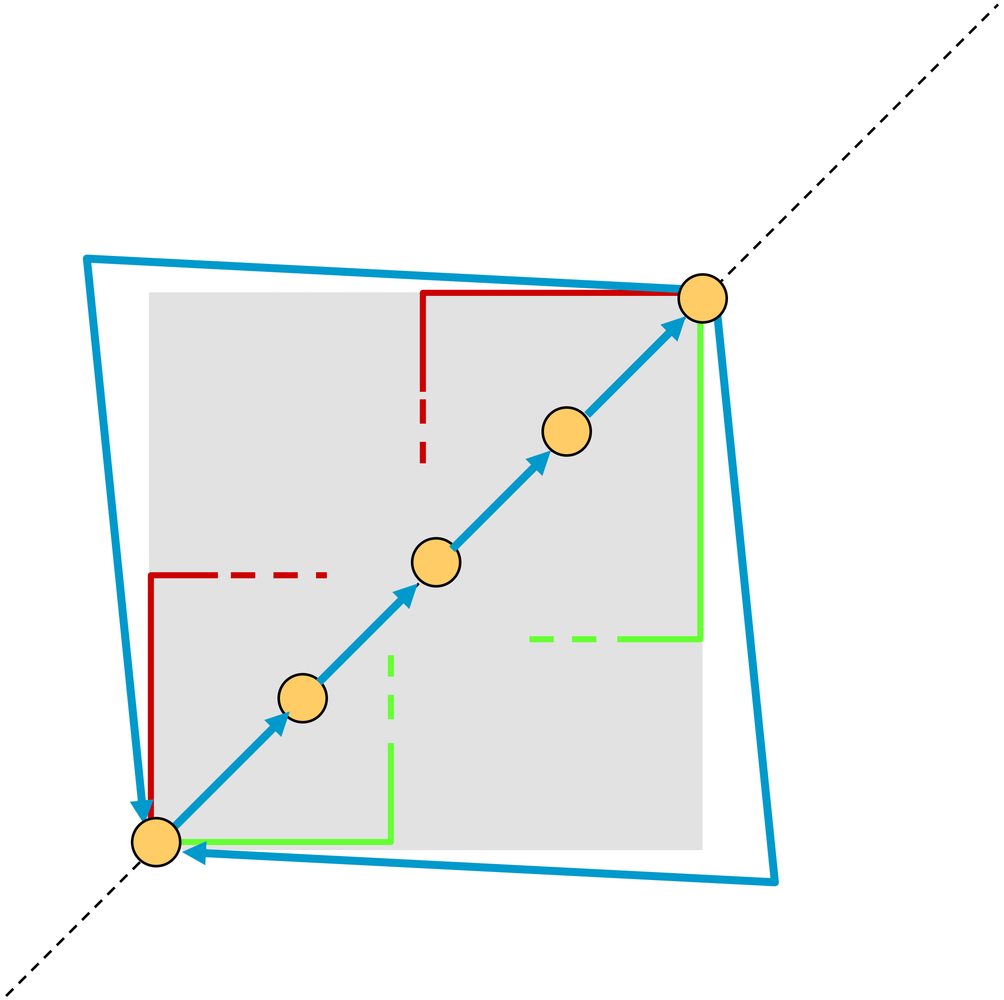
It is the topmost vertex of its blue cycle and its west-port is free

Drawing algorithms: 2-bend RAC

One more case:

there is no vertex of the blue cycle having a red/green edge that goes towards other cycles

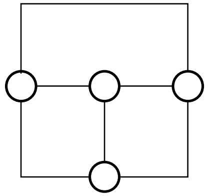
all the useful ports of the bottommost/topmost vertex are occupied, but the cycle is not connected with other cycles



Drawing algorithms: Summary

Graph	Bends per edge	Area	Citation
Any	3	$O(n^4)$	Didimo, Eades, Liotta 2011
Any	4	$O(n^3)$	Di Giacomo et al. 2011
Δ=6	2	$O(n^2)$	Angelini et al. 2011
Δ=3	1	$O(n^2)$	Angelini et al. 2011
Any	3	$O((n+m)^2)$	Fink et al. 2012
Planar	4	$O(\Delta^{0.5}n^{1.5})$	Angelini et al. 2012
NIC-plane	1	$O(n^2)$	Chaplick et al. 2018

additional

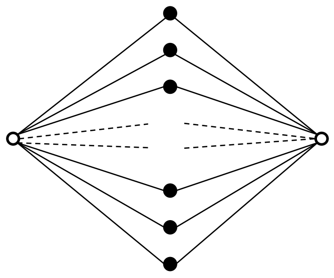


But what about 0-bend RAC drawing algorithms?

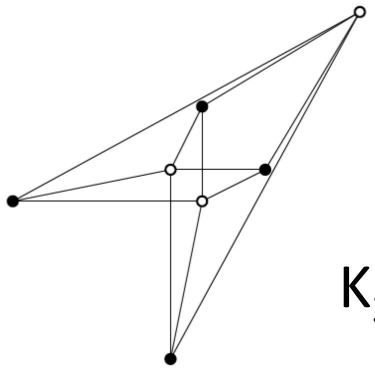
0-bend RAC drawability

- Testing whether a graph has a 0-bend drawing is NP-hard
  - *E. N. Argyriou, M. A. Bekos, A. Symvonis*: The Straight-Line RAC Drawing Problem is NP-Hard. J. Graph Algorithms Appl. 16(2) (2012)

- Testing whether a complete bipartite graph has a 0-bend drawing can be done in O(1) time ($K_{2,n}$, $K_{3,3}$, $K_{3,4}$)
  - *W. Didimo, P. Eades, G. Liotta:* A characterization of complete bipartite RAC graphs. Inf. Process. Lett. 110(16) (2010)



$K_{2,n}$

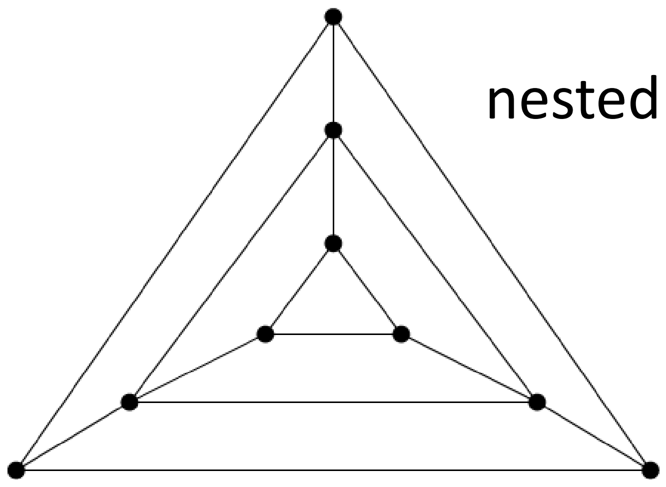


$K_{3,4}$

0-bend RAC drawings of planar graphs

- **Question:** Can we allow right angle crossings to improve the area requirement of straight-line planar drawings?

- **Remind:** straight-line planar drawings may require $\Omega(n^2)$ area
  - *H. de Fraysseix, J. Pach, R. Pollack*: How to draw a planar graph on a grid. Combinatorica 10(1) (1990)



nested triangles

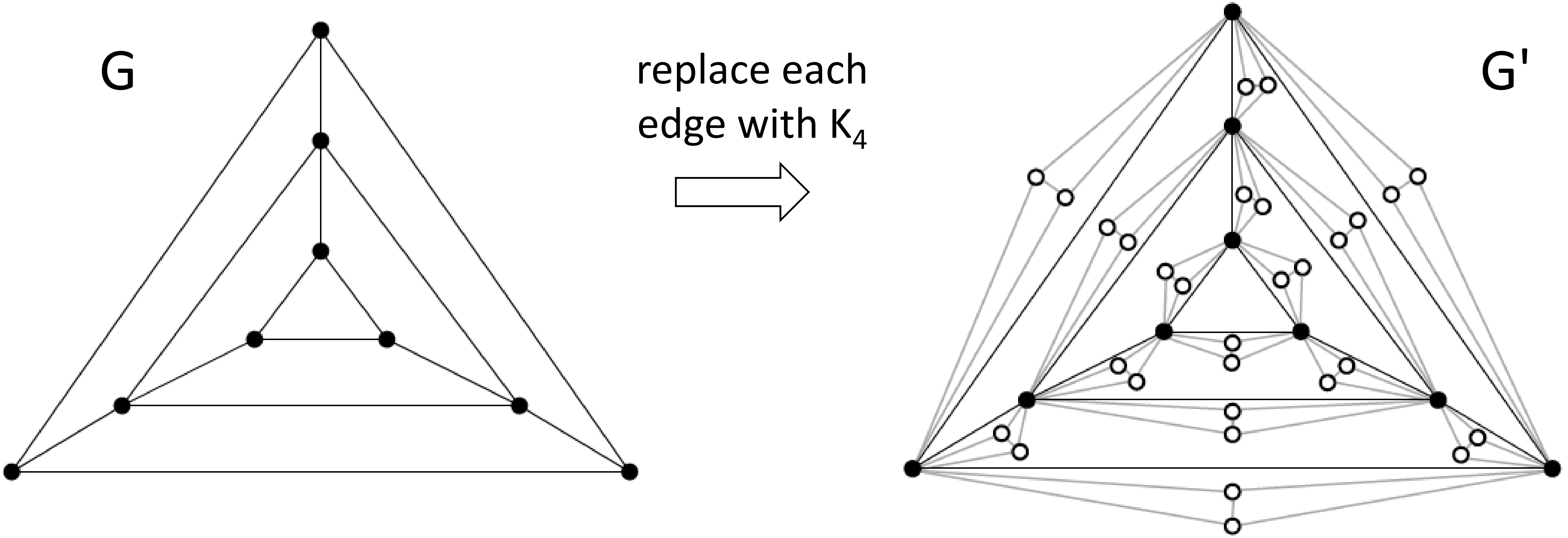
0-bend RAC drawings of planar graphs

- **Answer:** NO!

- **Area-Theorem.** There exist infinitely many planar graphs for which every 0-bend RAC drawing requires quadratic area
  - *P. Angelini, L. Cittadini, G. Di Battista, W. Didimo, F. Frati, M. Kaufmann, A. Symvonis*: On the Perspectives Opened by Right Angle Crossing Drawings. J. Graph Algorithms Appl. 15(1) (2011)

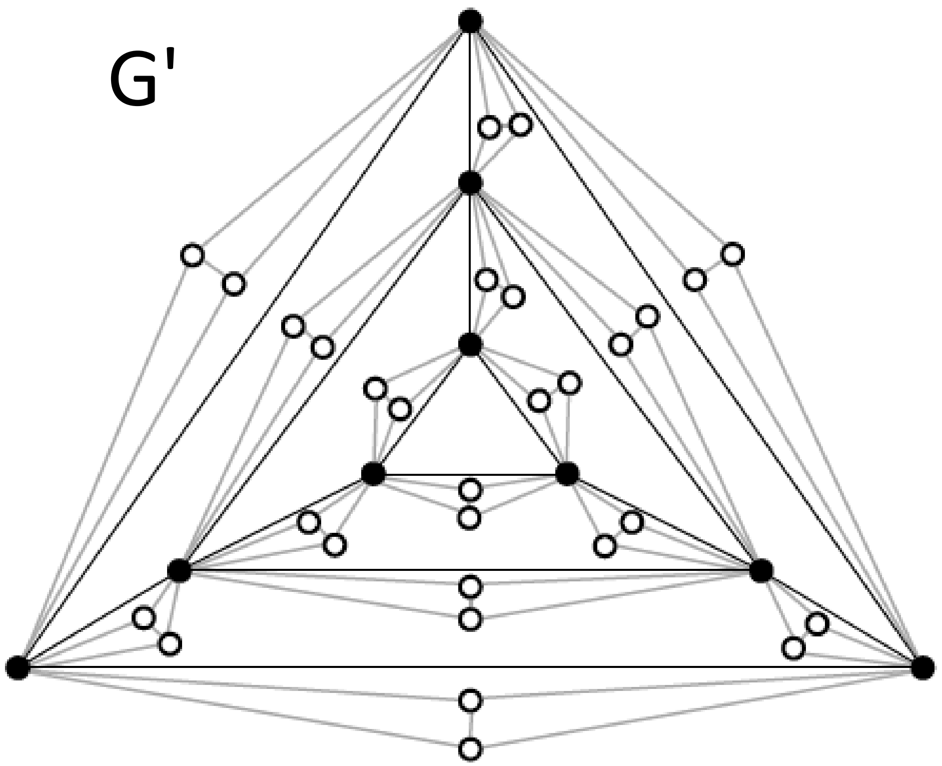
0-bend RAC drawings of planar graphs

- **Proof:**



0-bend RAC drawings of planar graphs

- **Proof:**



Uncrossability-Lemma: in any 0-bend RAC drawing of G' no two edges of G cross

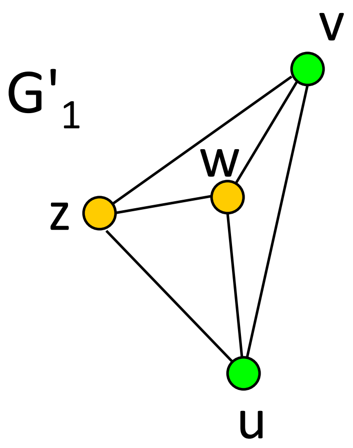
⇓

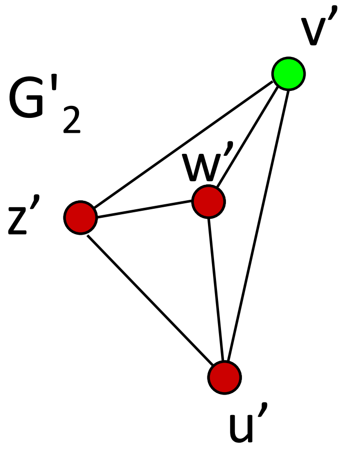
Consequence: the area of any 0-bend RAC drawing G' is not smaller than the area of every 0-bend planar drawing of G

0-bend RAC drawings of planar graphs

Uncrossability-Lemma (Proof)

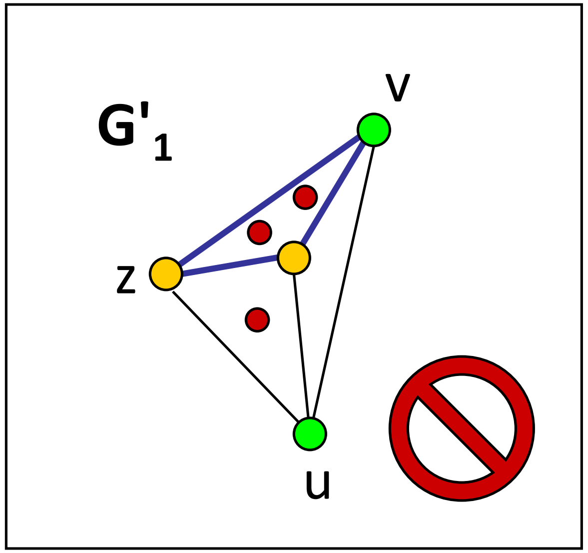
- **Stronger claim**: no two different K_4 in G' cross each other in a 0-bend RAC drawing of G'



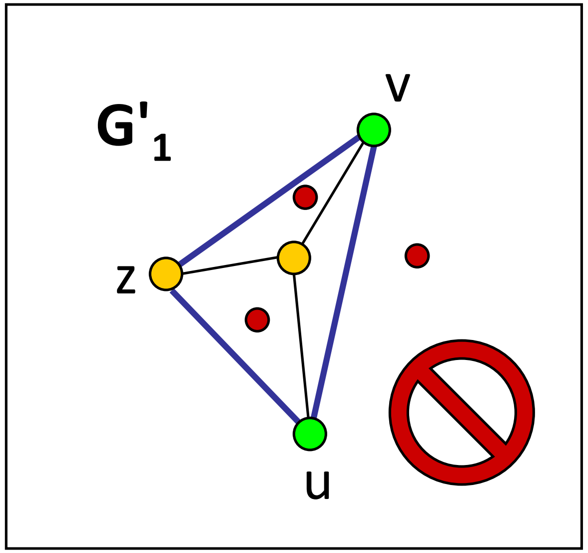


G'$_2$ contains at least three vertices (which form a cycle) that do not belong to **G'$_1$**; denote these three vertices with the red color

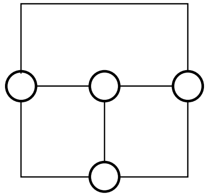
0-bend RAC drawings of planar graphs



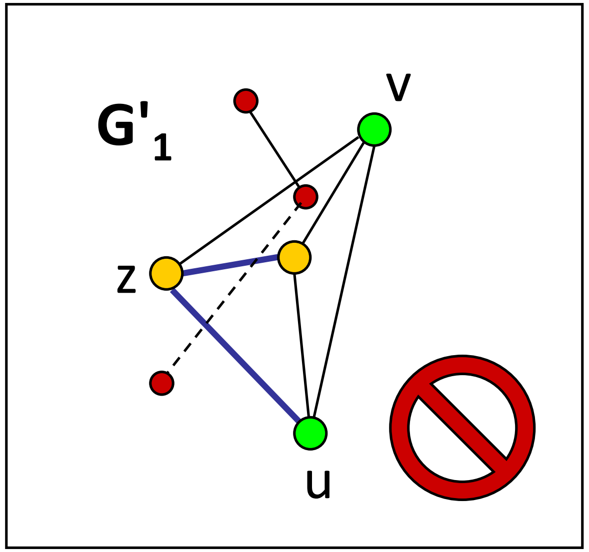
By the triangle property, we cannot have two red vertices inside an internal face and the other one outside this face



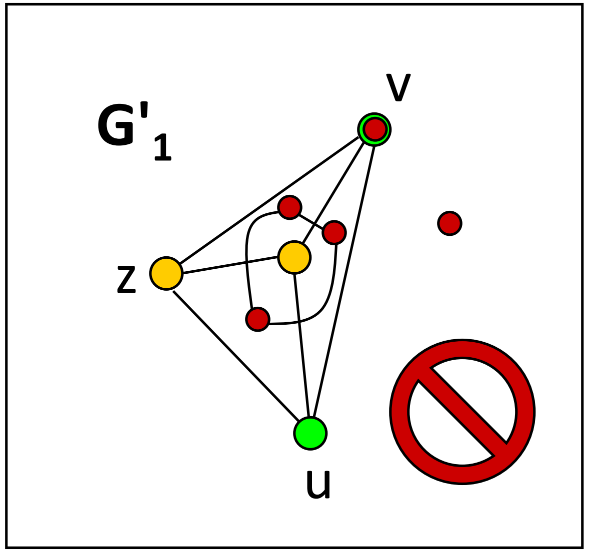
For the same reason we cannot have two vertices inside (u,v,z) and one outside



0-bend RAC drawings of planar graphs

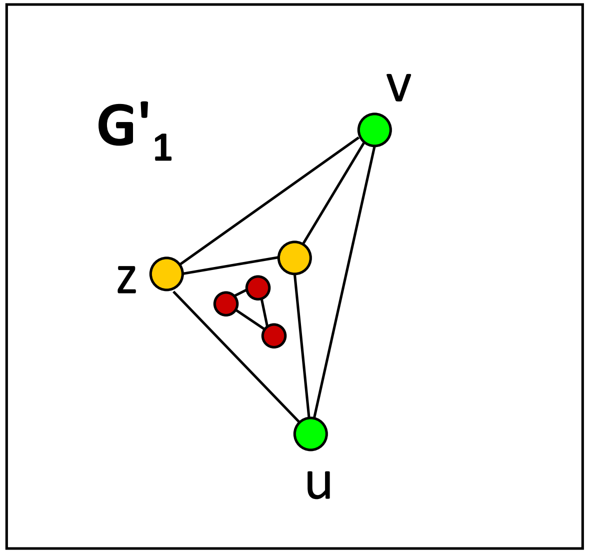


Two vertices outside (u,v,z) and one inside violate the fan property



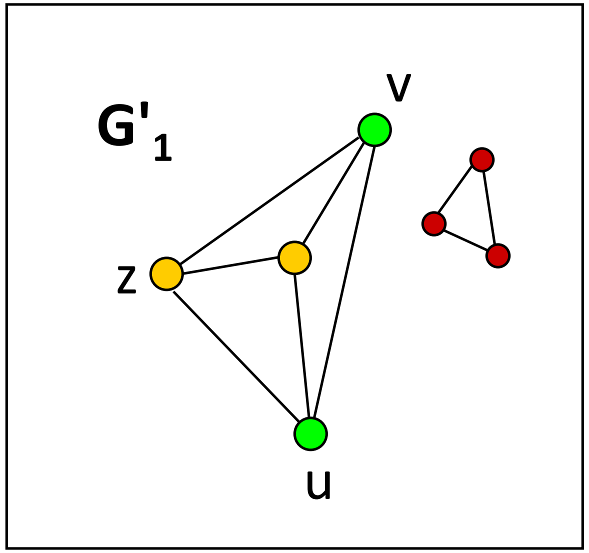
One vertex inside each internal face implies either the violation of the triangle property or the violation of the fan property when we consider the fourth vertex

0-bend RAC drawings of planar graphs



subcase 1: all the three red vertices are in the same internal face **f**

— the fourth vertex must be inside f or on its boundary, otherwise the triangle property is violated (no crossing)

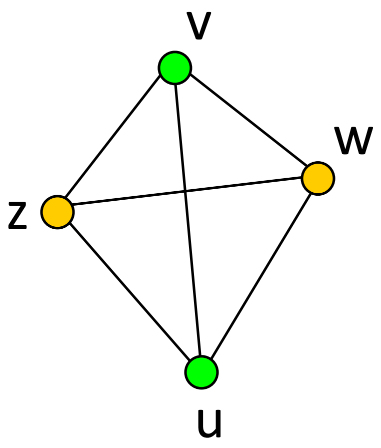


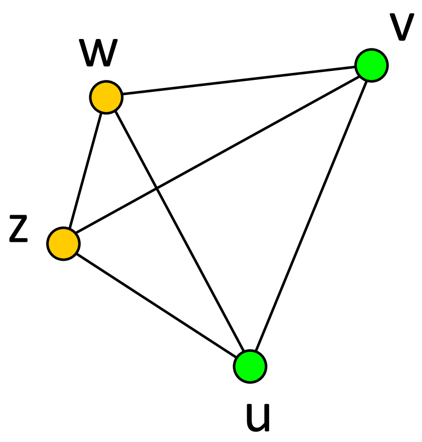
subcase 2: all the three red vertices are in the external face

— the fourth vertex must be in the external face or on its boundary, otherwise the fan property is violated (no crossing)

0-bend RAC drawings of planar graphs

Similar case analyses can be done for the other possible embeddings of $\mathbf{G'_1}$





Drawing algorithms: Open Problems

- **Problem DA1.** What is the complexity of testing whether a graph admits a 1-bend or a 2-bend RAC drawing?
- **Problem DA2.** Is it possible to realize any n-vertex graph as a k-bend RAC drawing in $O(n^2)$ area, for some $k \geq 3$?
- **Problem DA3.** Is it possible to draw every 3-graph as a 0-bend RAC drawing?
- **Problem DA4.** Design polynomial-time heuristics for computing RAC drawings with few bends in total or with few bent edges

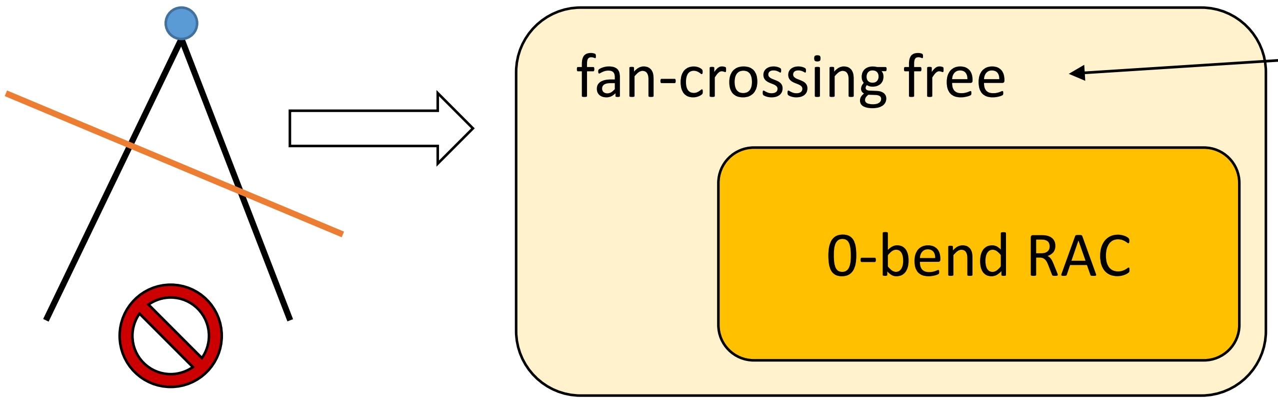
Inclusion Relationships

- **Question**: Are there interesting inclusion relationships between RAC drawable graphs and other classes of graphs that admit drawings with specific forbidden types of crossings?

Inclusion Relationships

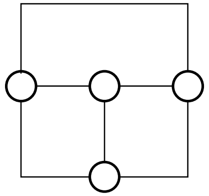
- **Question**: Are there interesting inclusion relationships between RAC drawable graphs and other classes of graphs that admit drawings with specific forbidden types of crossings?

- **Immediate**: 0-bend RAC drawable graphs are fan-crossing free



same density as 1-planar graphs (4n-8)

O. Cheong, S. Har-Peled, H. Kim, H. Kim: On the number of edges of fan-crossing free graphs. Algorithmica 73(4) (2015)



Inclusion Relationships: RAC and 1-planar

- **Question**: What is the relationship between 0-bend RAC drawable graphs and 1-planar graphs?

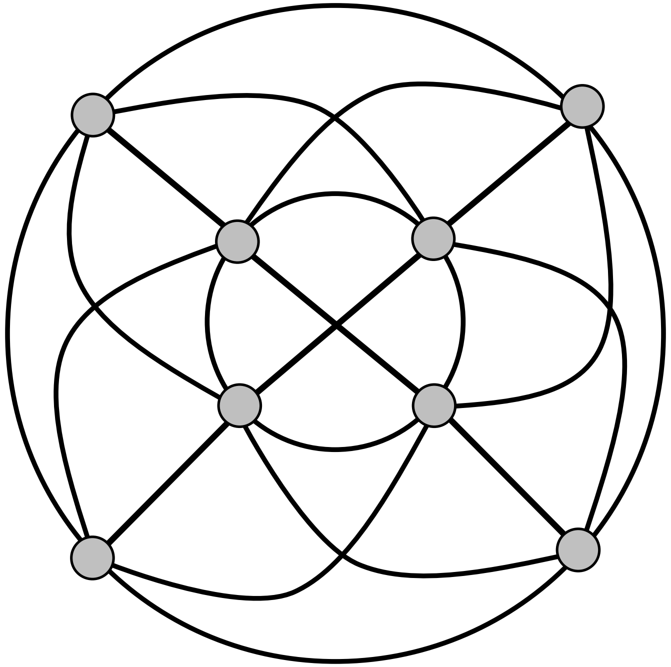
- **Remind:** a 1-planar graph is drawable with at most 1 crossing per edge:
  - 1 planar graphs have at most 4n-8 edges (tight) [*J. Pach, G. Tóth*: Graphs Drawn with Few Crossings per Edge. Combinatorica 17(3) (1997)]

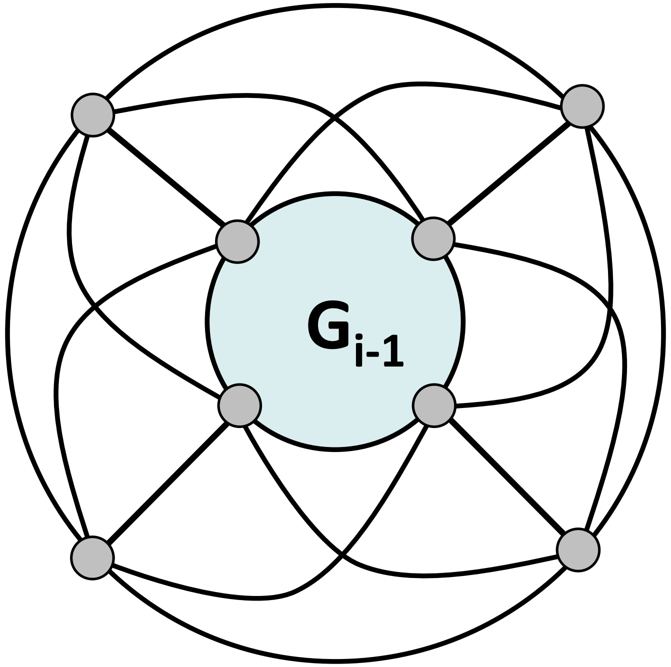
- **Observation:** What about 1-planar graphs with no more than 4n-10 edges?
  - *P. Eades, G. Liotta*: Right angle crossing graphs and 1-planarity. Discrete Applied Mathematics 161(7-8) (2013)

Inclusion Relationships: RAC and 1-planar

- This family of graphs is 1-planar but *not* 0-bend RAC

G_0

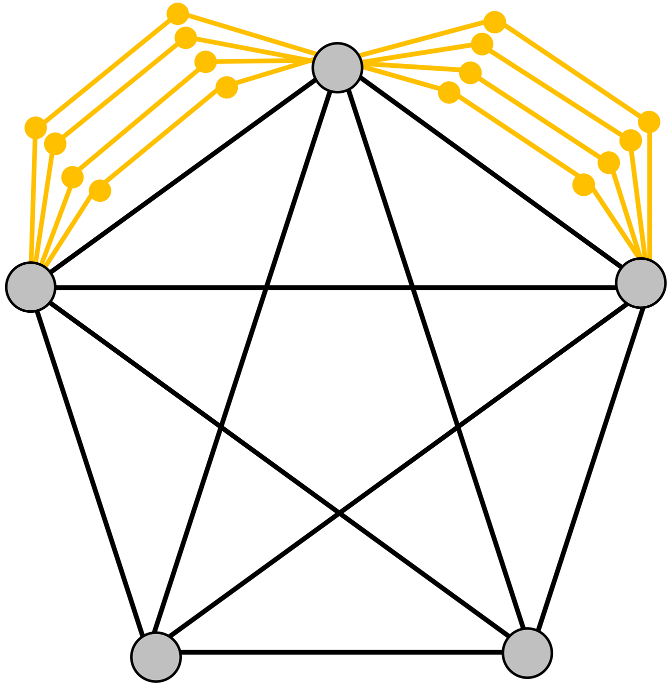




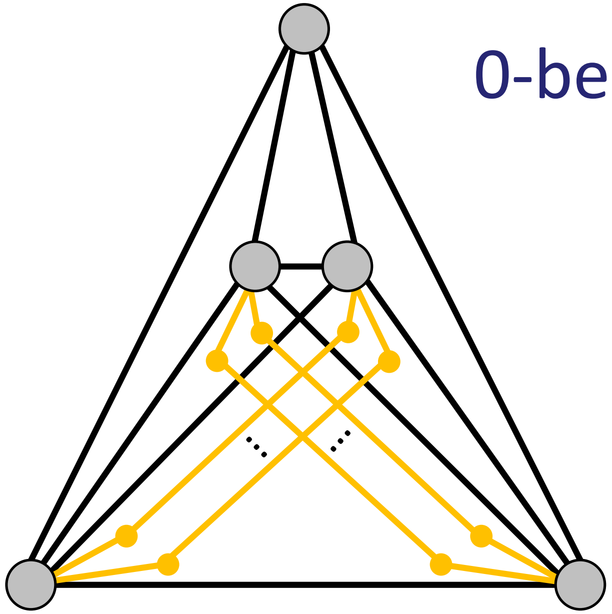
G_o has **n=8** vertices and **4n-10=22** edges;
for **i≥0**, **G_i** has **n=8+4i** vertices and **4n-10** edges

Inclusion Relationships: RAC and 1-planar

- This graph is 0-bend RAC but *not* 1-planar



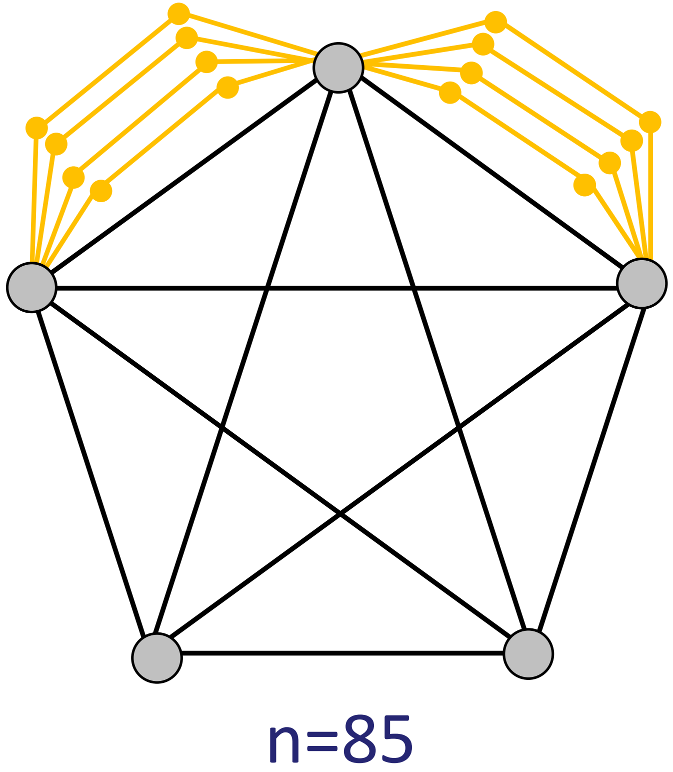
n=85

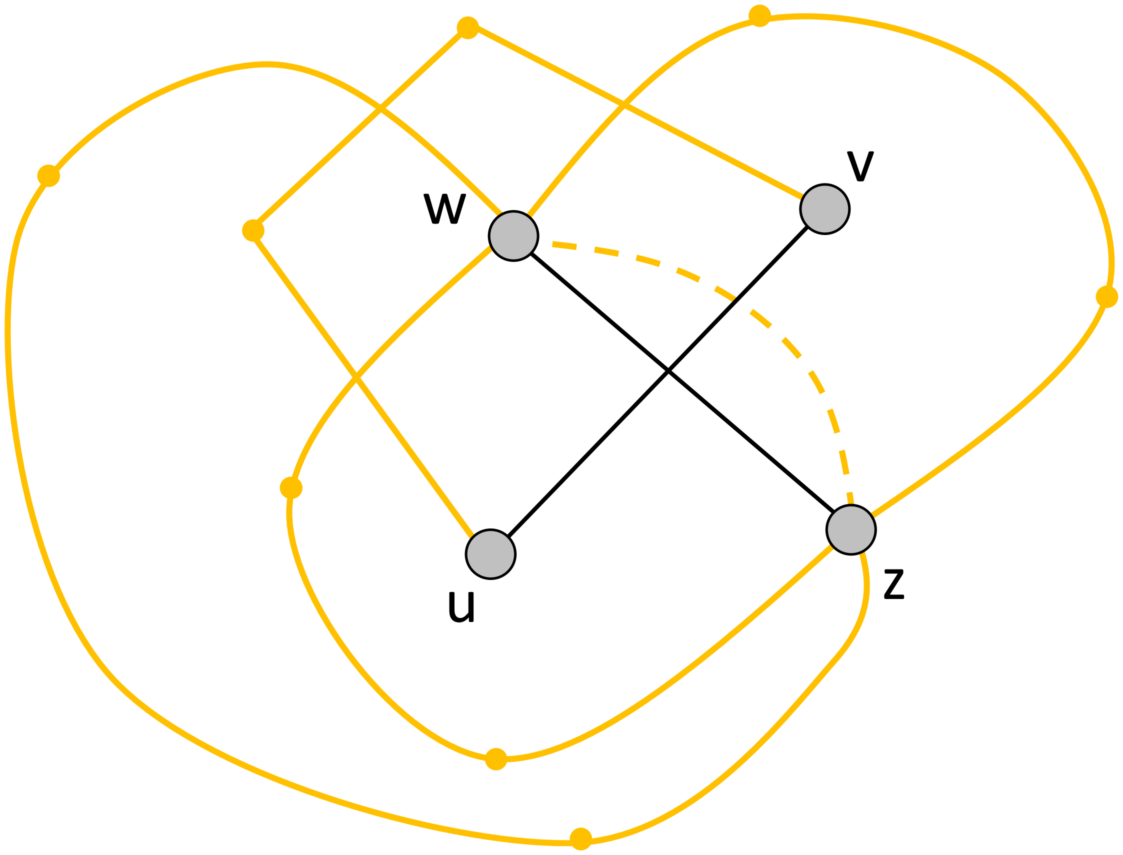


0-bend RAC drawing

Inclusion Relationships: RAC and 1-planar

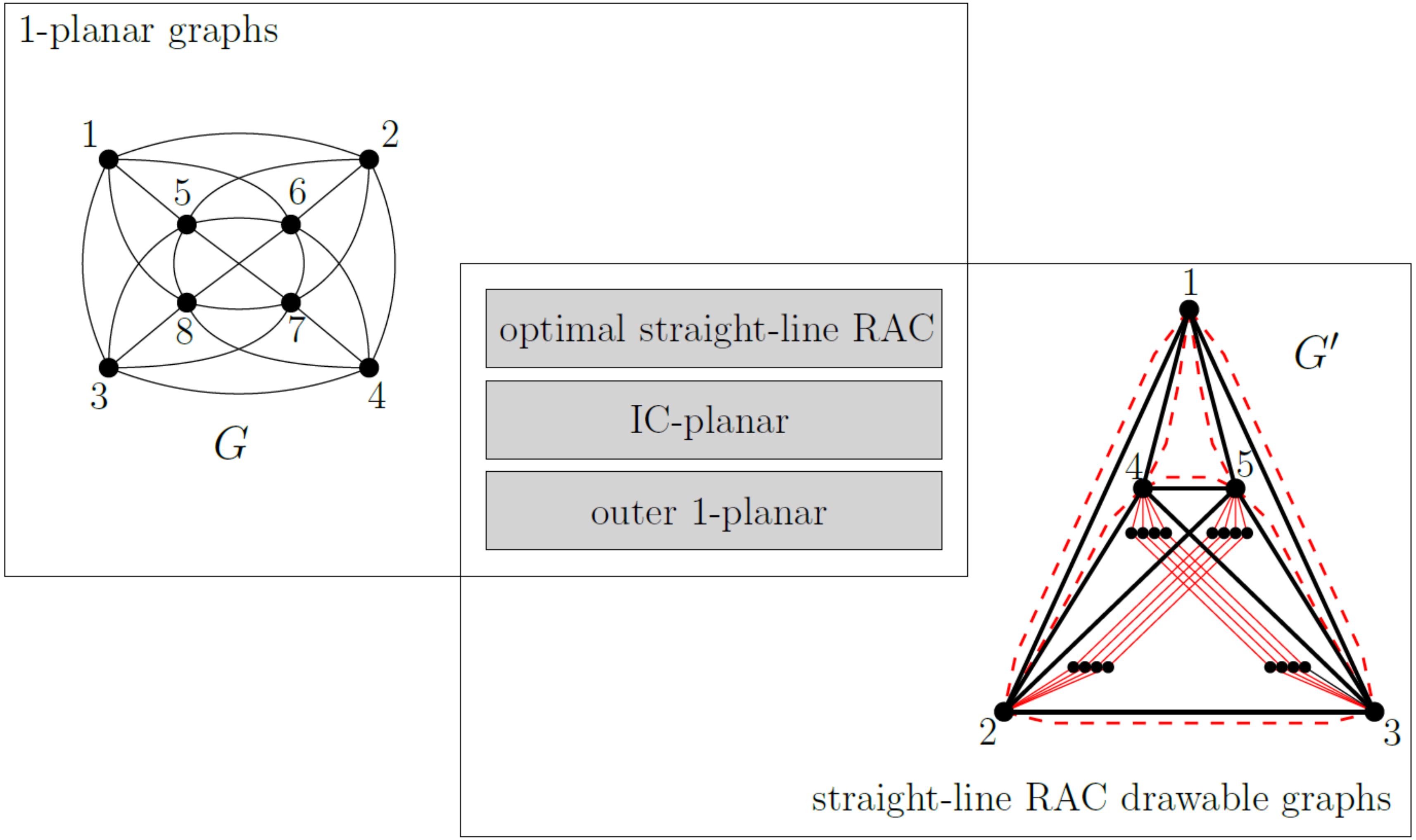
- This graph is 0-bend RAC but *not* 1-planar





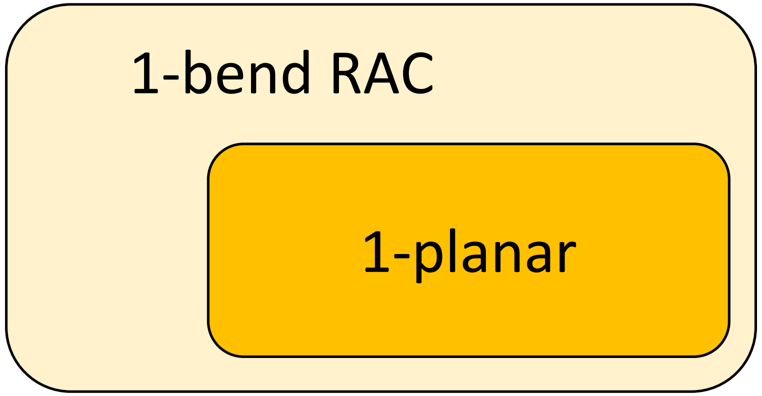
Inclusion Relationships: RAC and 1-planar

Summary

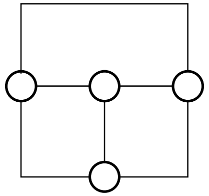


Inclusion Relationships: 1-bend RAC and 1-planar

- **Question**: What about 1-bend RAC and 1-planar?

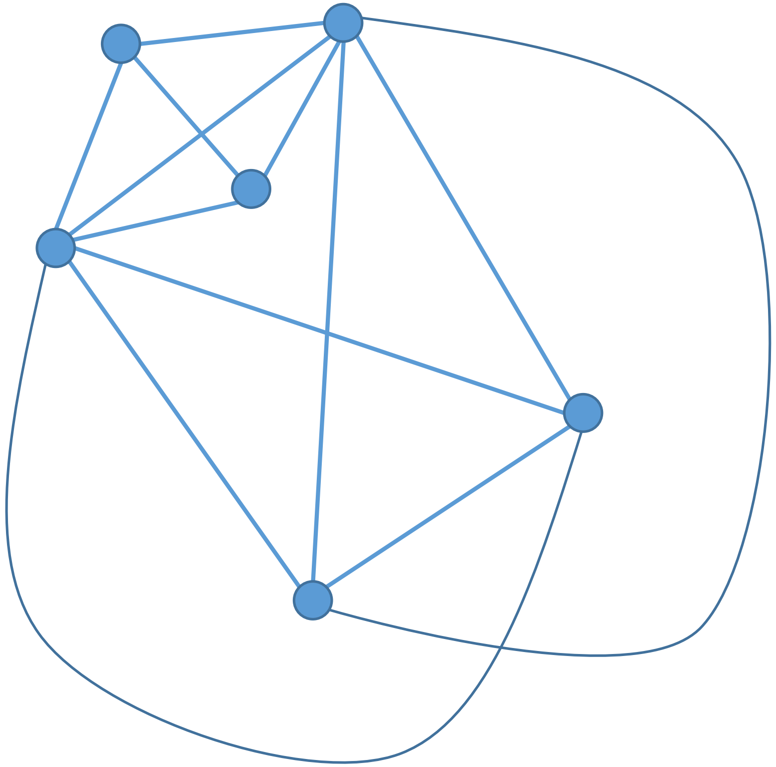


M. A. Bekos, W. Didimo, G. Liotta, S. Mehrabi, F. Montecchiani: On RAC drawings of 1-planar graphs. Theor. Comput. Sci. 689 (2017)



Some definitions

1-plane graph
(not necessarily simple)



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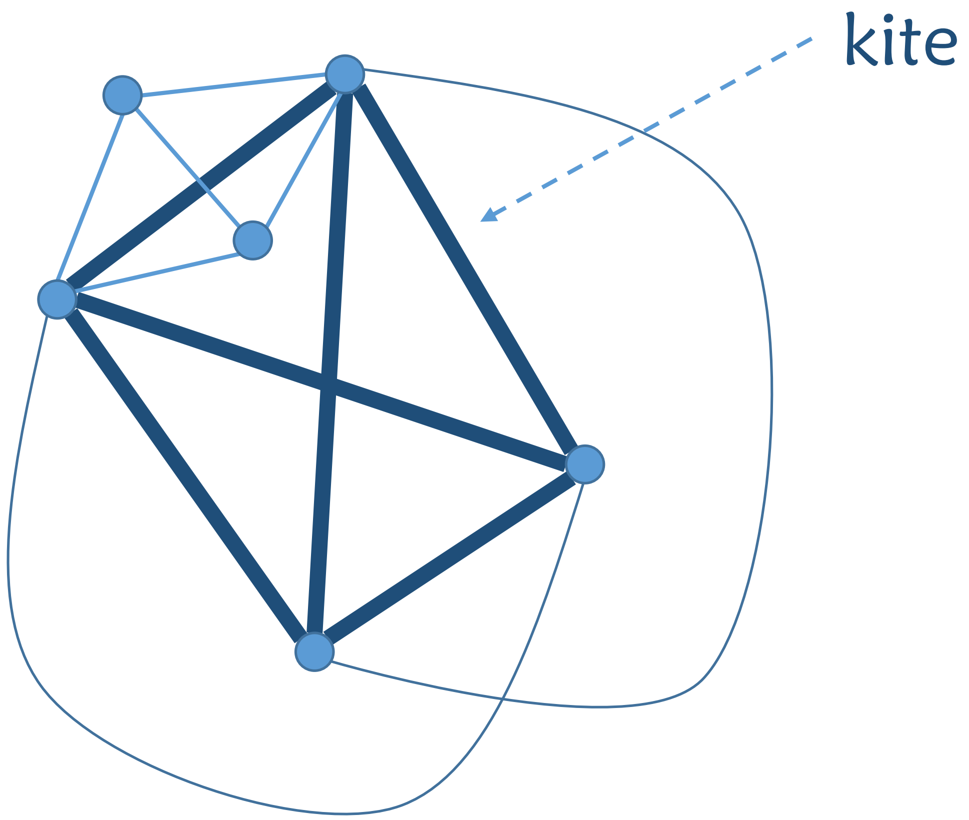

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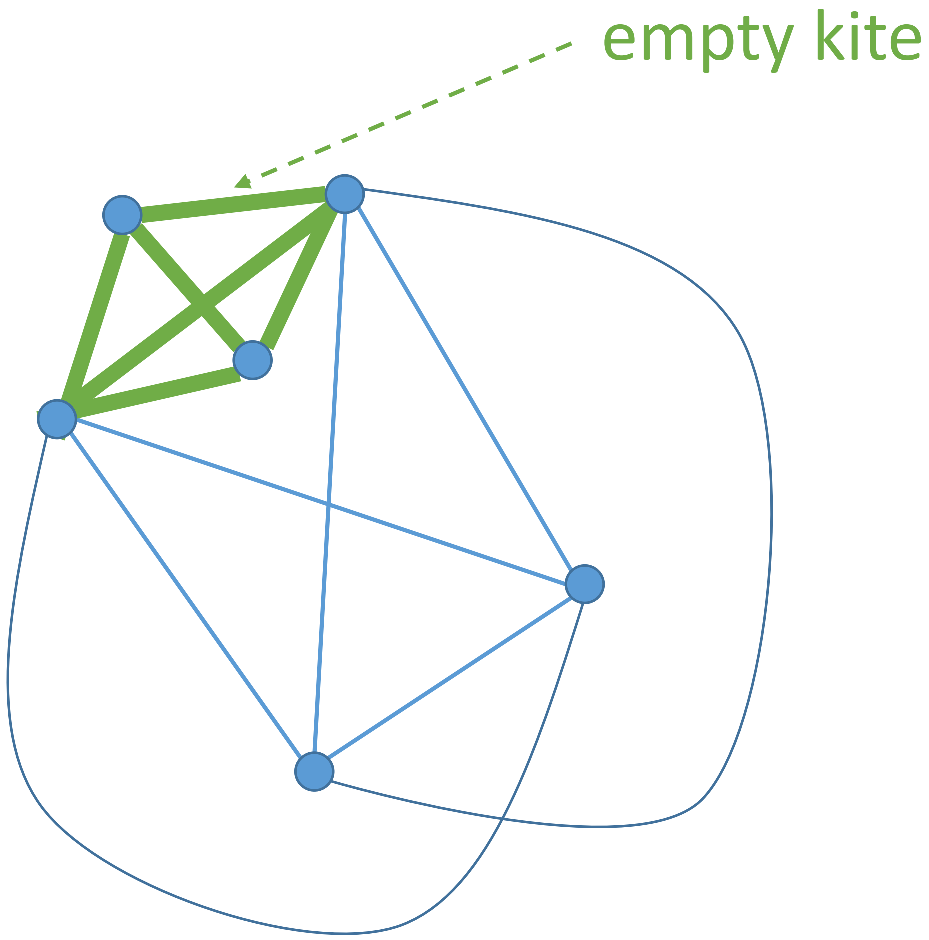

Some definitions

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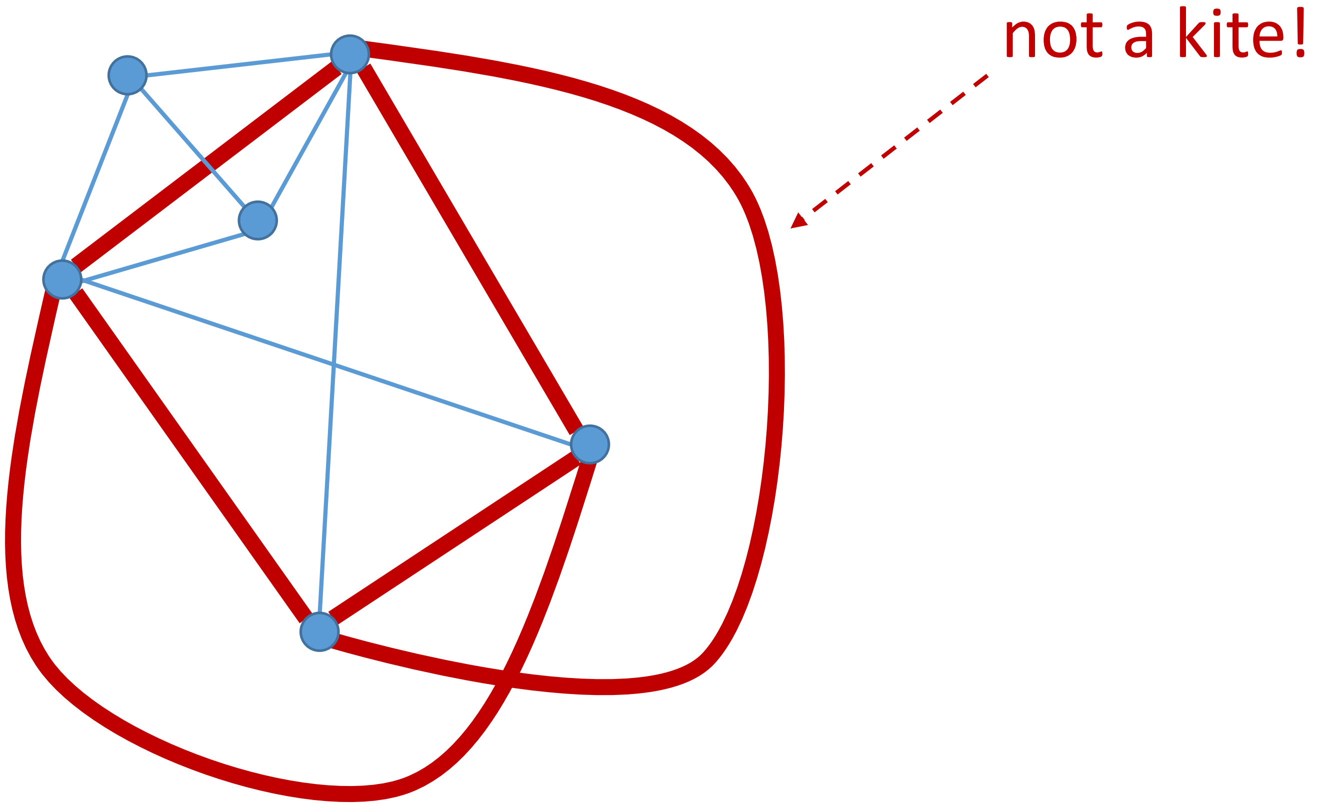
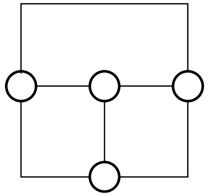
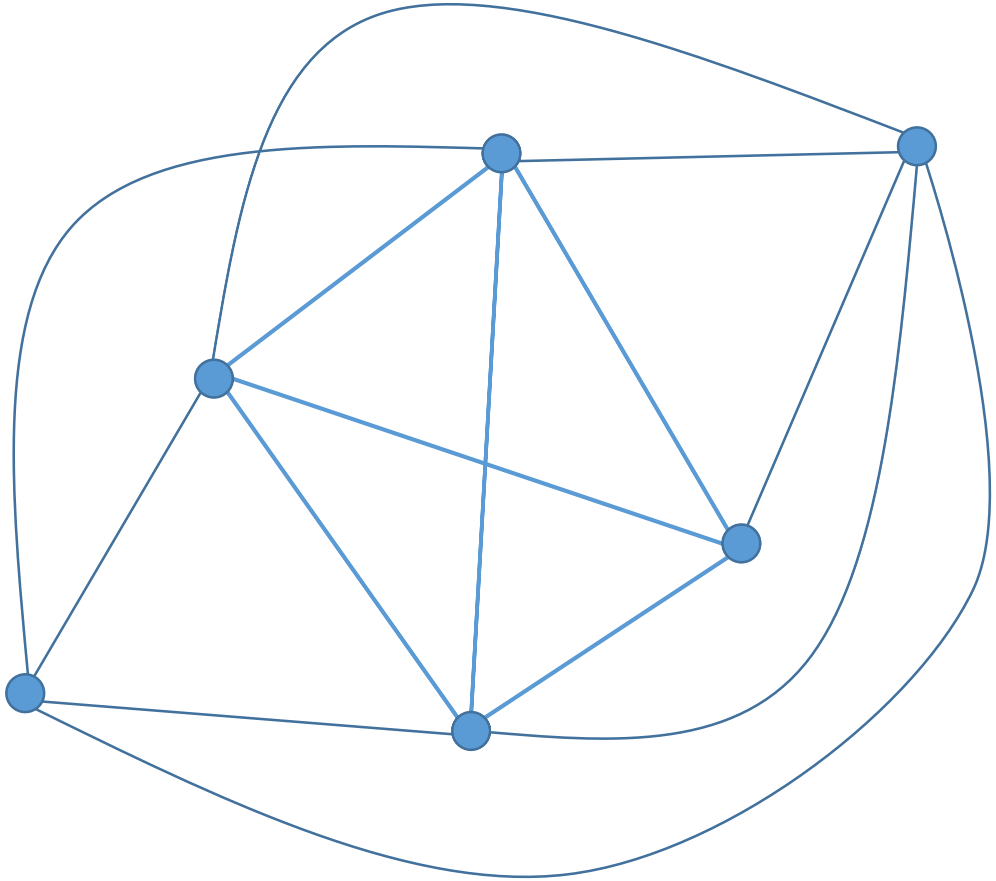




Observation

triangulated 1-plane graph
(not necessarily simple)

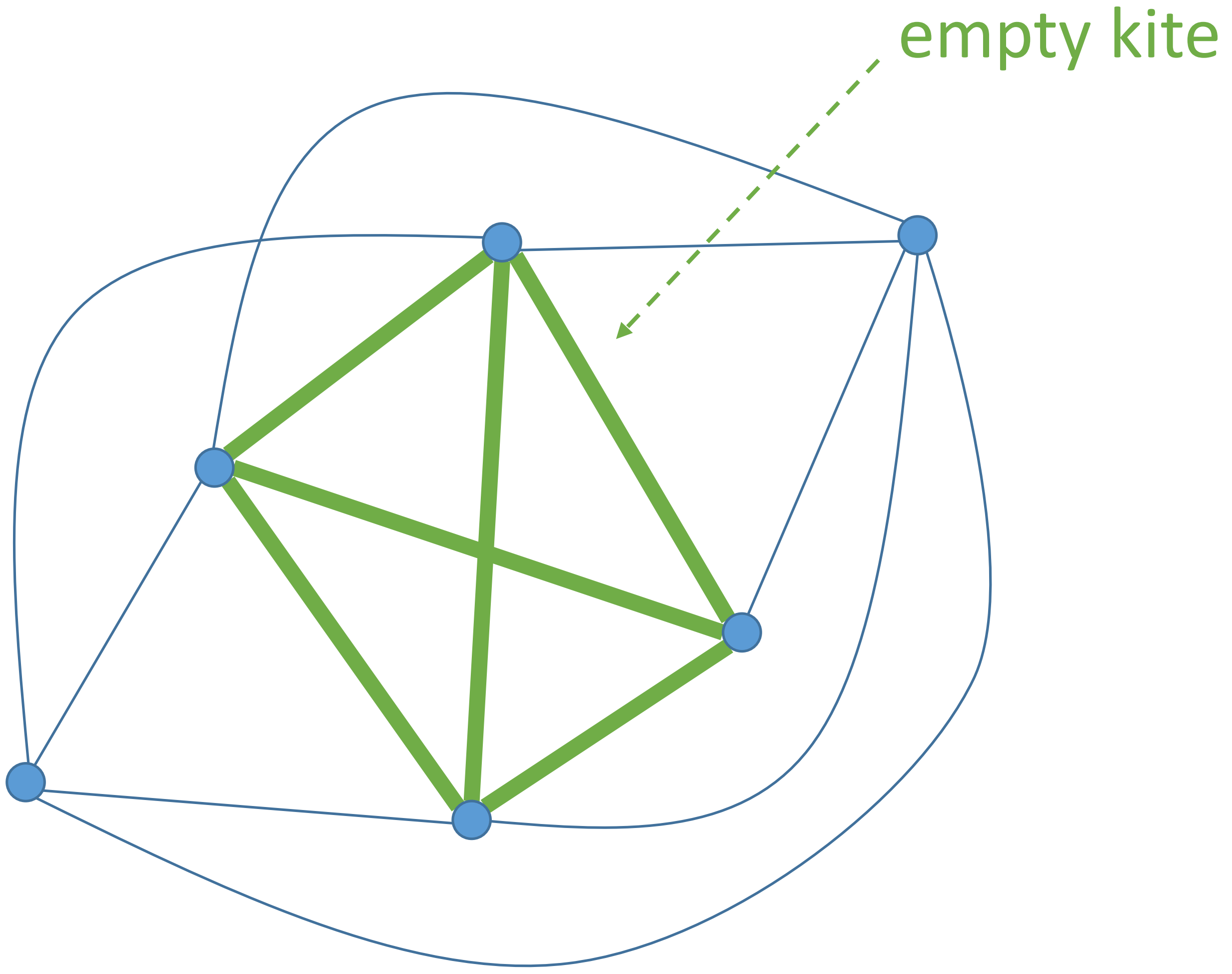


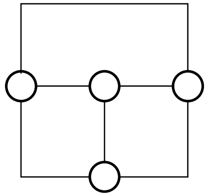
Observation

triangulated 1-plane graph
(not necessarily simple)

⇓

every pair of crossing edges forms an empty kite except possibly for a pair of crossing edges on the outer face





Observation

triangulated 1-plane graph
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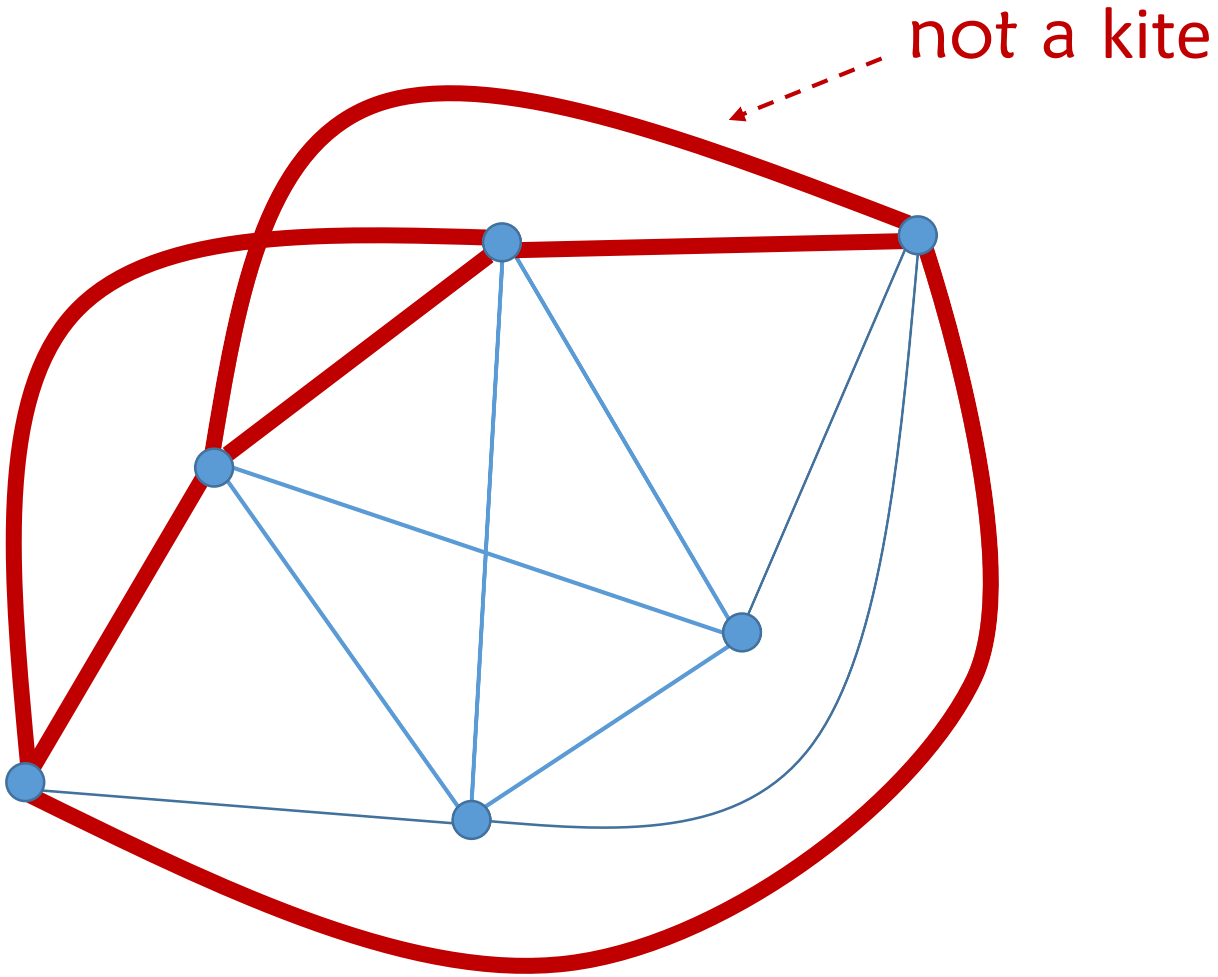
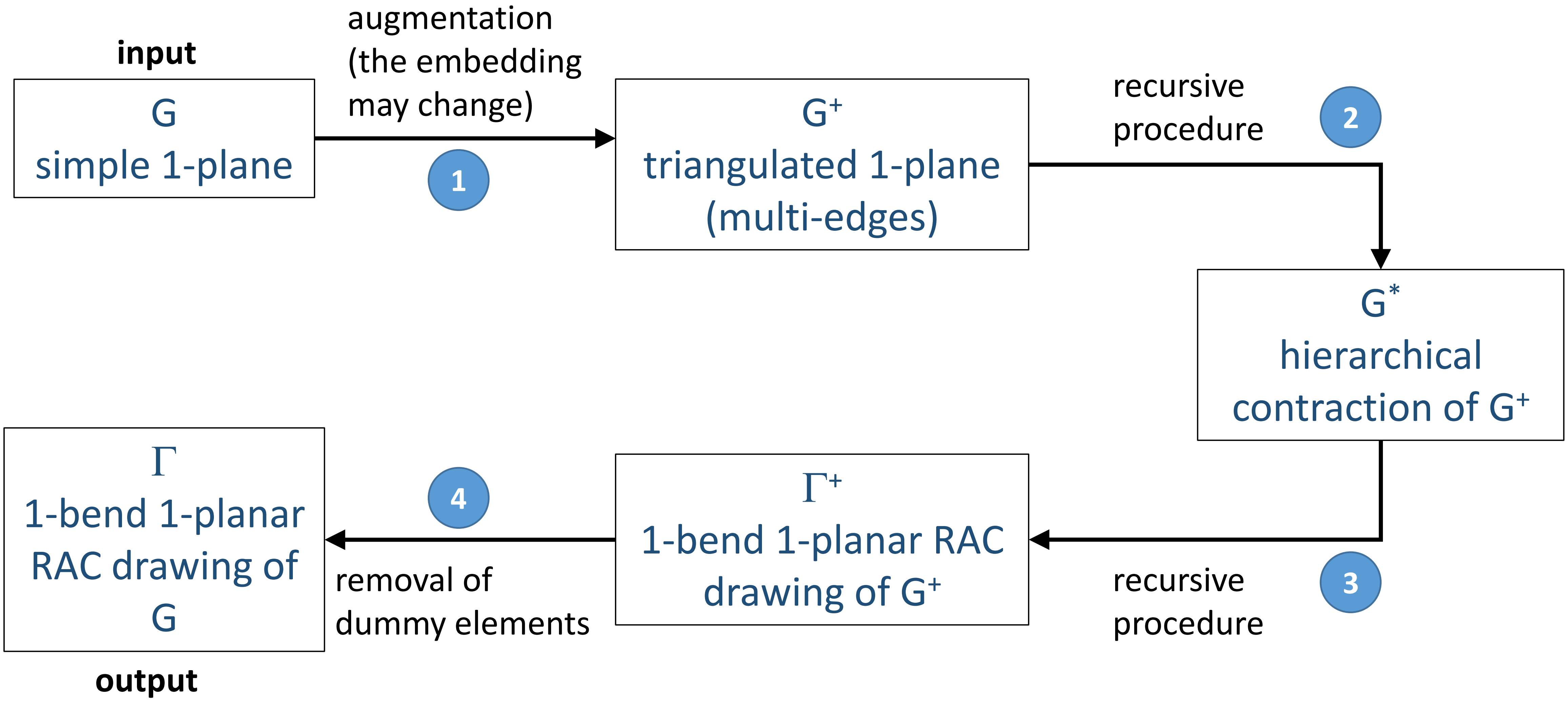
⬇

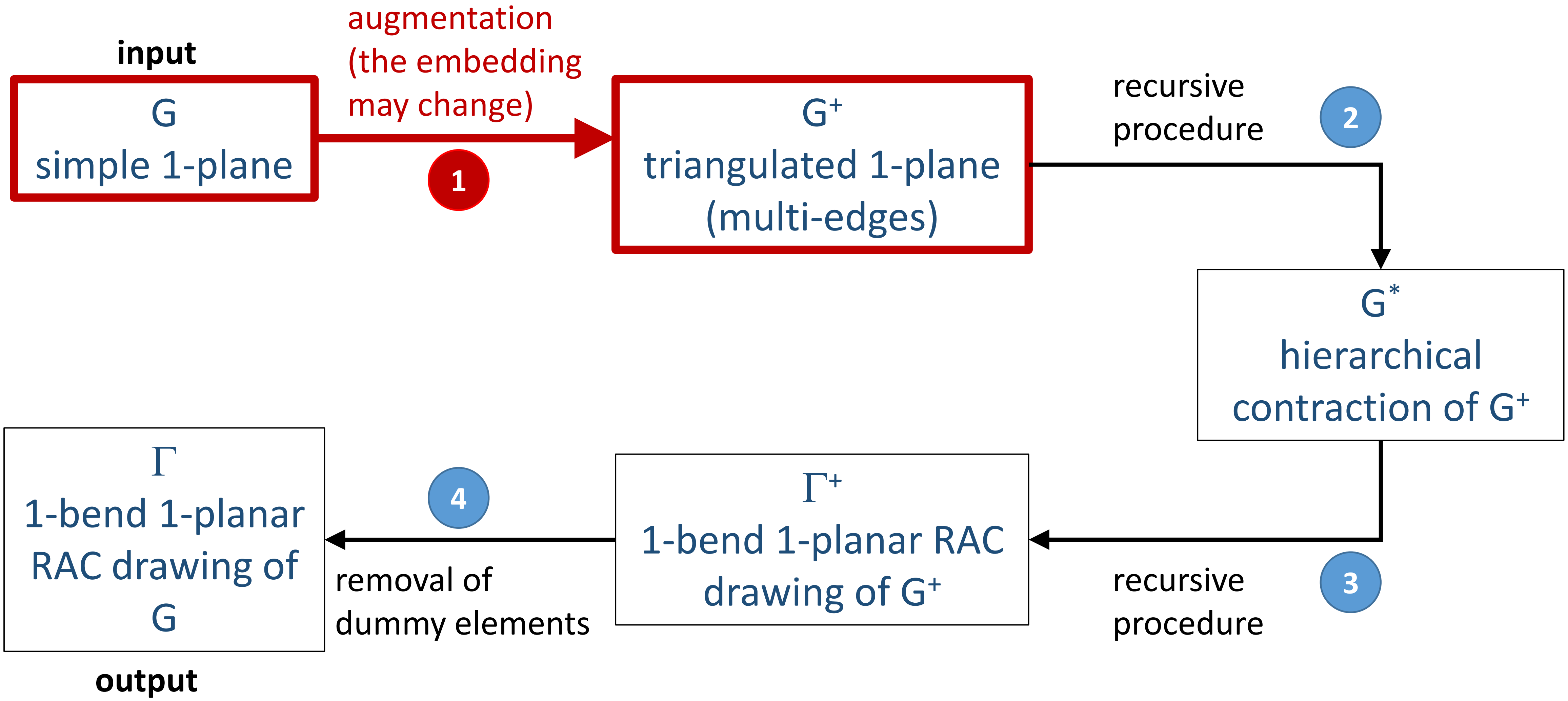
every pair of crossing edges forms an empty kite except possibly for a pair of crossing edges on the outer face



Algorithm Outline

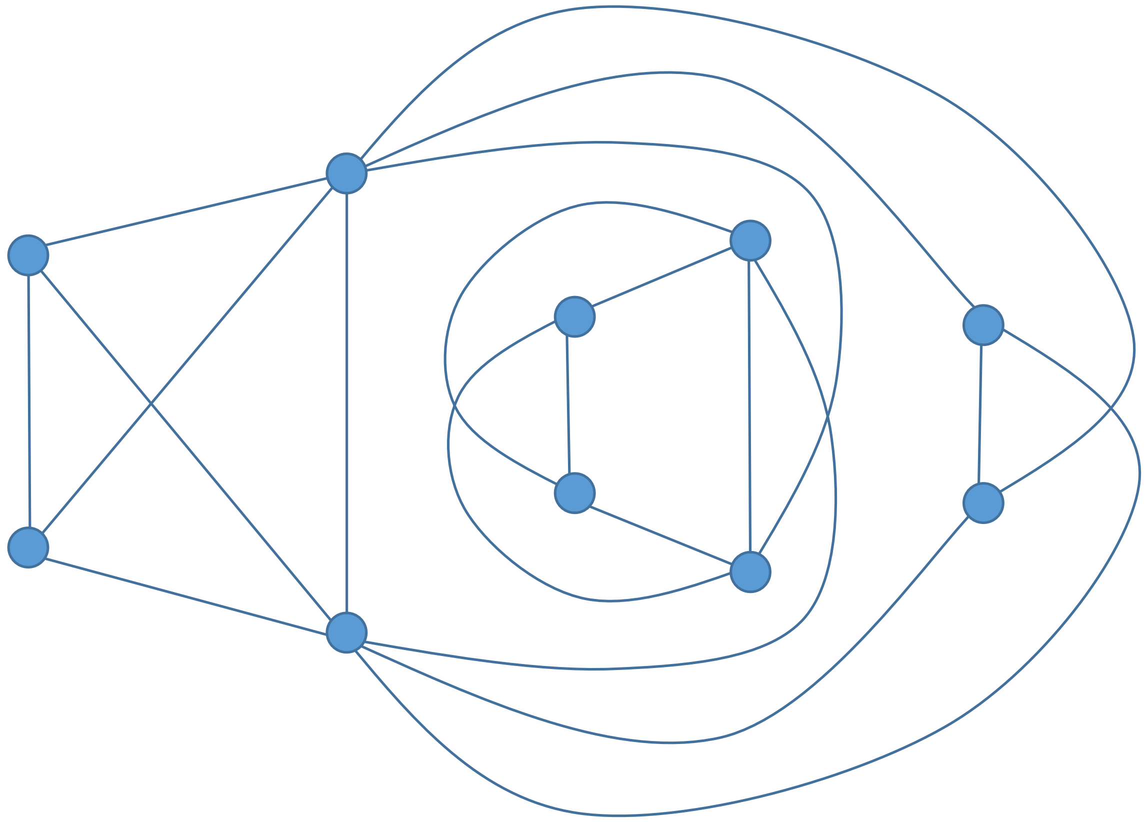


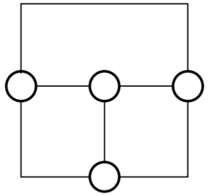
Algorithm Outline



Augmentation

G
simple 1-plane

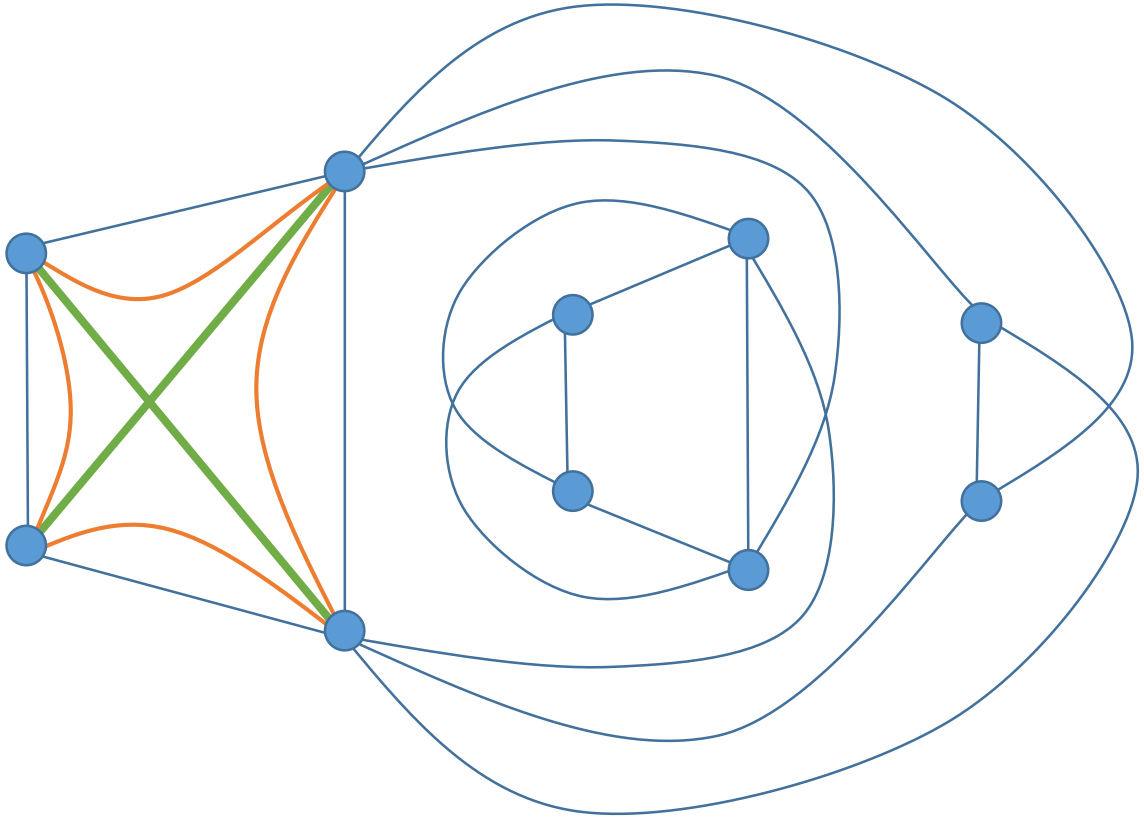




Augmentation

G

simple 1-plane



for each

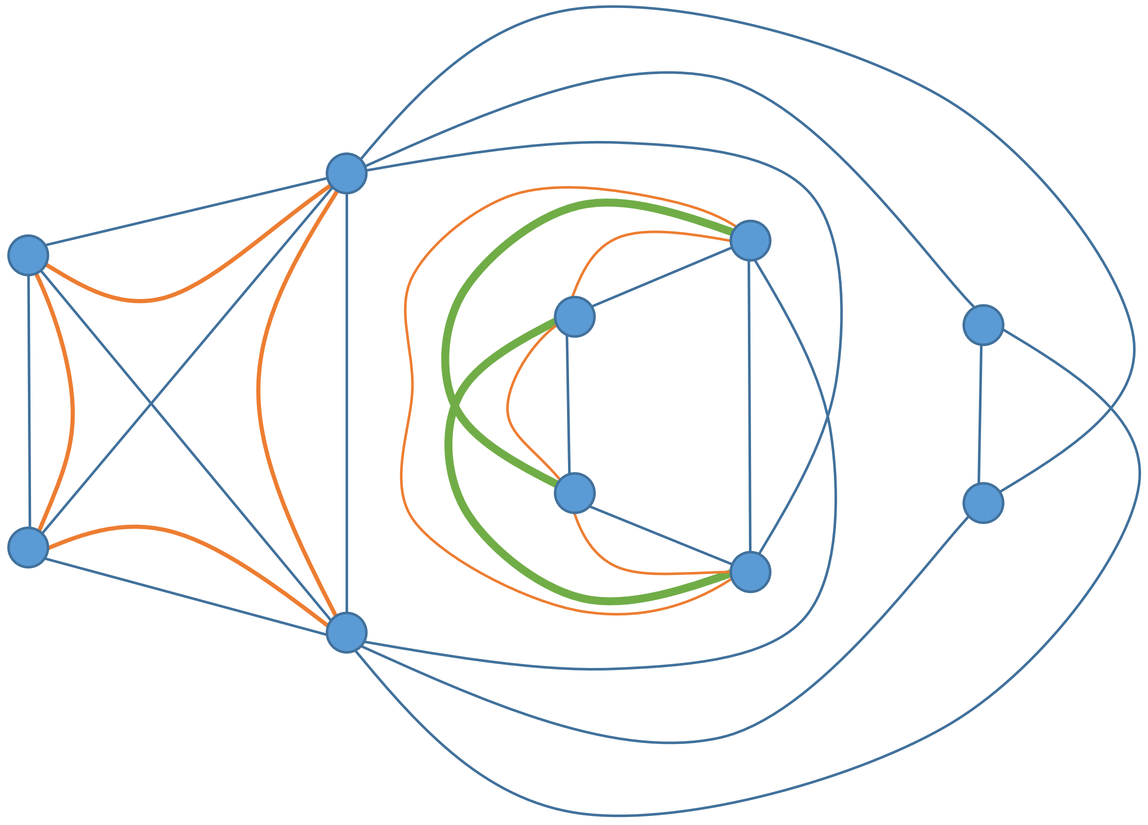
pair of crossing edges

add an enclosing

4-cycle

Augmentation

G
simple 1-plane



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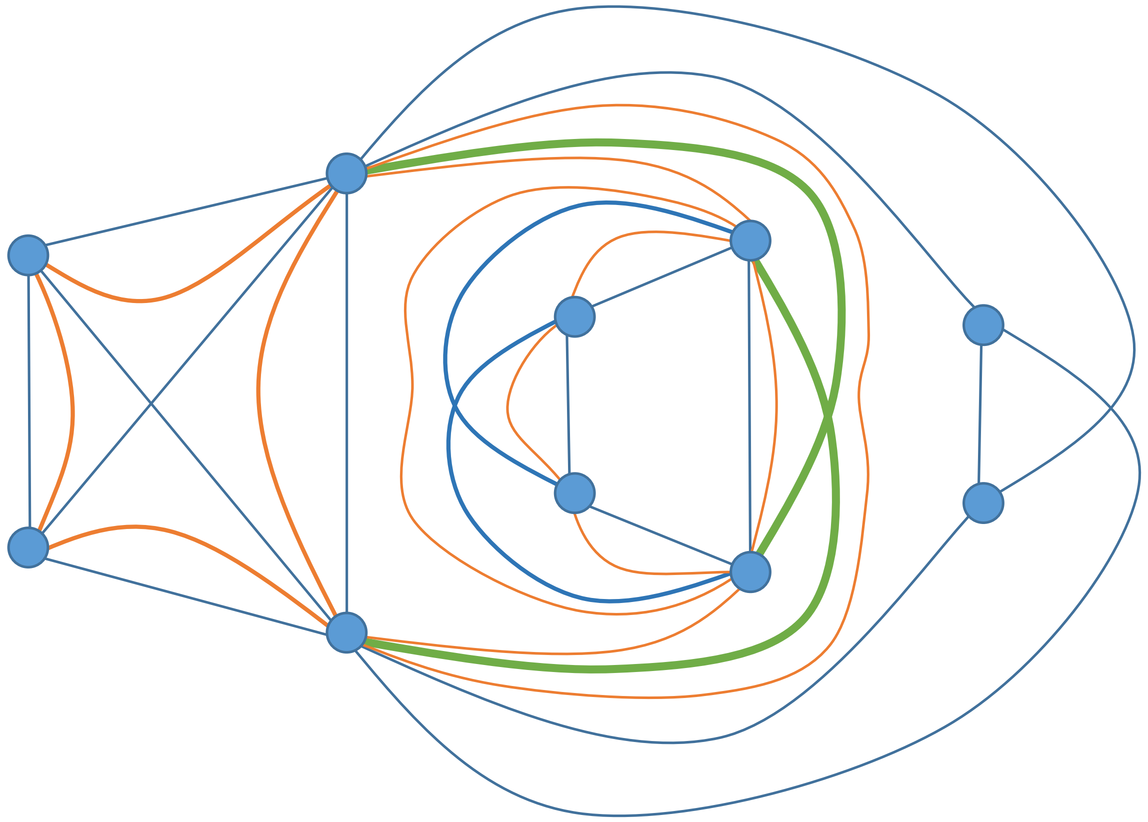
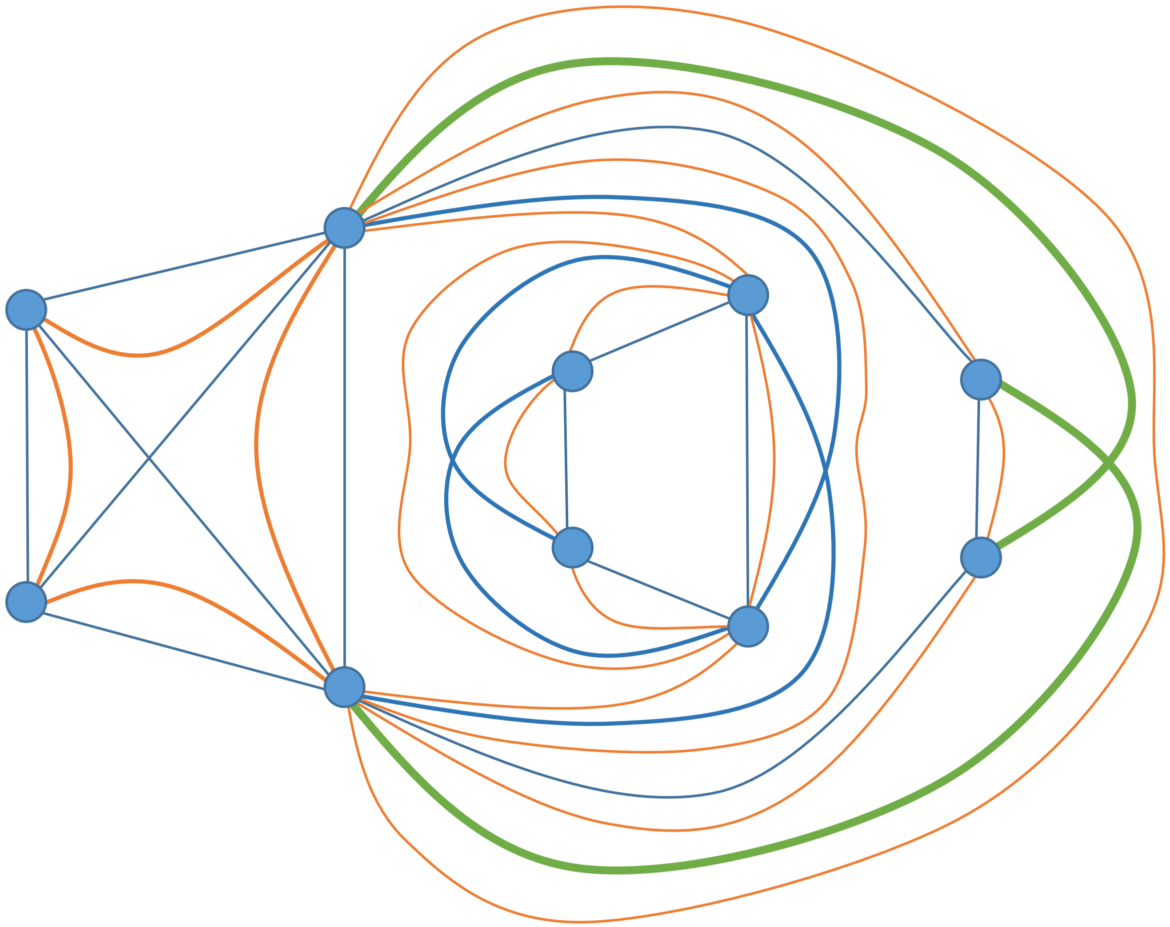
G
simple 1-plane



for each
pair of crossing edges
add an enclosing
4-cycle

Augmentation

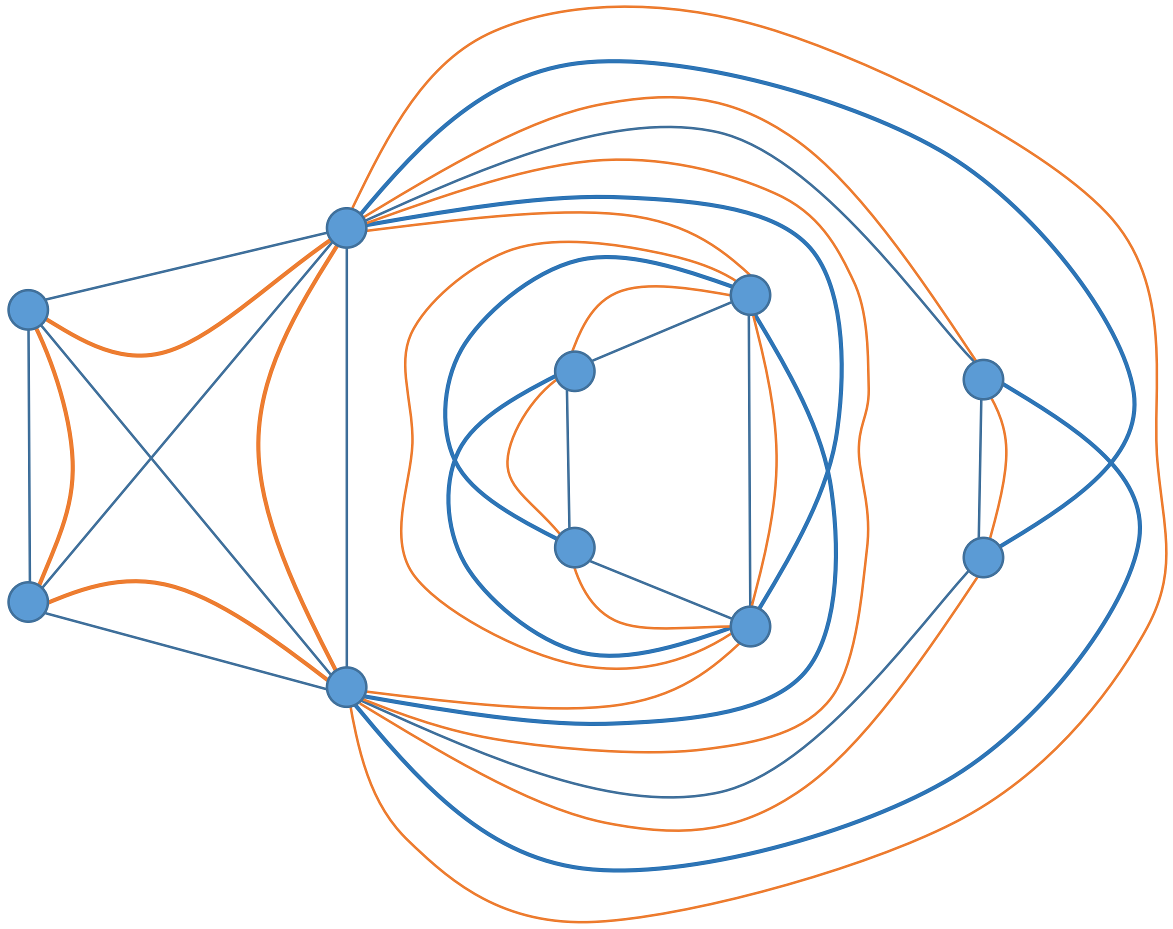
G
simple 1-plane



for each
pair of crossing edges
add an enclosing
4-cycle

Augmentation

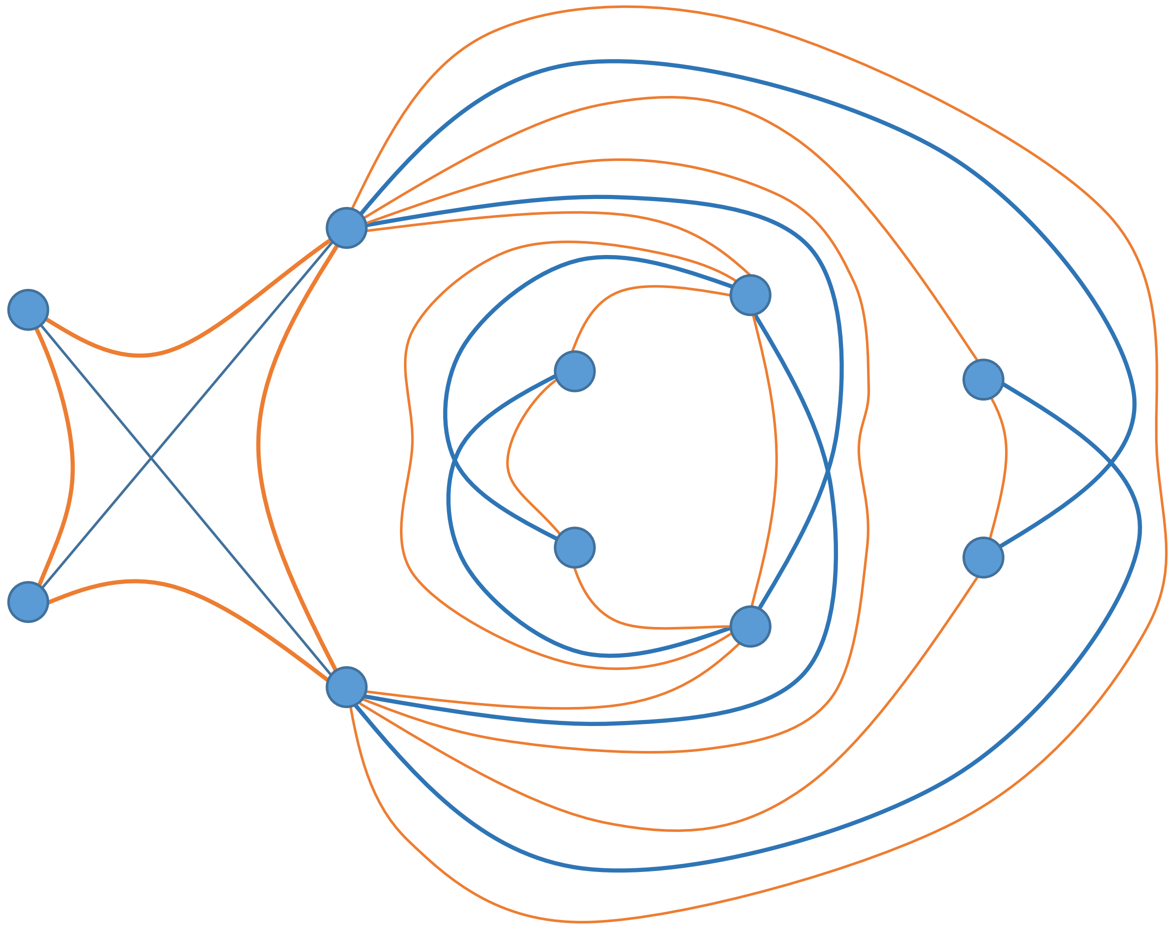
G
simple 1-plane

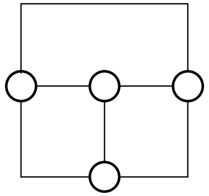


remove those multiple edges that belong to the input graph

Augmentation

G
simple 1-plane

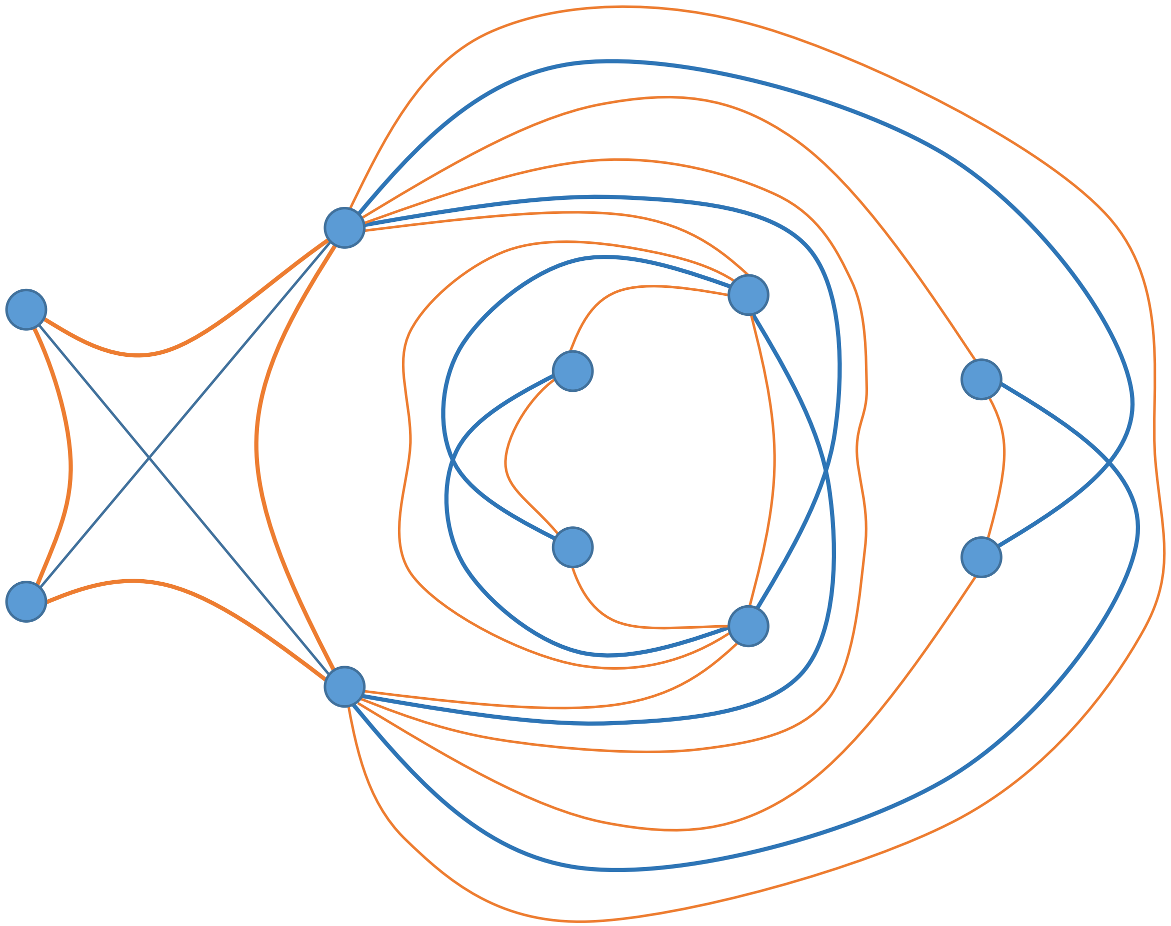




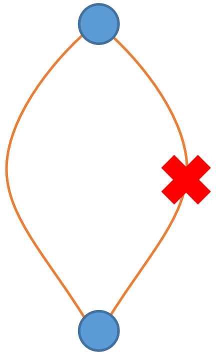
Augmentation

G

simple 1-plane

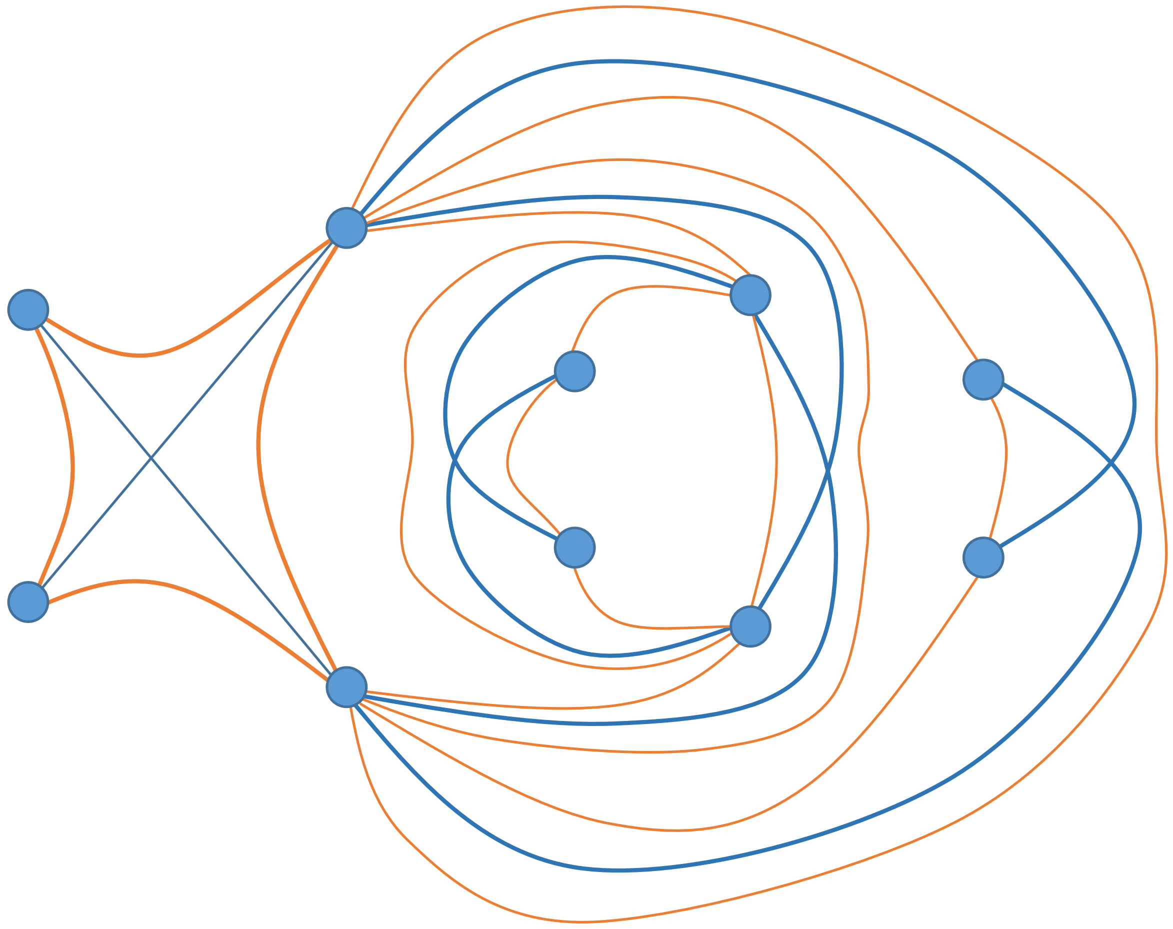


remove one (multiple) edge from each face of degree two, if any

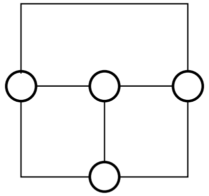


Augmentation

G
simple 1-plane

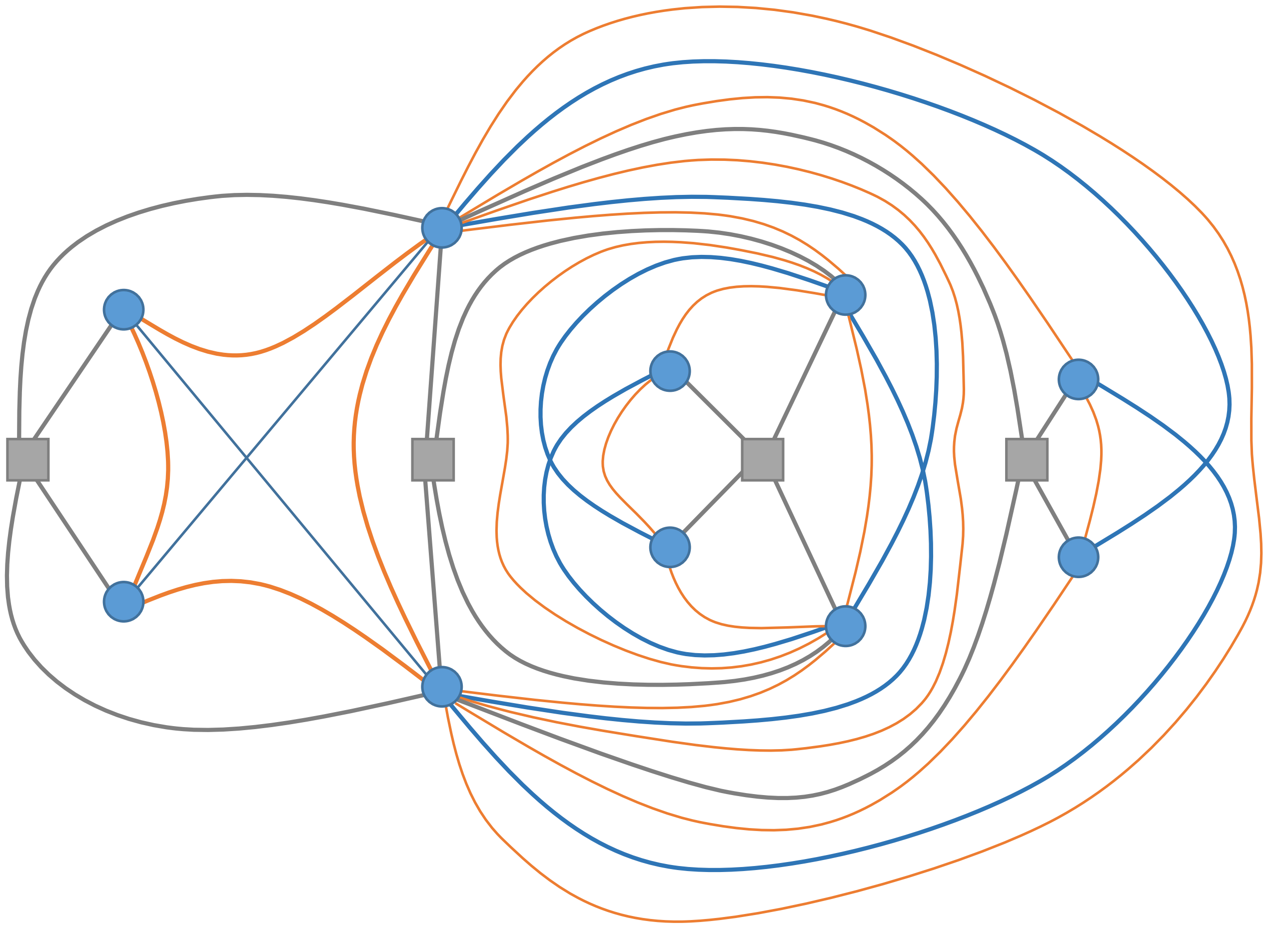


triangulate faces of degree > 3 by inserting a **star** inside them

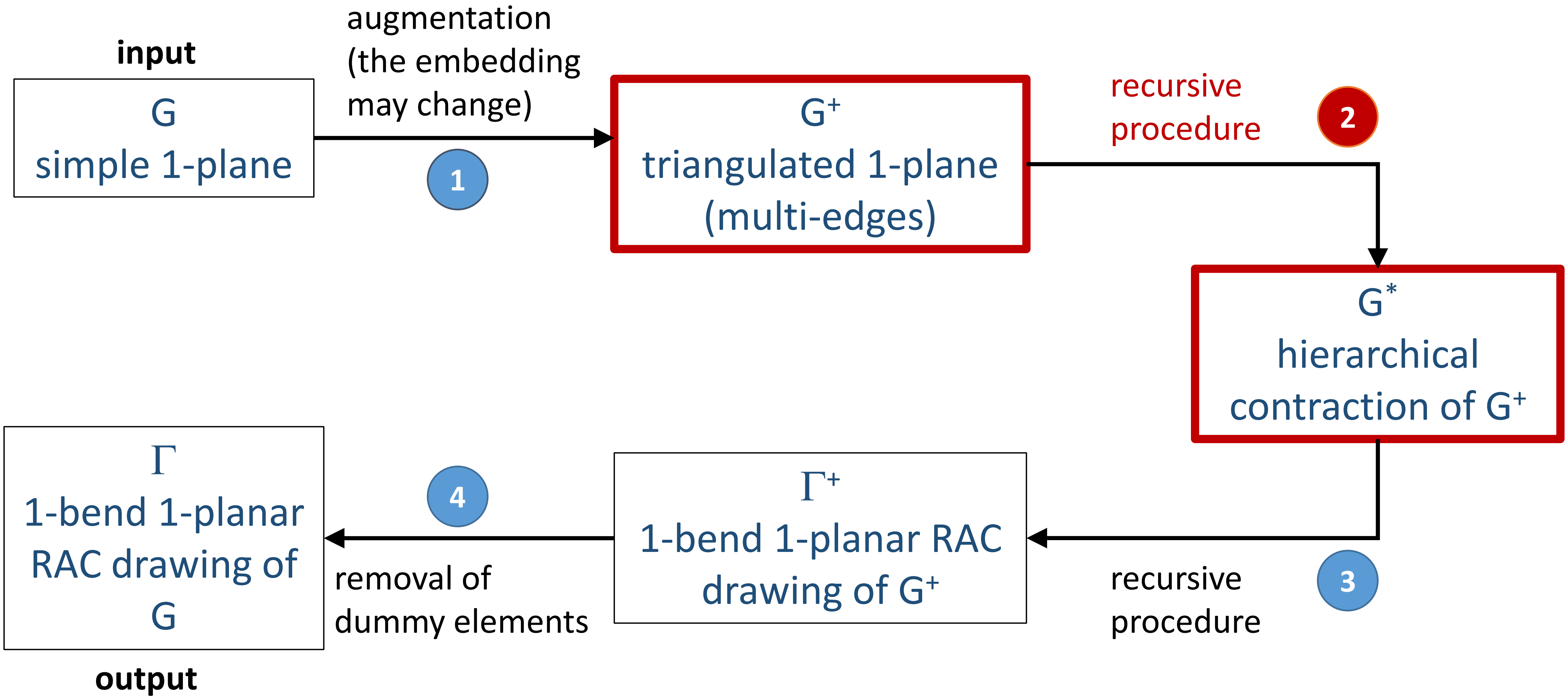


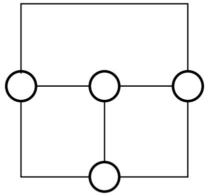
Augmentation

G⁺
triangulated 1-plane

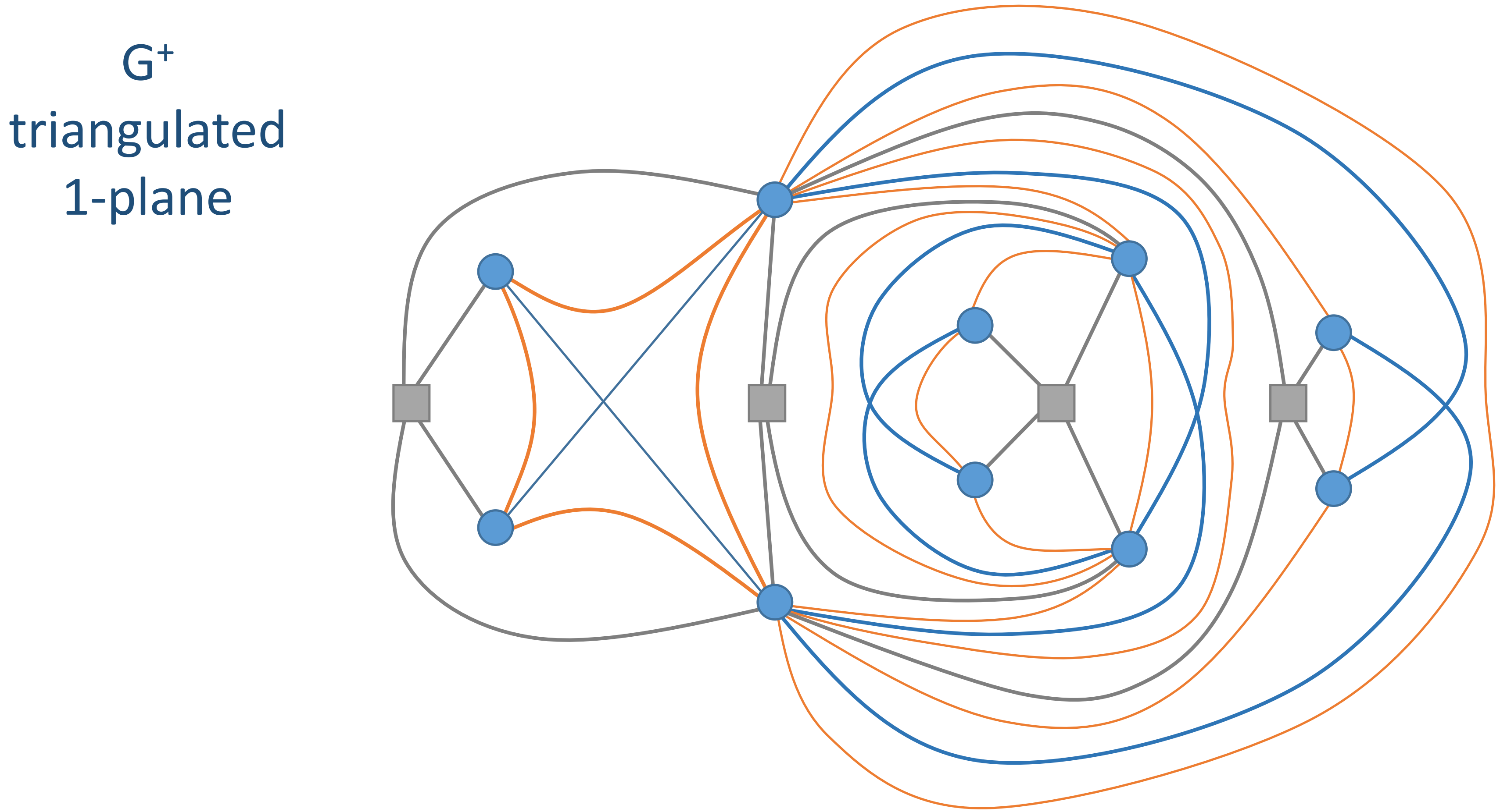


Algorithm Outline





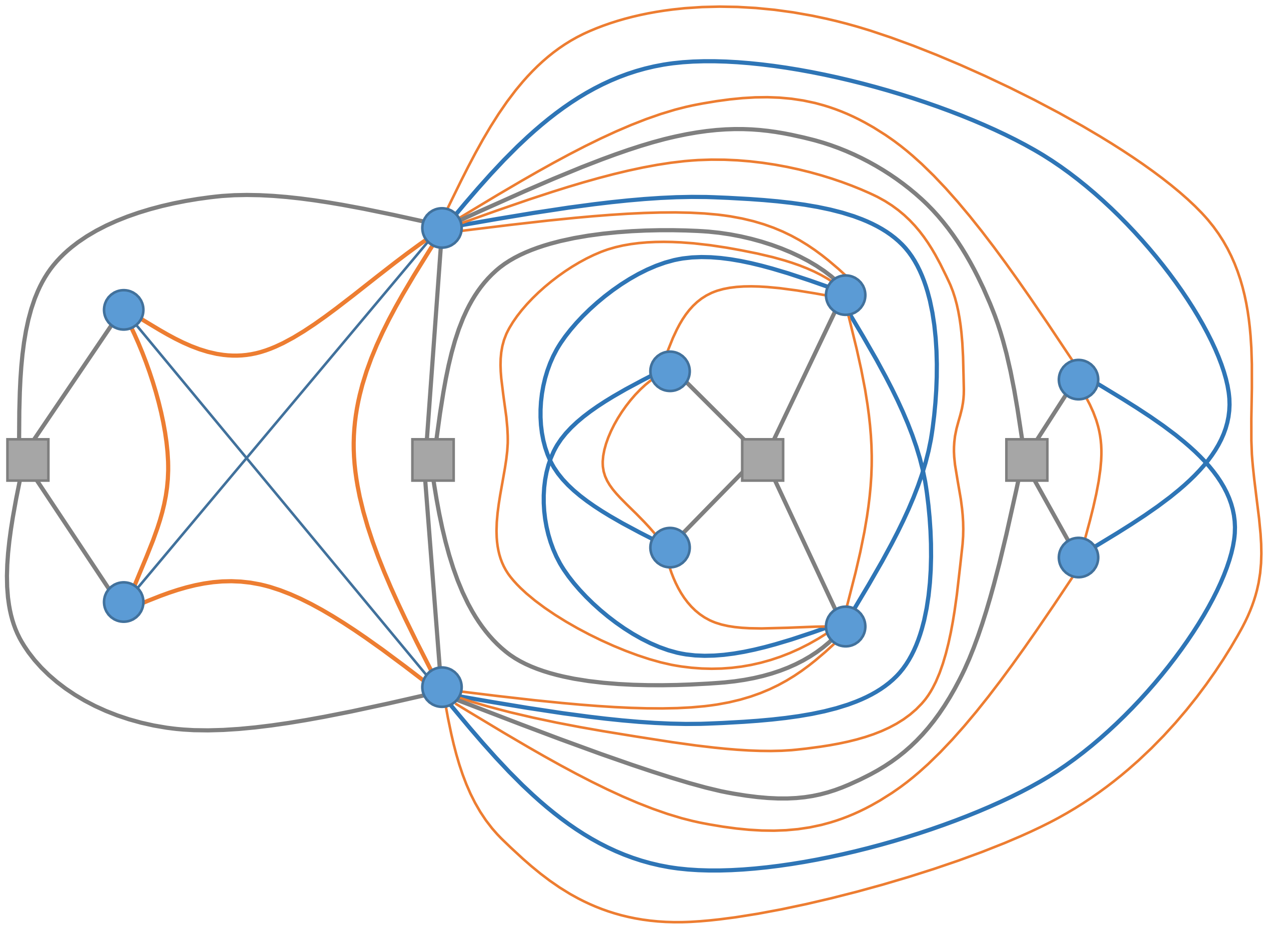
Property of G^+



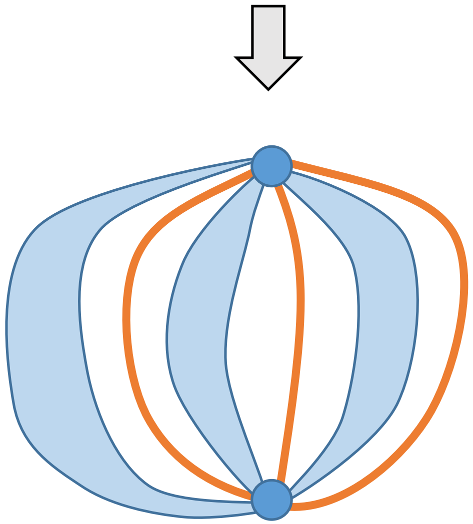
- triangular faces
- multiple edges never crossed
- only empty kites

Property of G^+

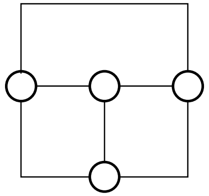
G^+
triangulated
1-plane



- triangular faces
- multiple edges never crossed
- only empty kites

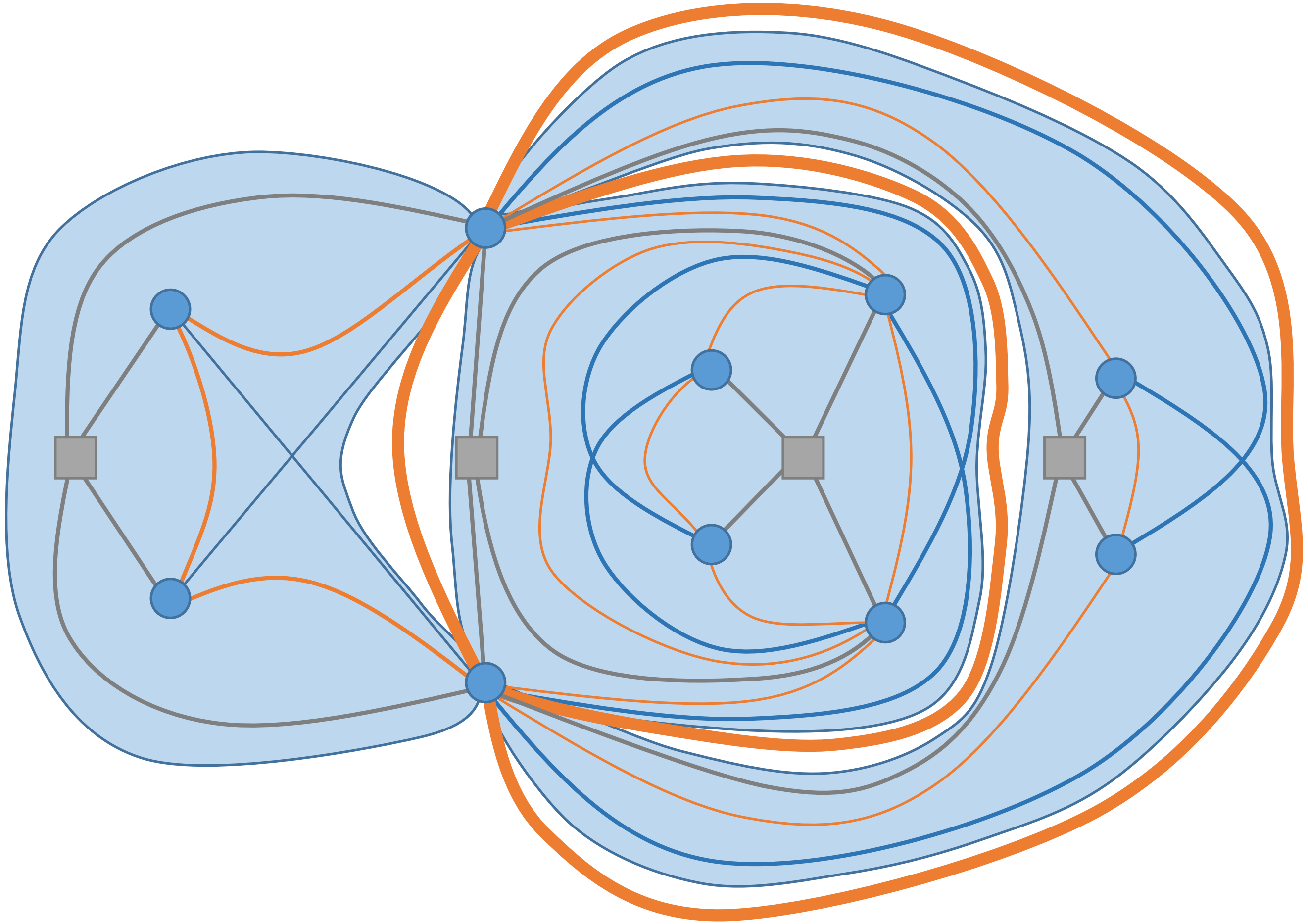


structure of each separation pair

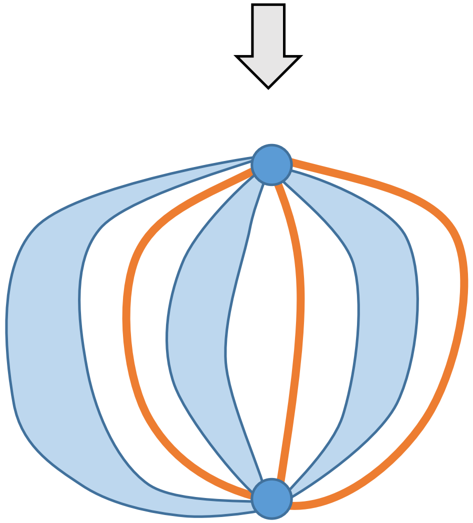


Property of G⁺

G⁺
triangulated
1-plane



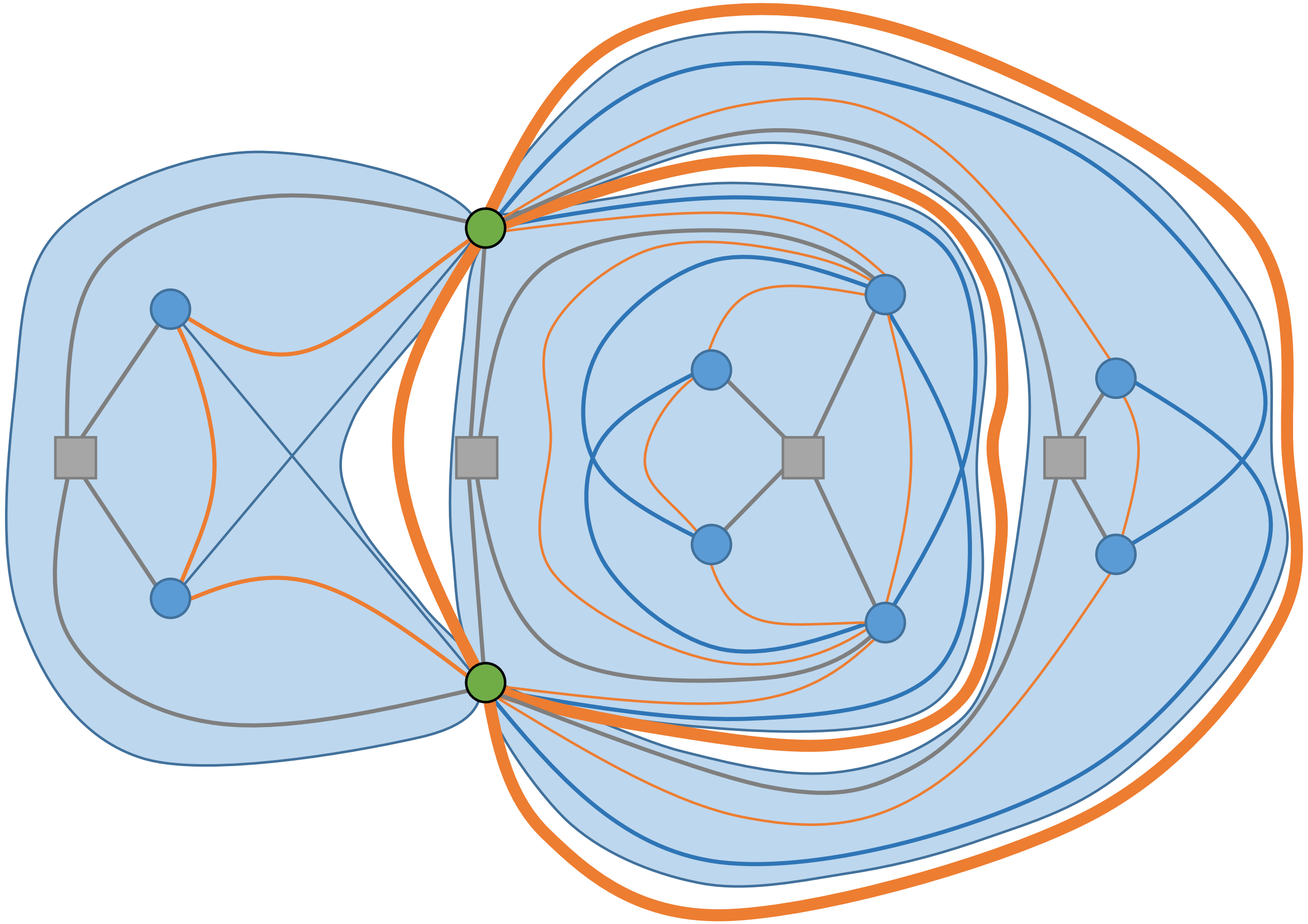
- triangular faces
- multiple edges never crossed
- only empty kites



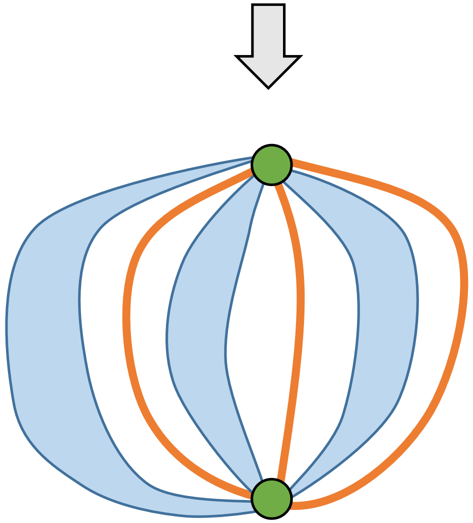
structure of each separation pair

Hierarchical contraction

G^+
triangulated
1-plane



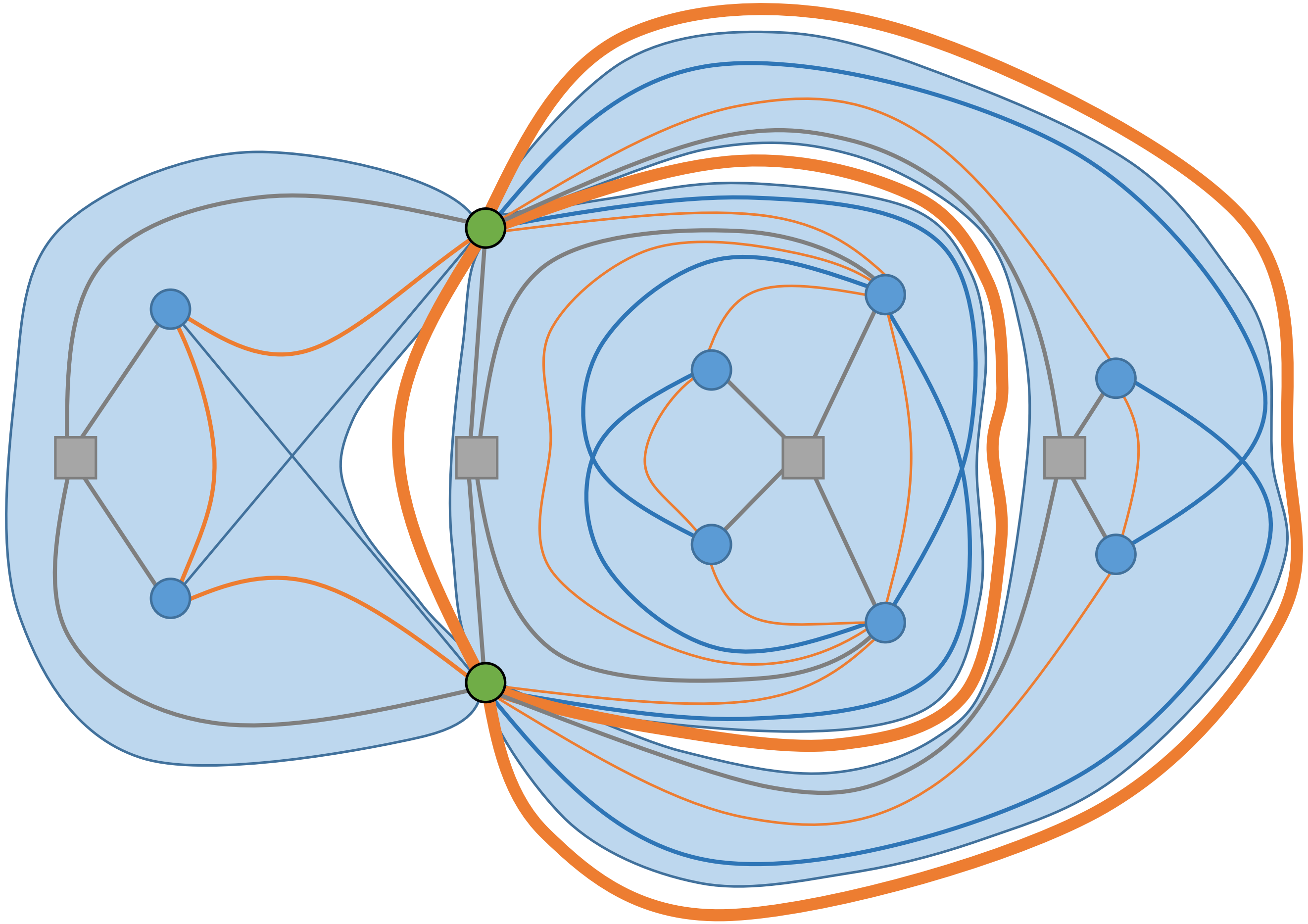
contract all inner components of each separation pair into a **thick edge**



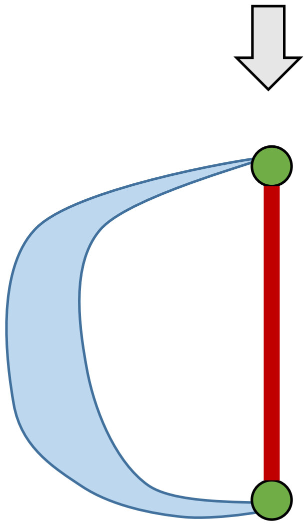
structure of each separation pair

Hierarchical contraction

G⁺
triangulated
1-plane



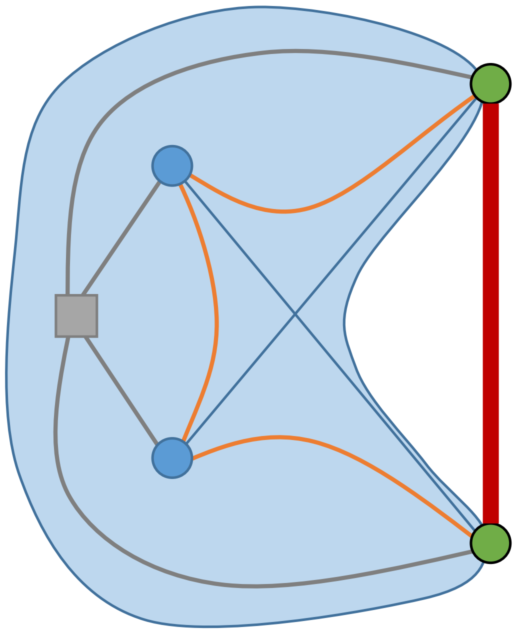
contract all inner components of each separation pair into a **thick edge**



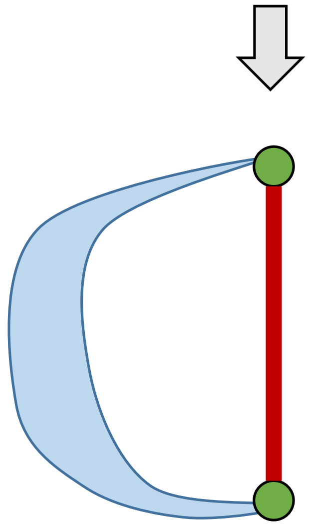
contraction

Hierarchical contraction

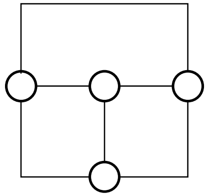
G⁺
triangulated
1-plane



contract all inner components of each separation pair into a **thick edge**

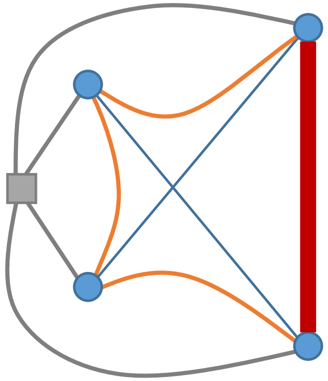


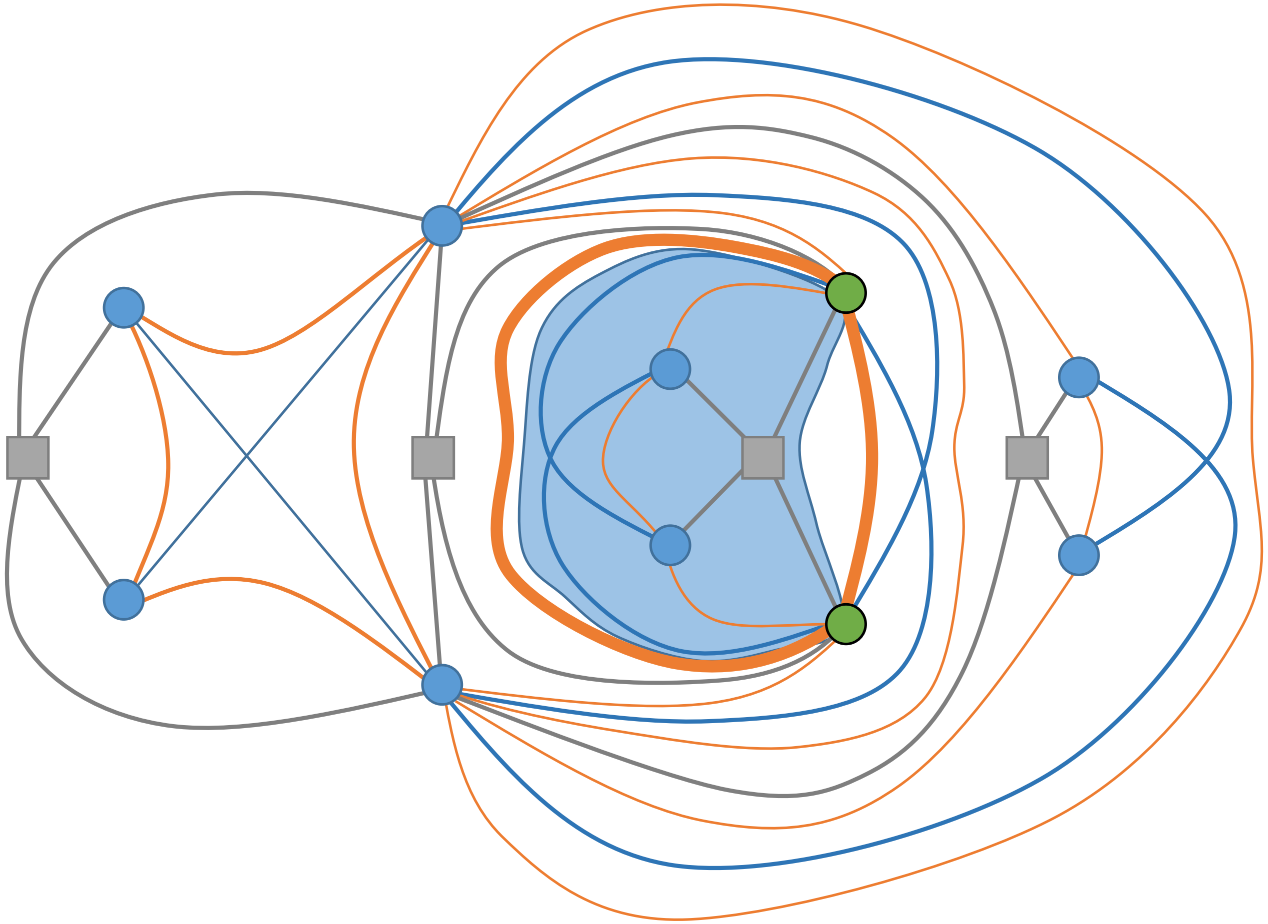
contraction

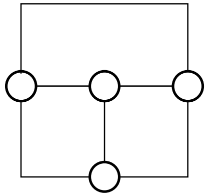


Hierarchical contraction

G^+
triangulated
1-plane

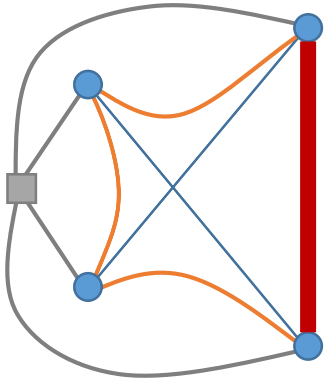


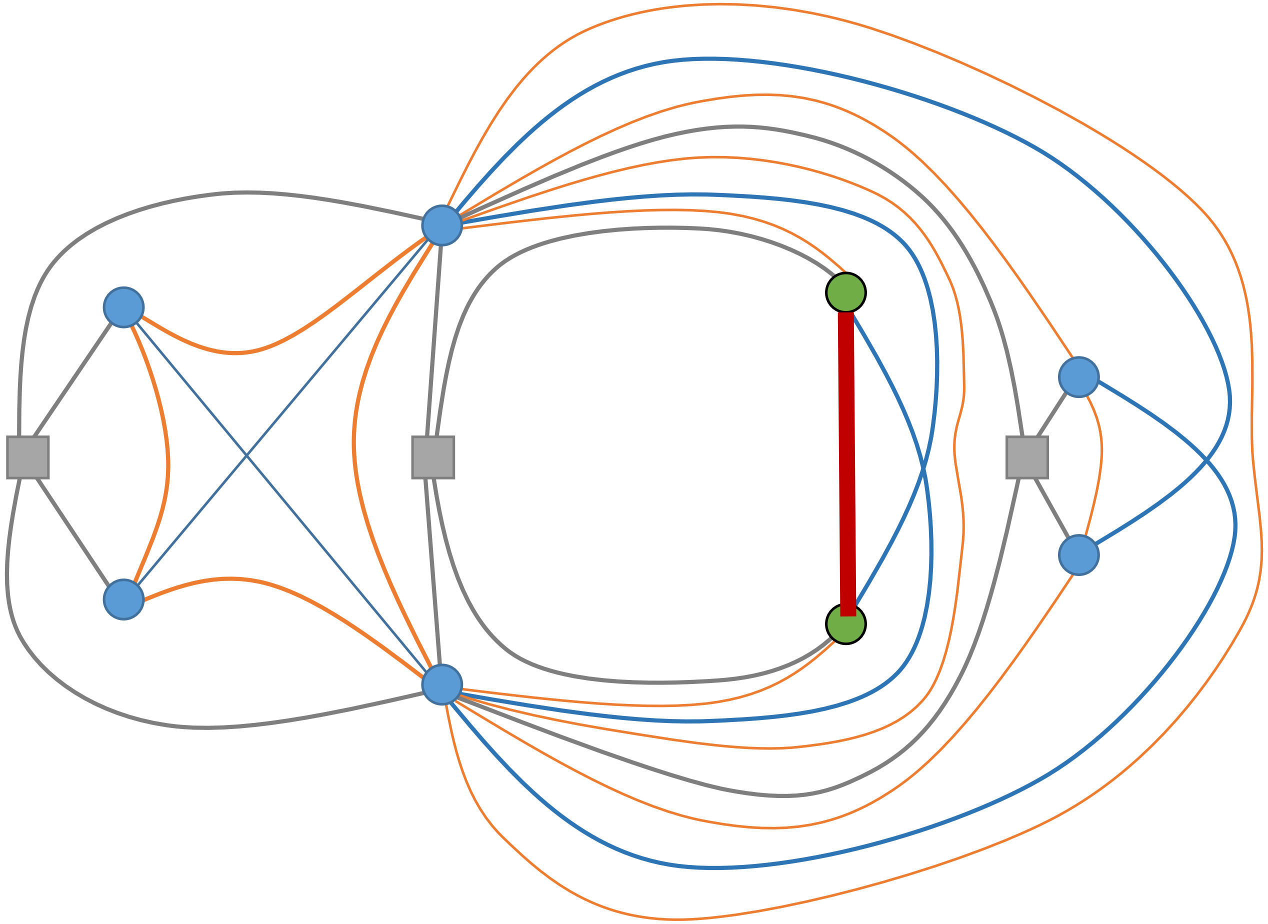




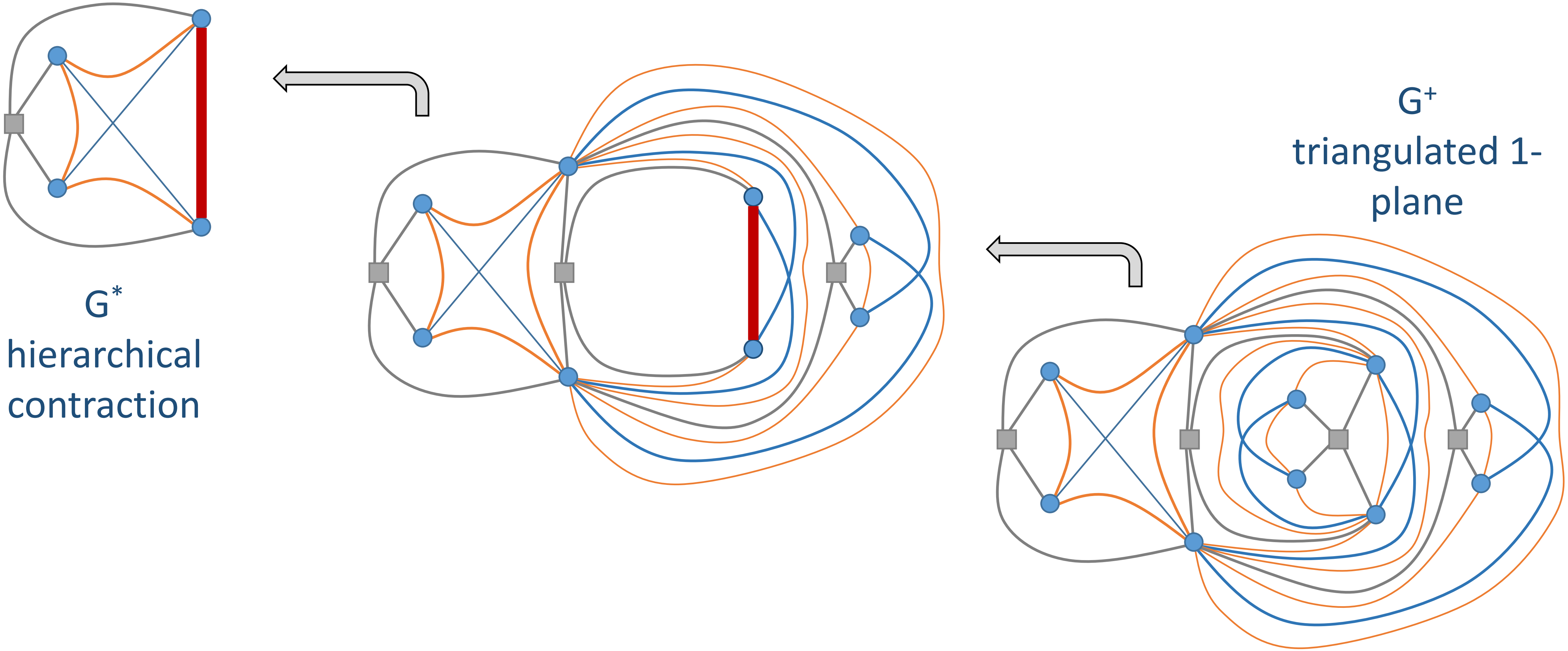
Hierarchical contraction

G^+
triangulated
1-plane

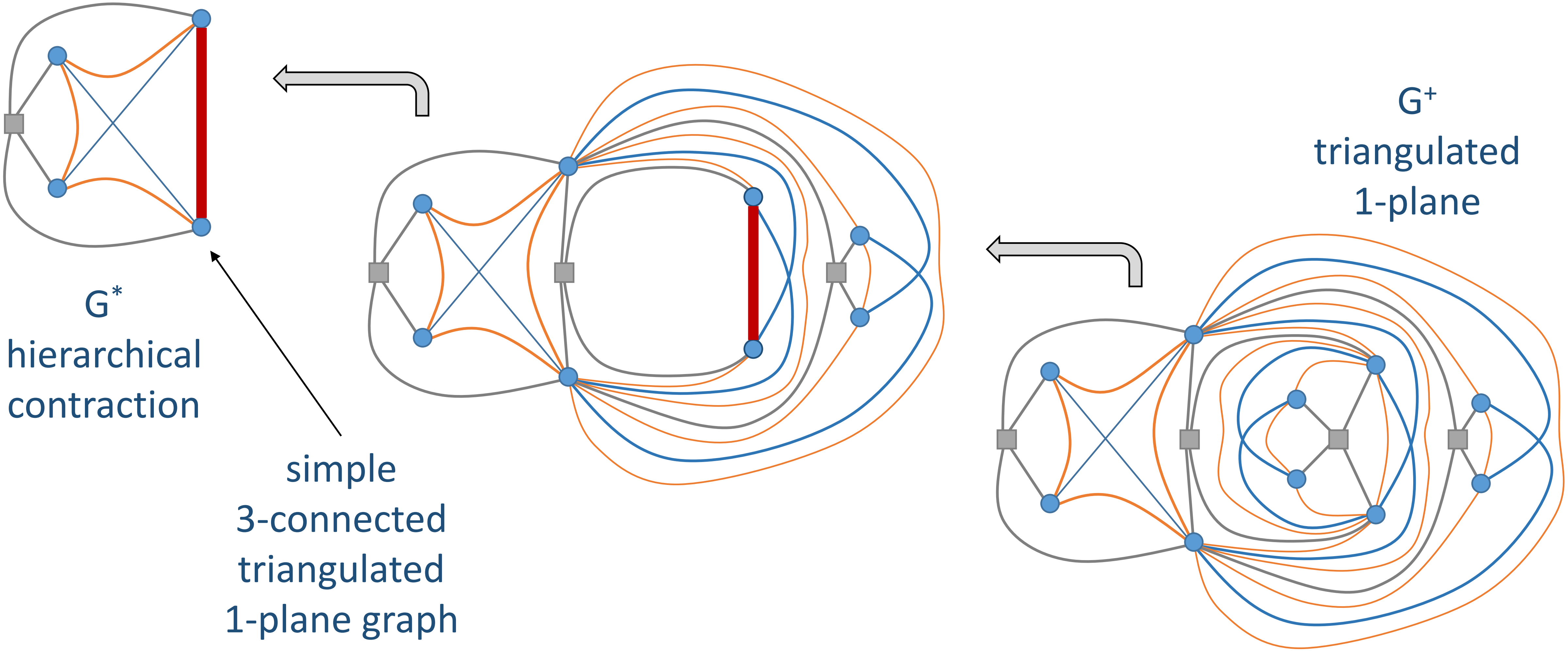




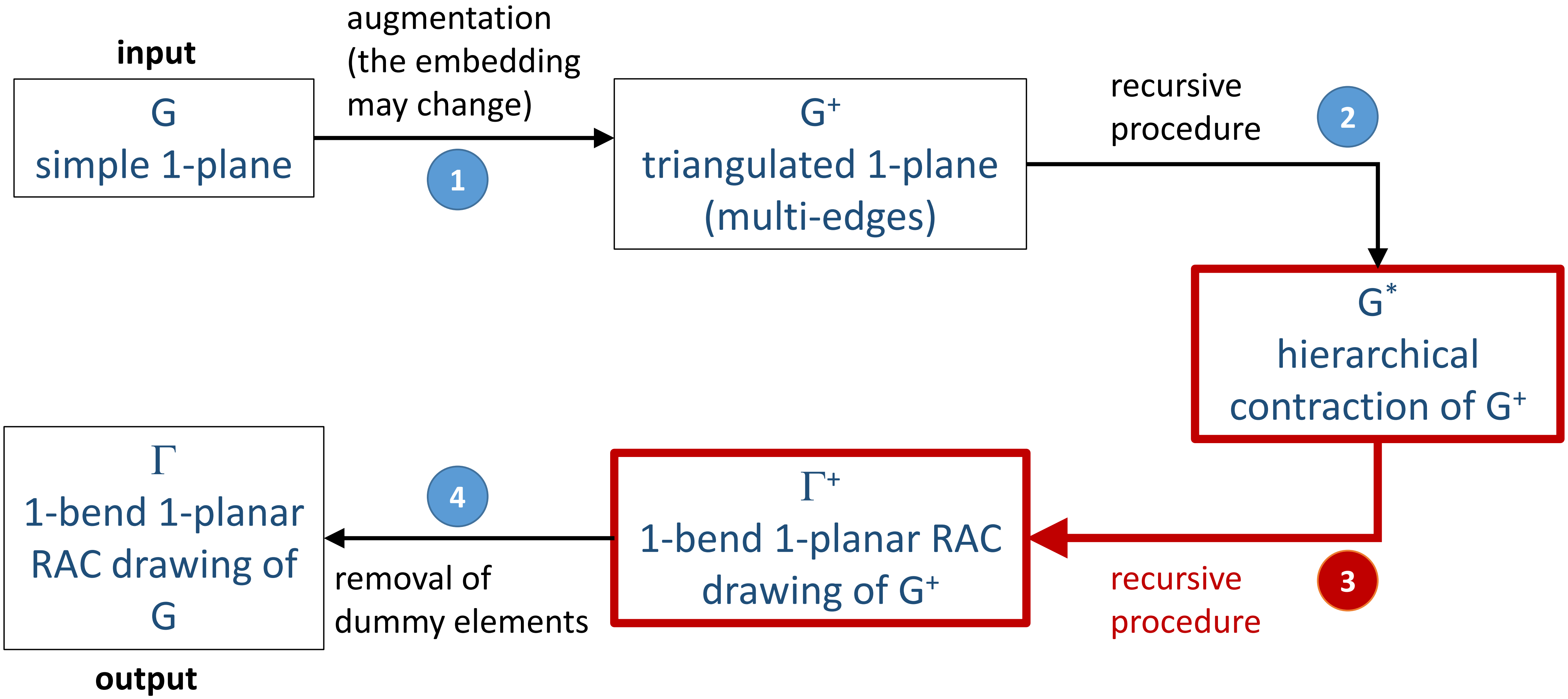
Hierarchical contraction



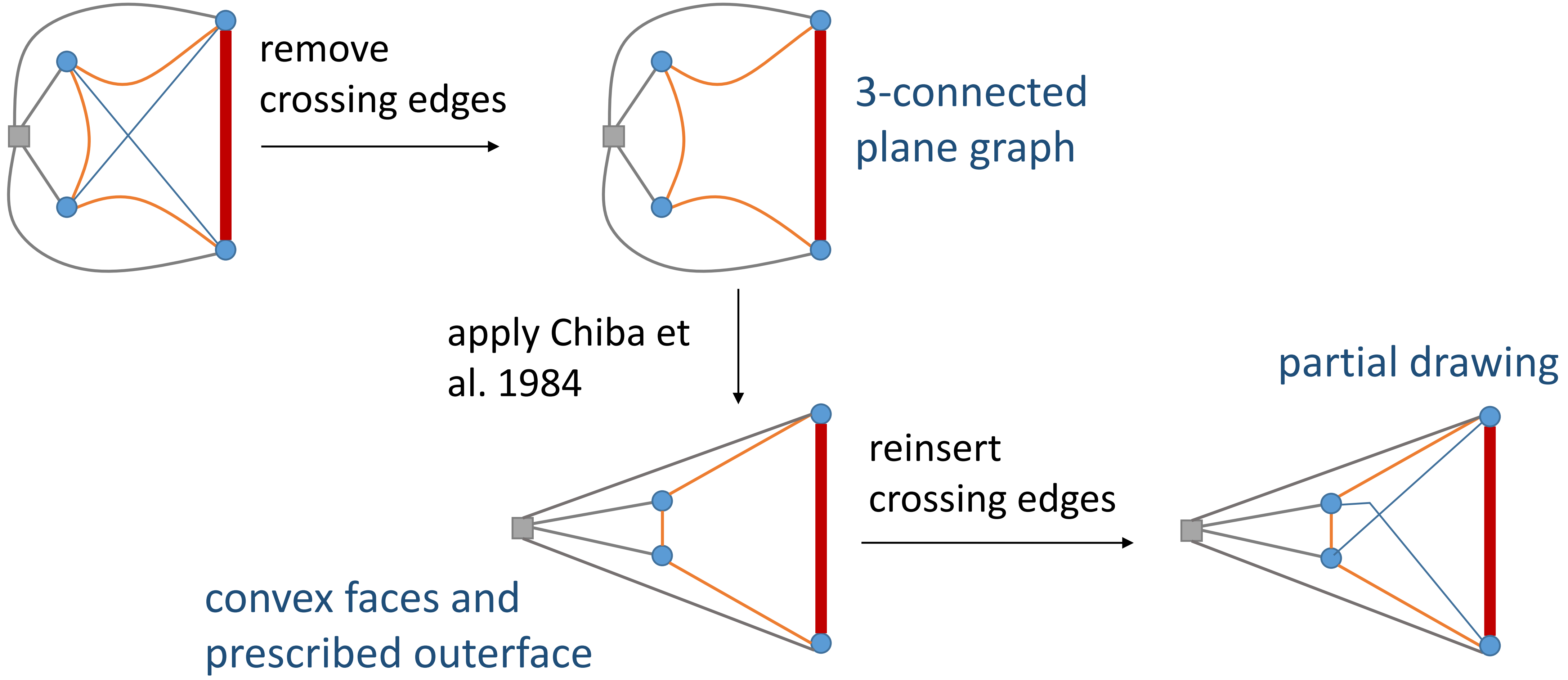
Hierarchical contraction

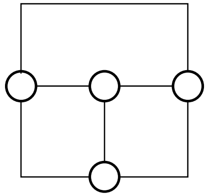


Algorithm Outline



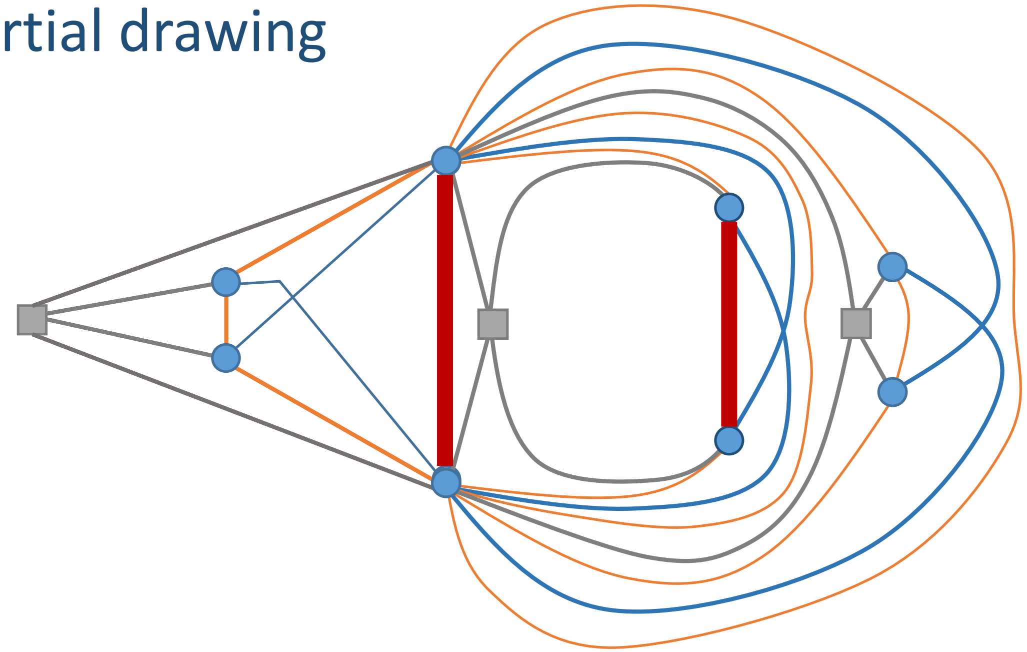
Drawing procedure

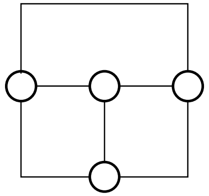




Drawing procedure

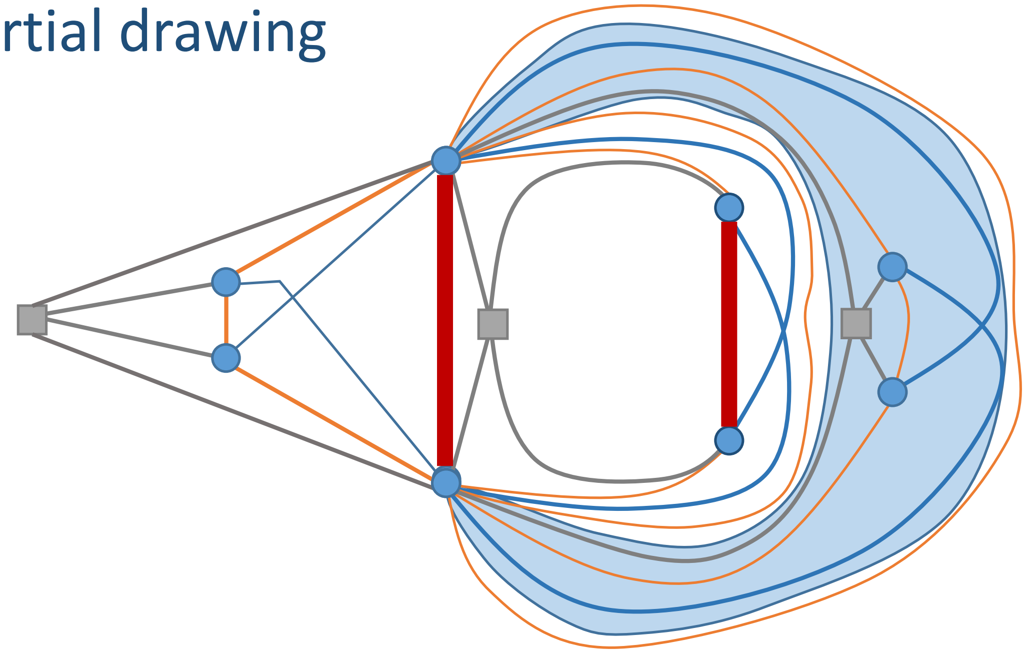
partial drawing

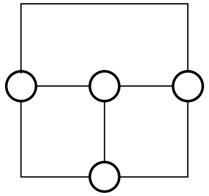




Drawing procedure

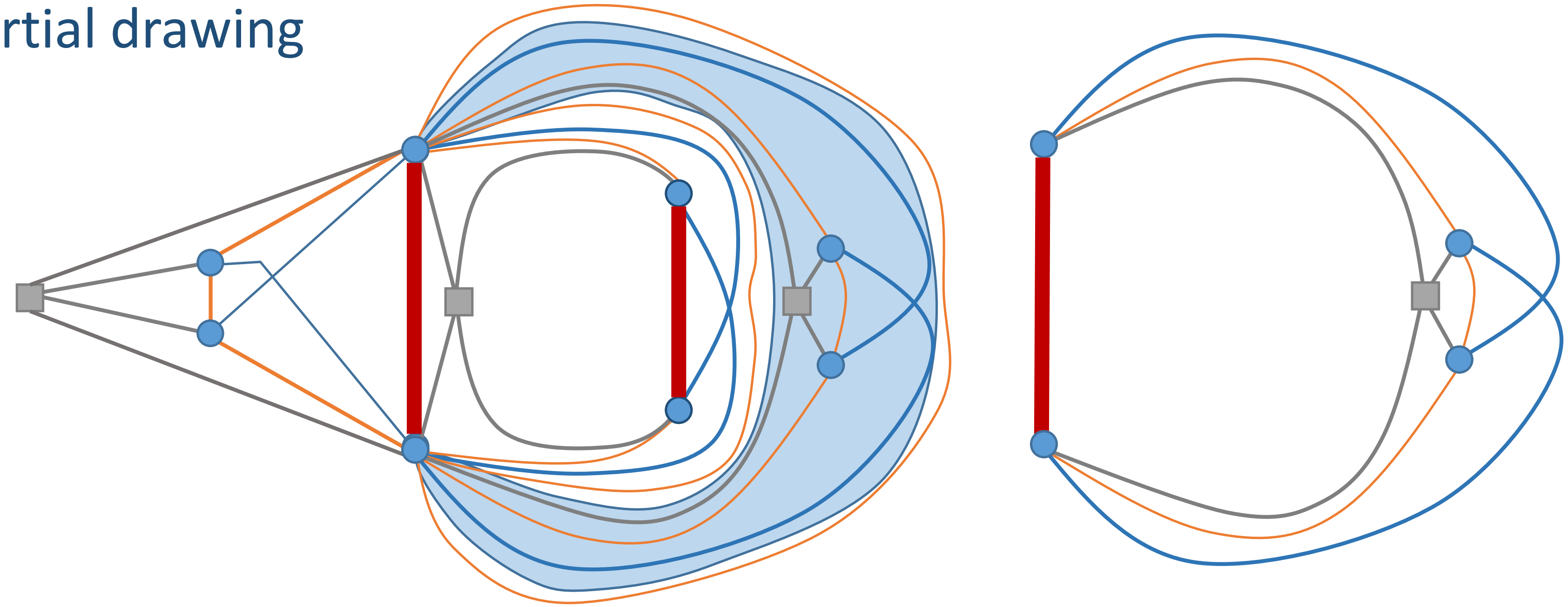
partial drawing

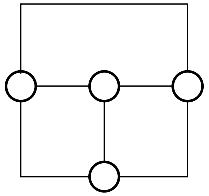




Drawing procedure

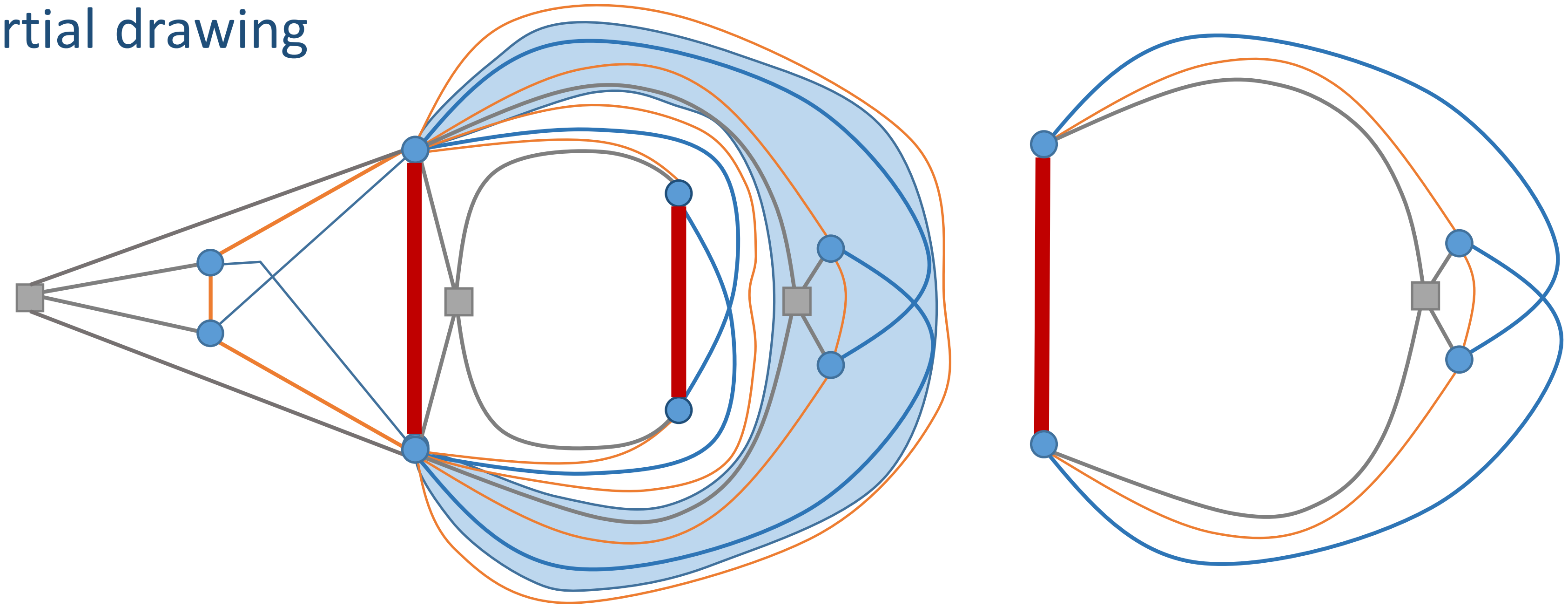
partial drawing



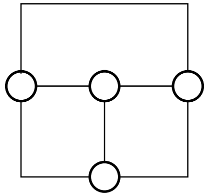


Drawing procedure

partial drawing

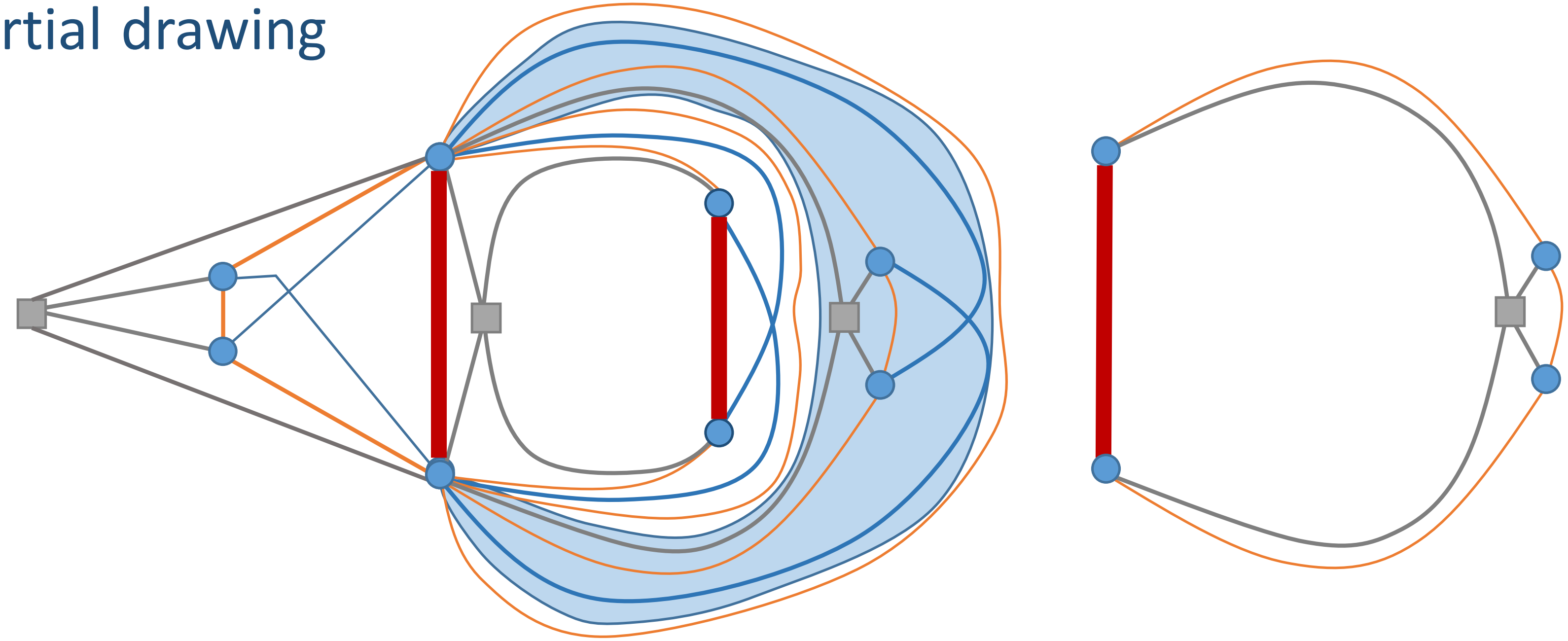


remove
crossing edges

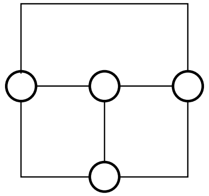


Drawing procedure

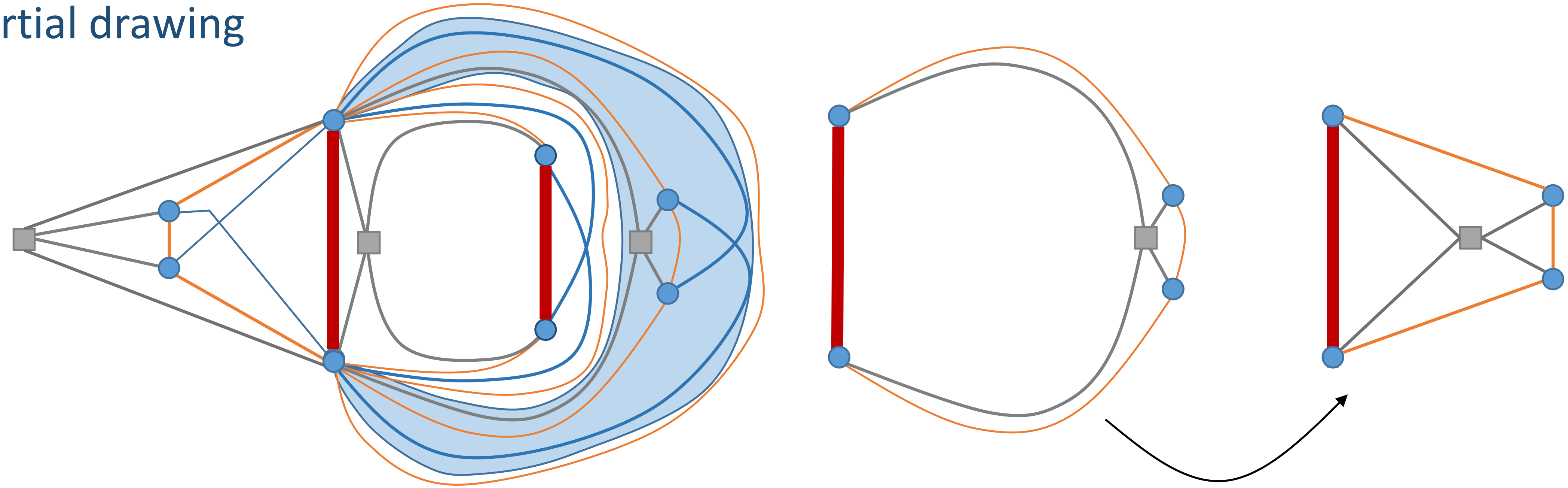
partial drawing



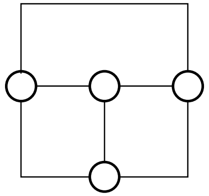
Drawing procedure



partial drawing

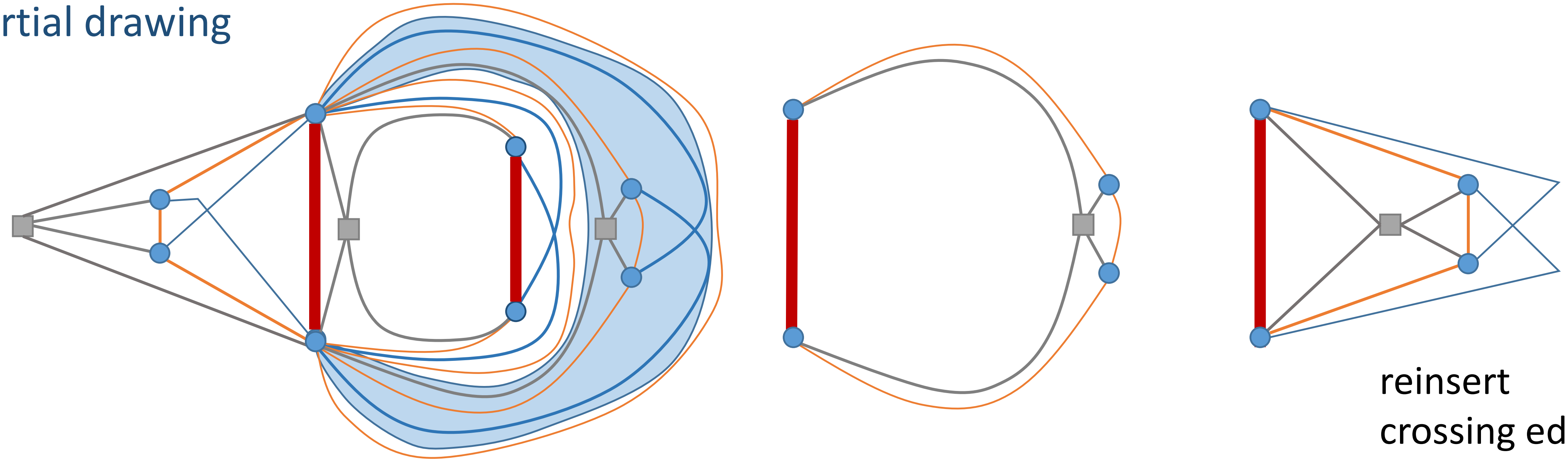


apply Chiba et al. 1984

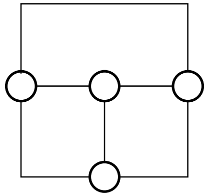


Drawing procedure

partial drawing

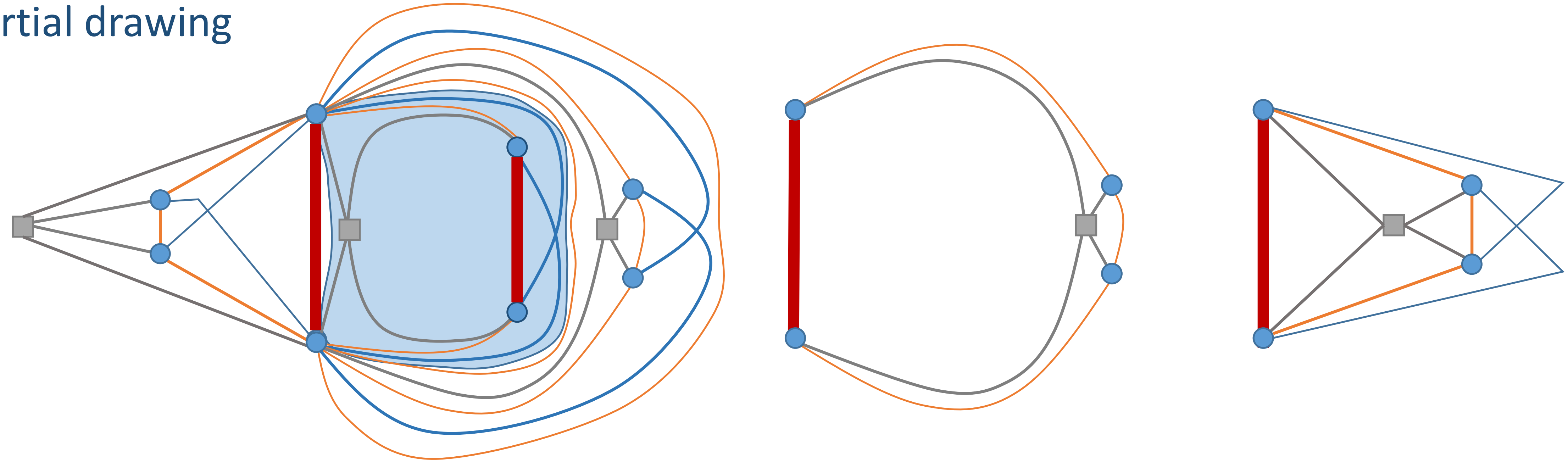


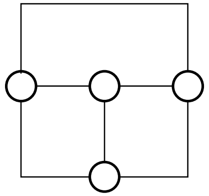
reinsert
crossing edges



Drawing procedure

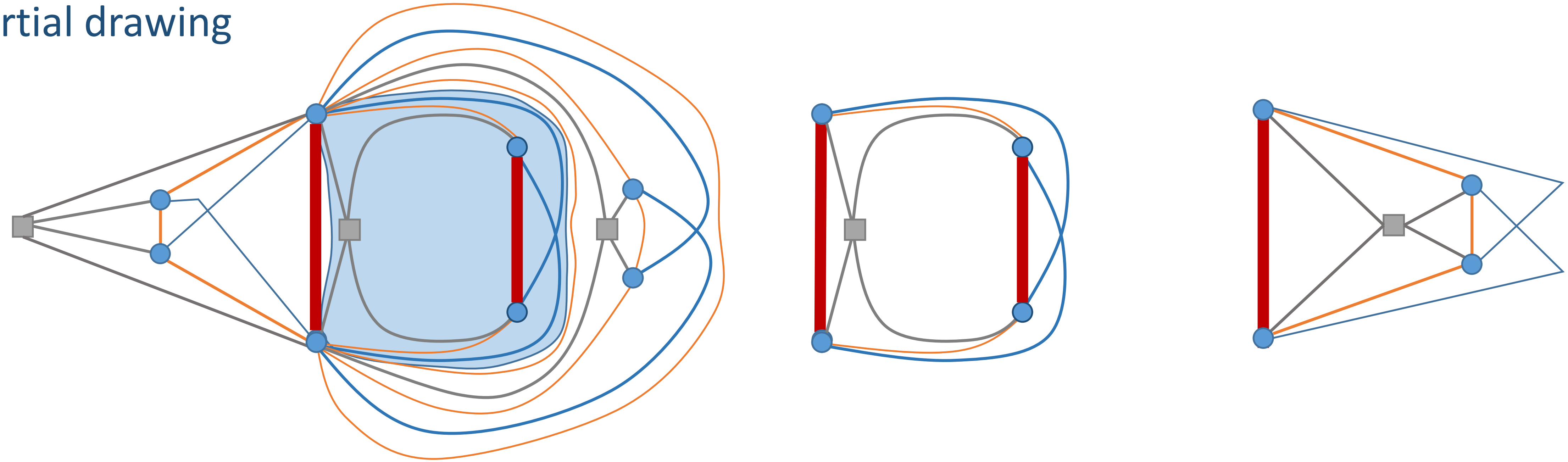
partial drawing

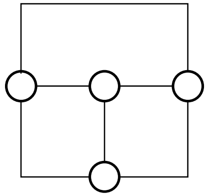




Drawing procedure

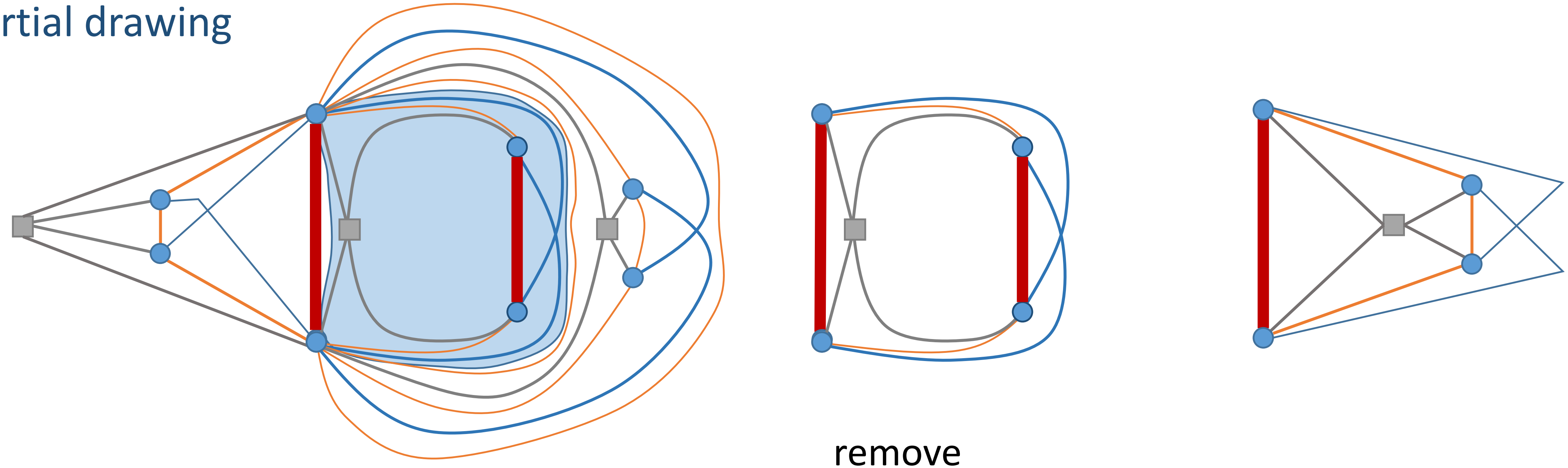
partial drawing



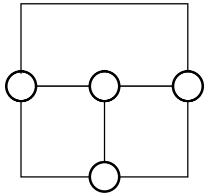


Drawing procedure

partial drawing

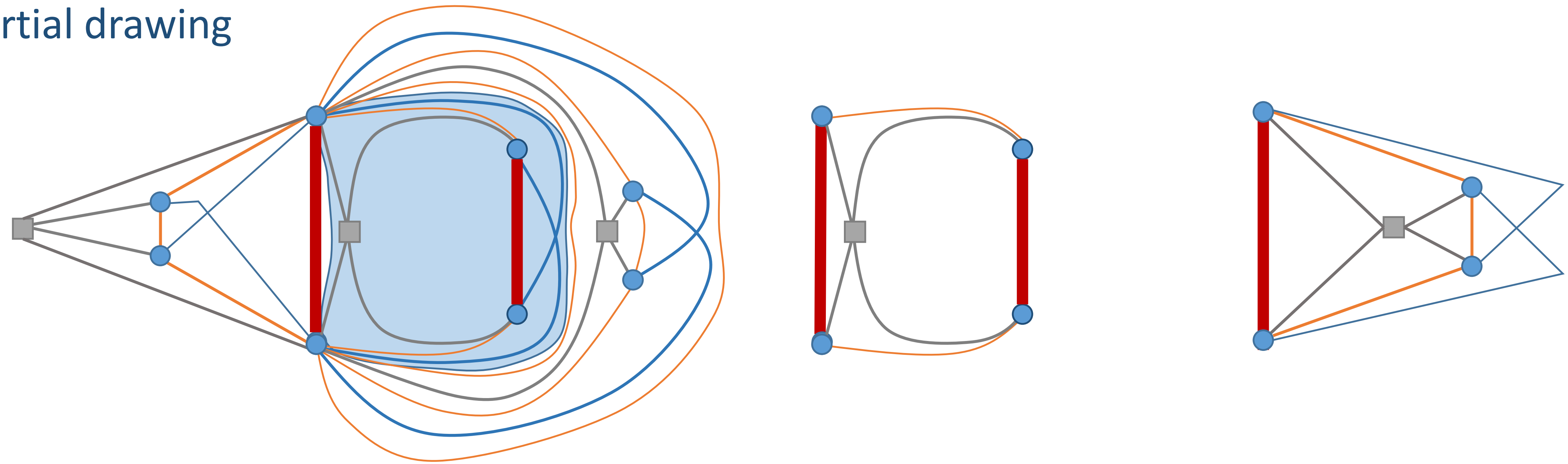


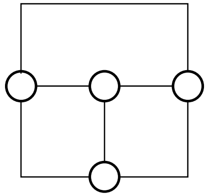
remove
crossing edges



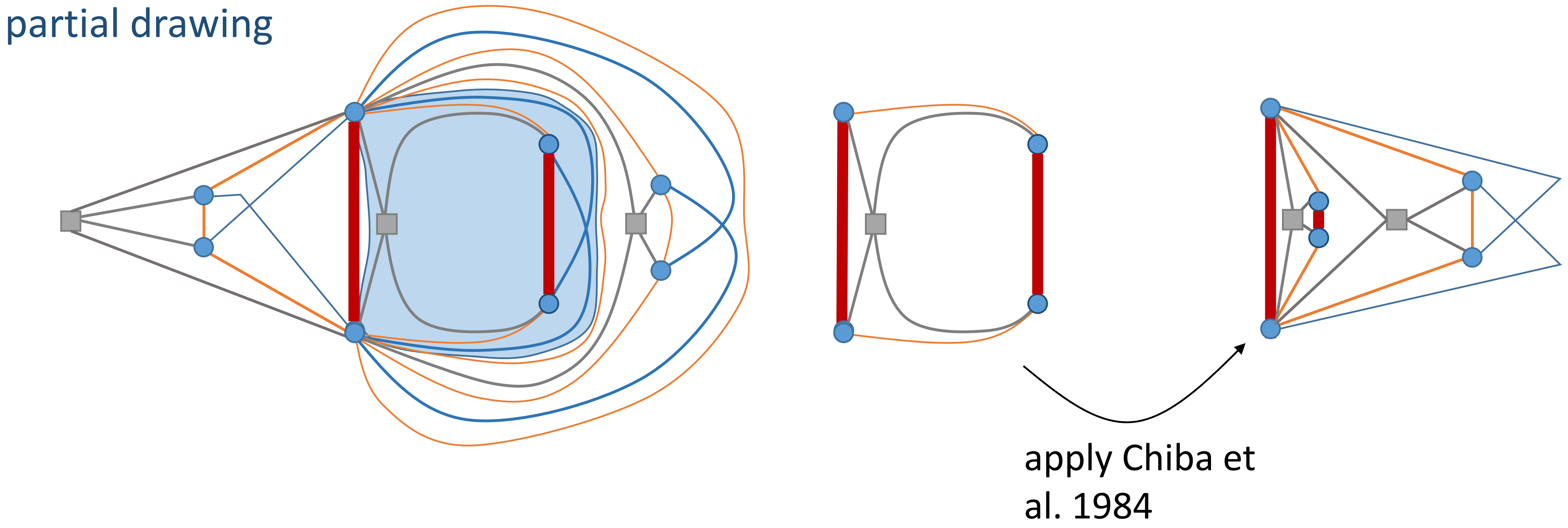
Drawing procedure

partial drawing



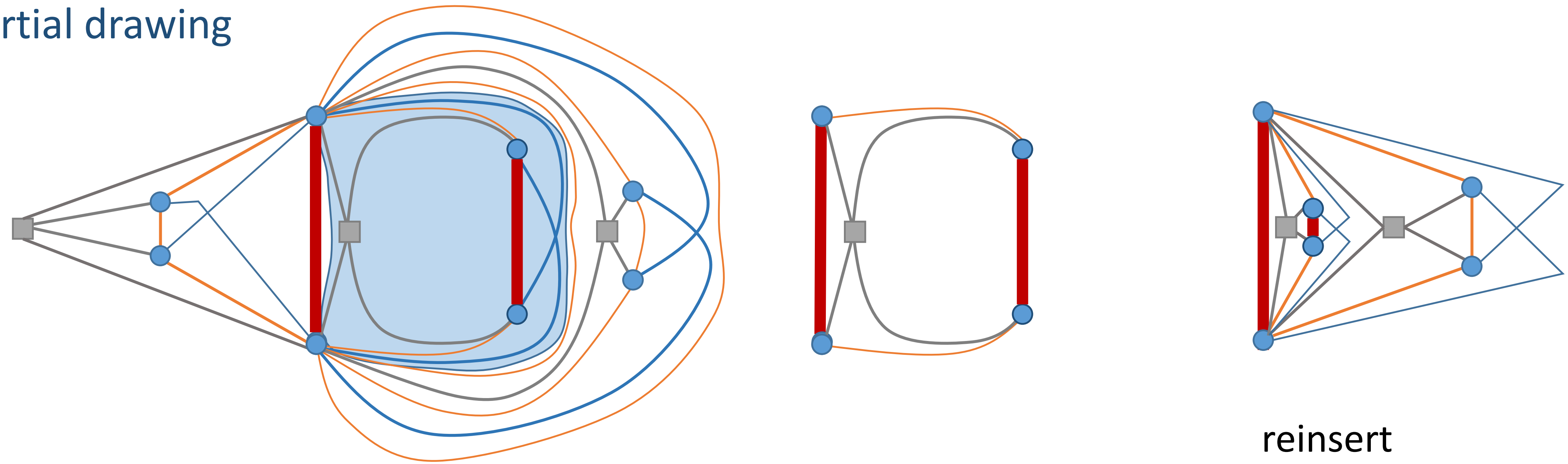


Drawing procedure

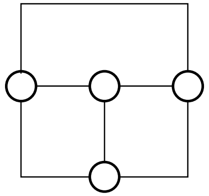


Drawing procedure

partial drawing

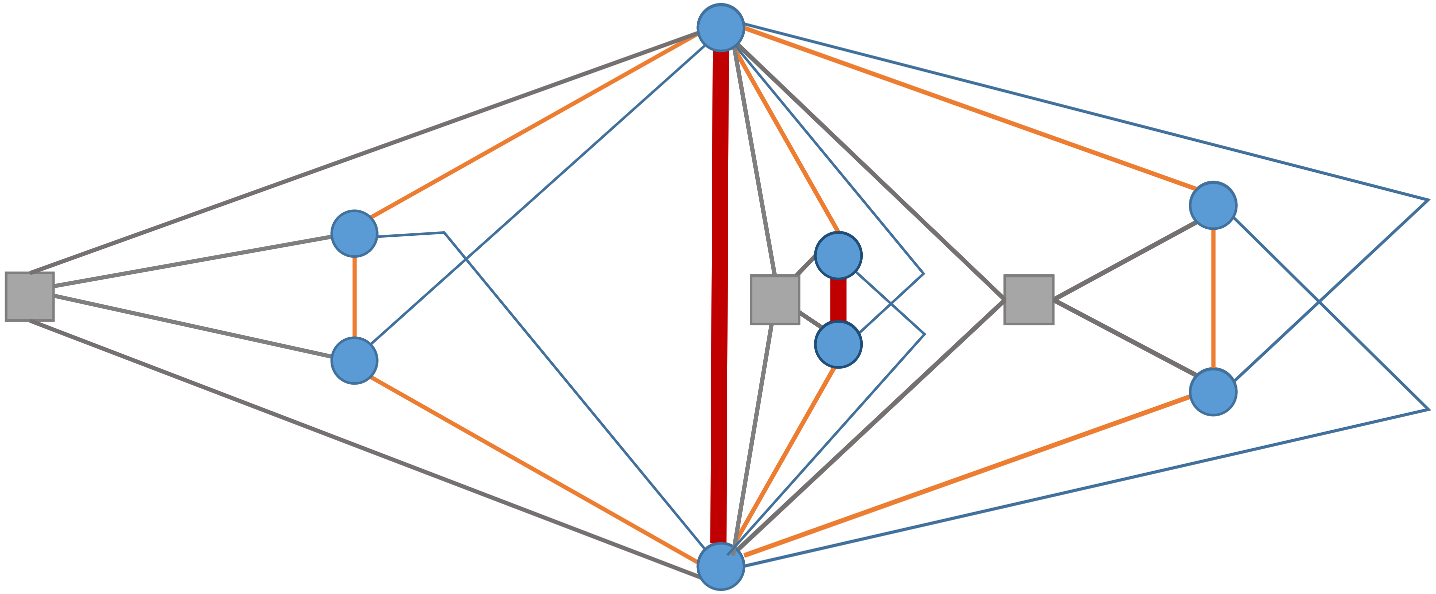


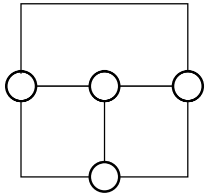
reinsert
crossing edges



Drawing procedure

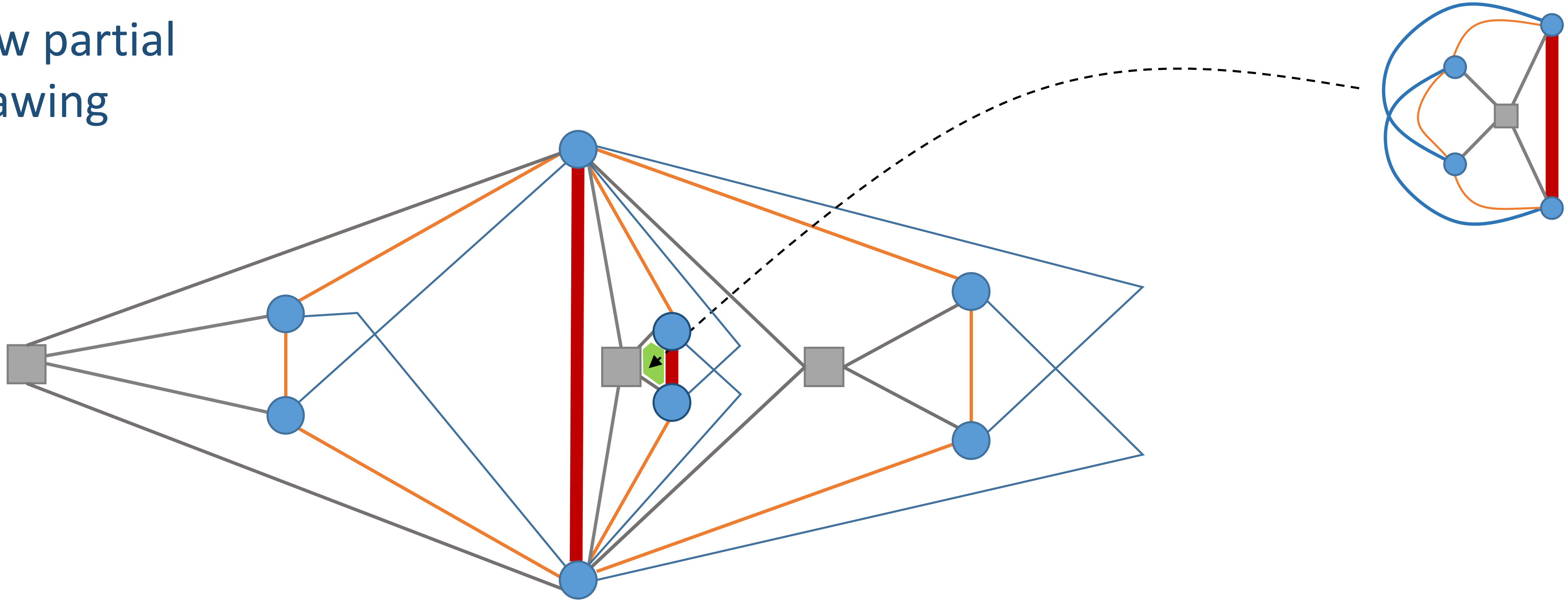
new partial
drawing





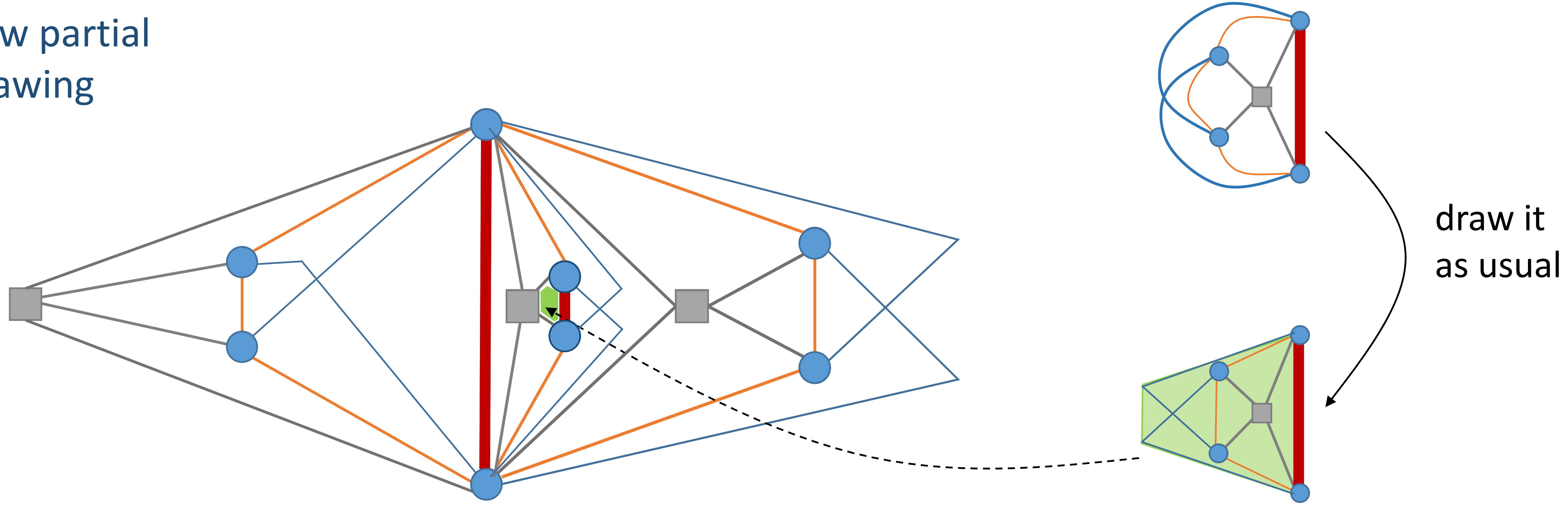
Drawing procedure

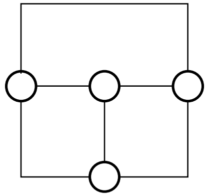
new partial drawing



Drawing procedure

new partial drawing

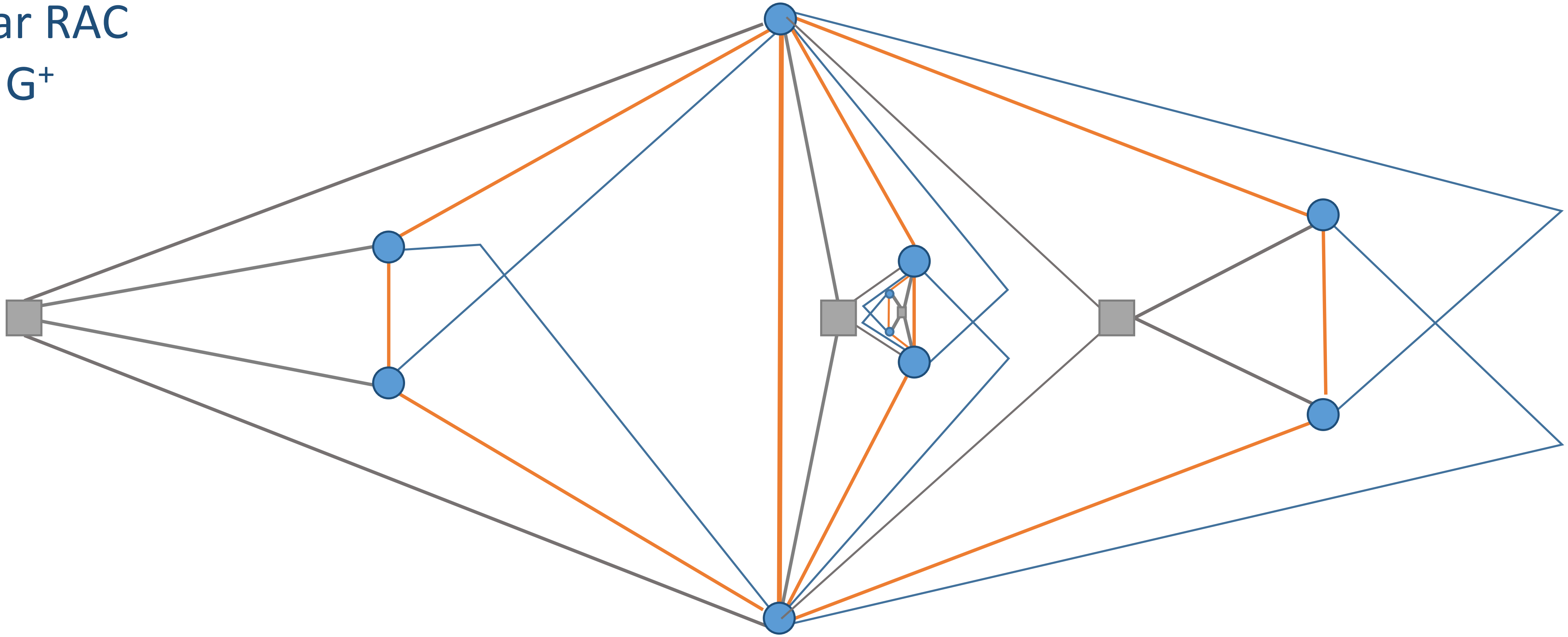




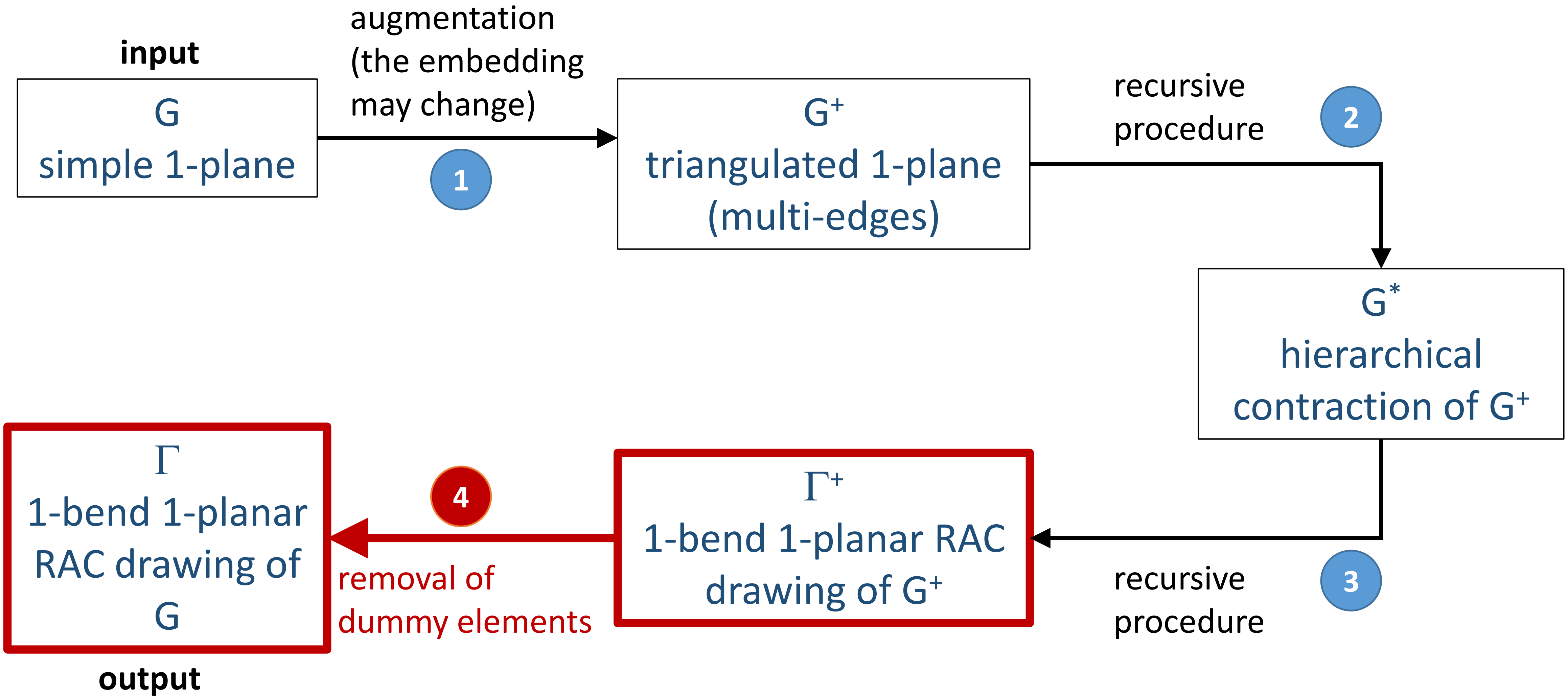
Drawing procedure

Γ^+

1-bend 1-planar RAC drawing of G^+



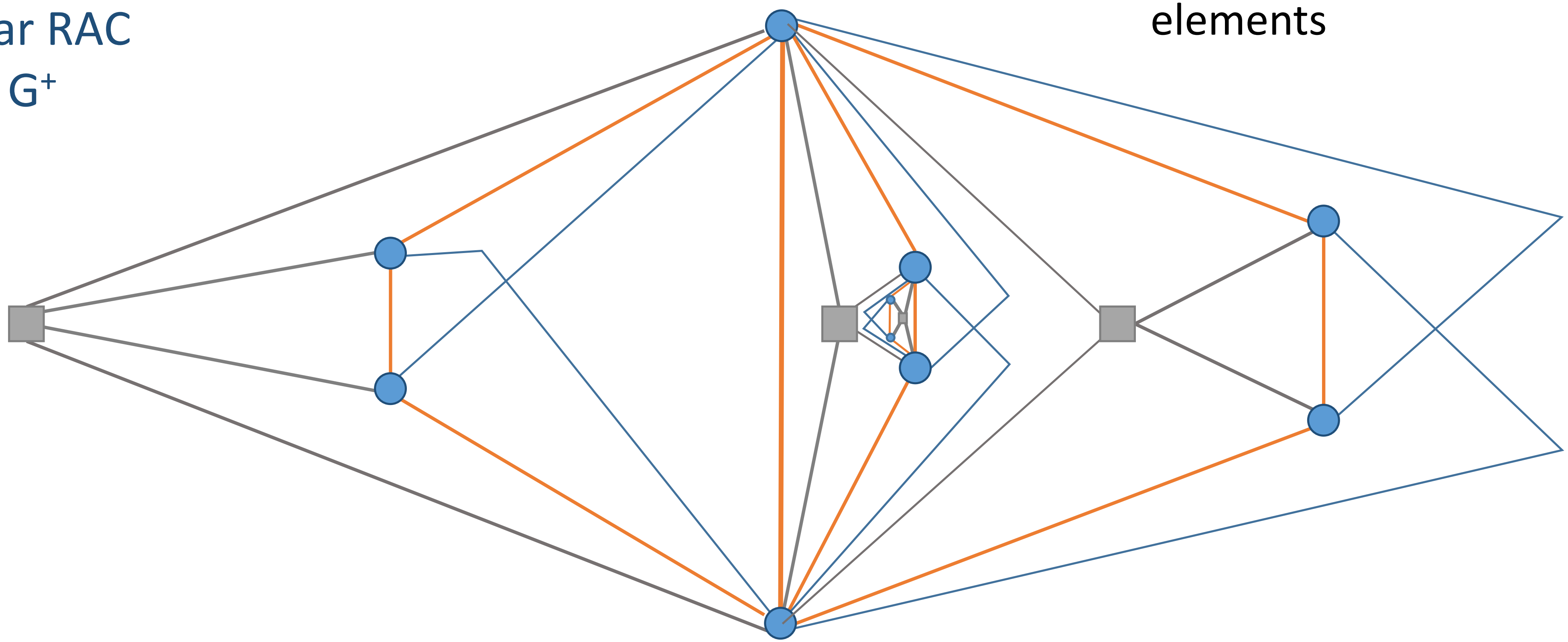
Algorithm Outline

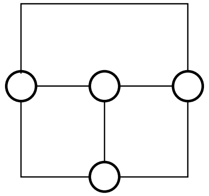


Drawing procedure

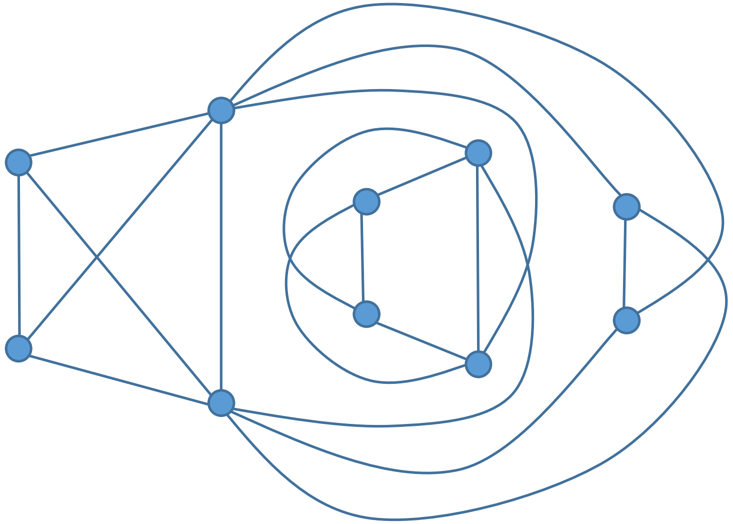
Γ^+
1-bend 1-planar RAC
drawing of G^+

remove
dummy
elements

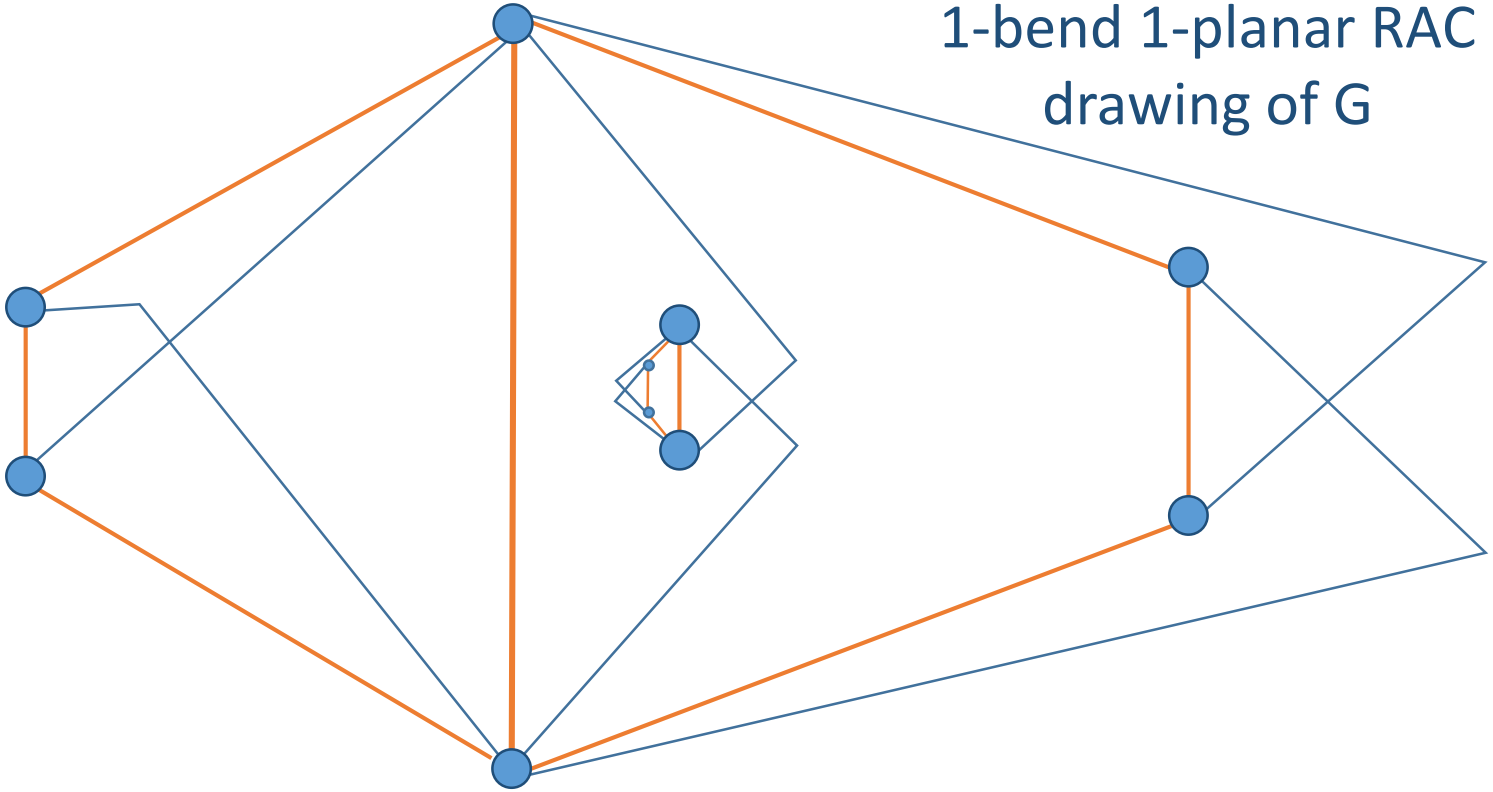




Drawing procedure



input graph G



Γ

1-bend 1-planar RAC drawing of G

Inclusion Relationships: RAC and 1-planar

- **Further advances**:
  - embedding preserving 1-bend 1-planar RAC
  - $O(n^2)$ area for 1-bend RAC NIC-plane
  - $O(n^9)$ area for 2-bend RAC 1-plane

S. Chaplick, F. Lipp, A. Wolff, J. Zink:

Compact Drawings of 1-Planar Graphs with Right-Angle Crossings and Few Bends. Graph Drawing 2018: 137-151

Inclusion Relationships: Open Problems

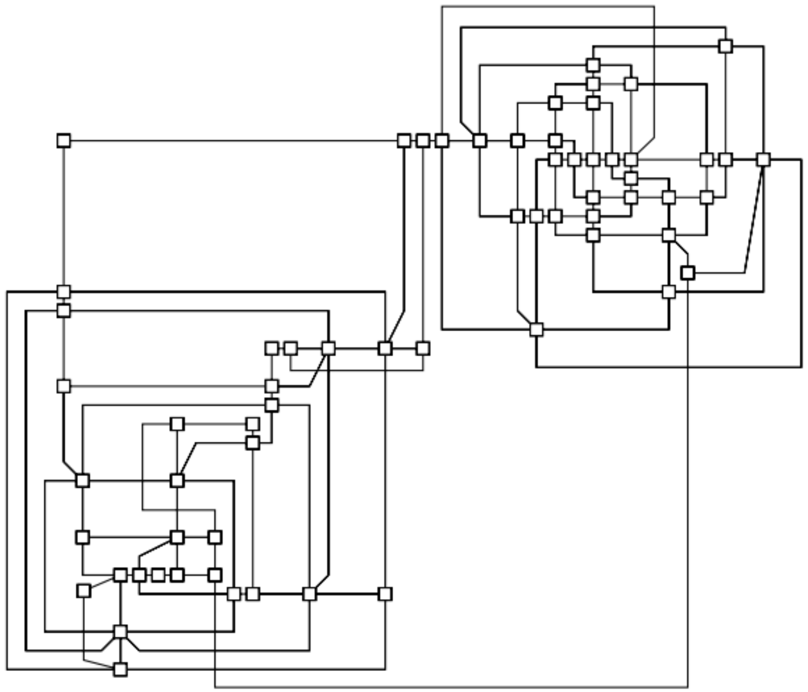
- **Problem IR1.** Are there fan-crossing free graphs with at most 4n-10 edges that are neither 1-planar nor 0-bend RAC drawable?

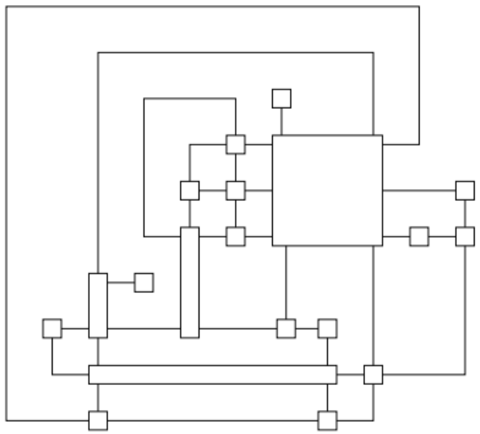
- **Problem IR2**. Characterize the 0-bend RAC drawable graphs that are 1-planar

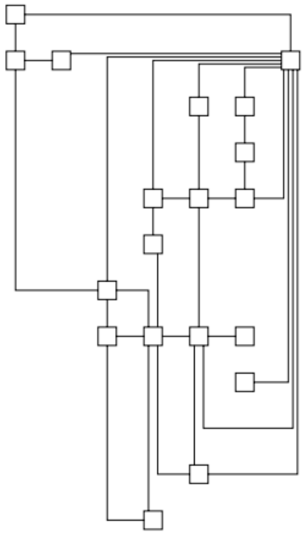
- **Problem IR3**. Characterize the 1-plane graphs that are 0-bend RAC drawable

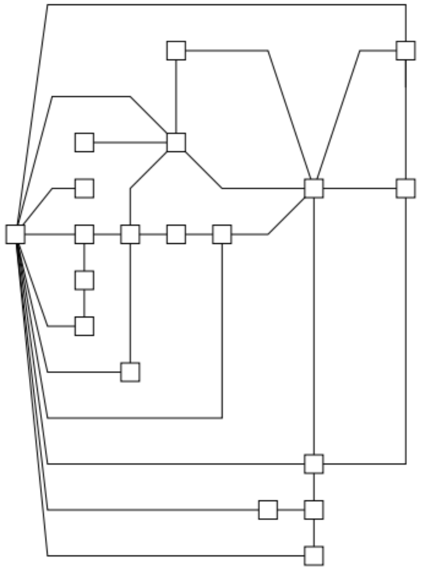
Other variants of orthogonal drawings

- Quasi-orthogonal drawings
  - *G. W. Klau and P. Mutzel*: Quasi-Orthogonal Drawing of Planar Graphs, Tech. Rep. Max–Planck–Institut fuer Informatik Saarbruecken, Germany (1998)



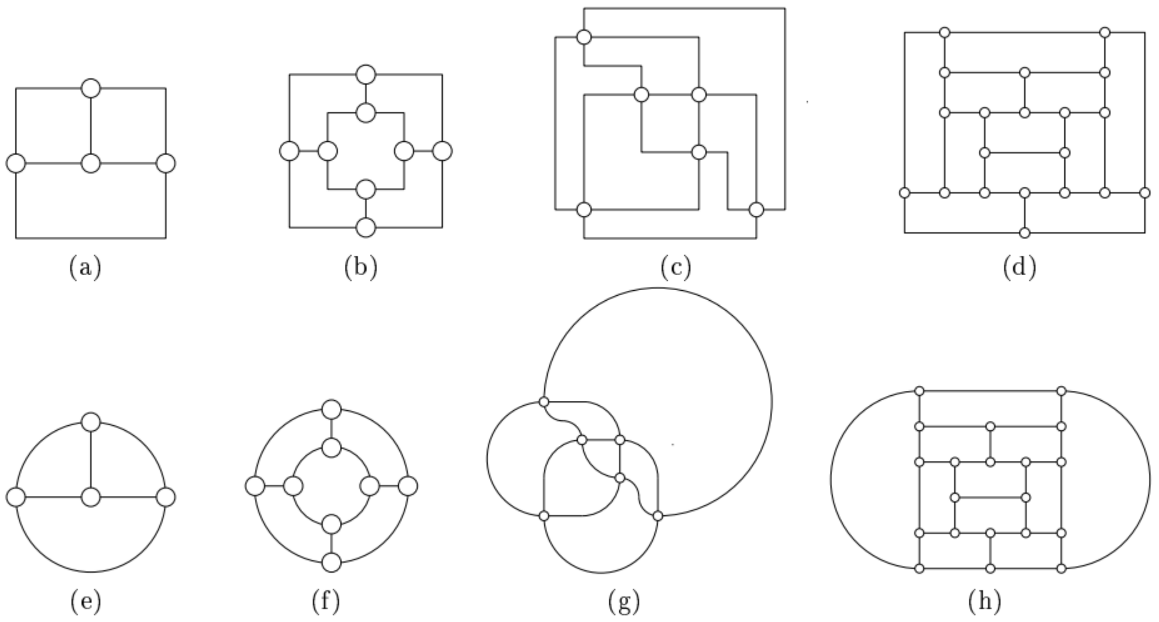






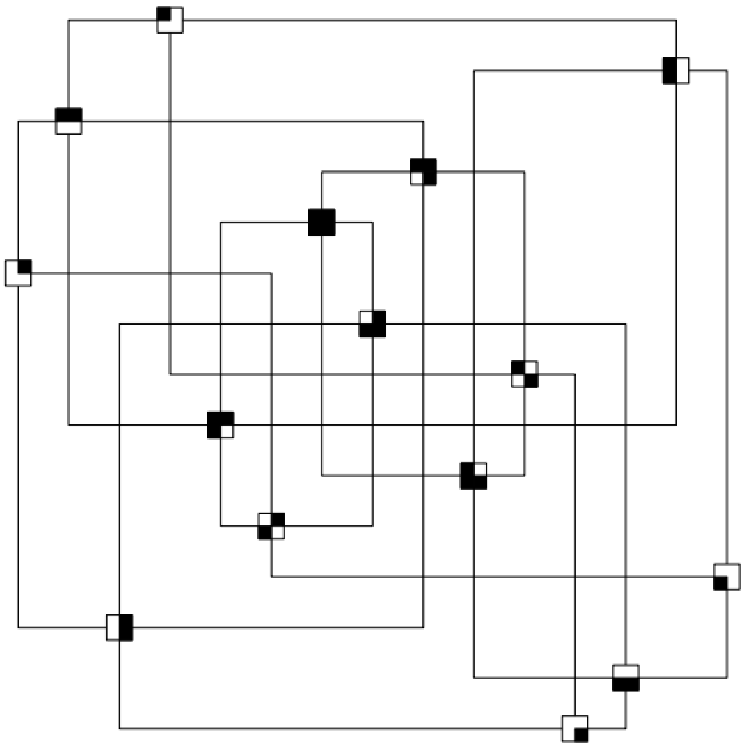
Other variants of orthogonal drawings

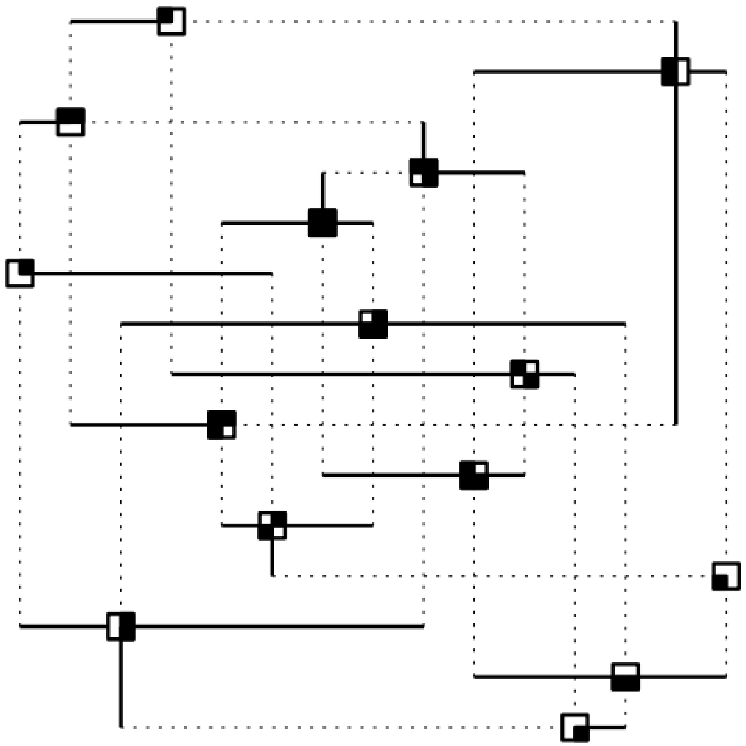
- Smooth orthogonal drawings
  - *M. A. Bekos, M. Kaufmann, S. G. Kobourov, A. Symvonis*: Smooth Orthogonal Layouts. J. Graph Algorithms Appl. 17(5) (2013)
  - *M. A. Bekos, H. Förster, M. Kaufmann*: On Smooth Orthogonal and Octilinear Drawings: Relations, Complexity and Kandinsky Drawings. Algorithmica 81(5) (2019)

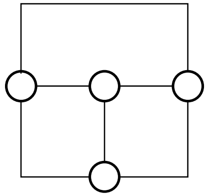


Other variants of orthogonal drawings

- 1-bend orthogonal partial edge drawings
  - *T. Bruckdorfer, M. Kaufmann, F. Montecchiani*: 1-Bend Orthogonal Partial Edge Drawing. J. Graph Algorithms Appl. 18(1) (2014)

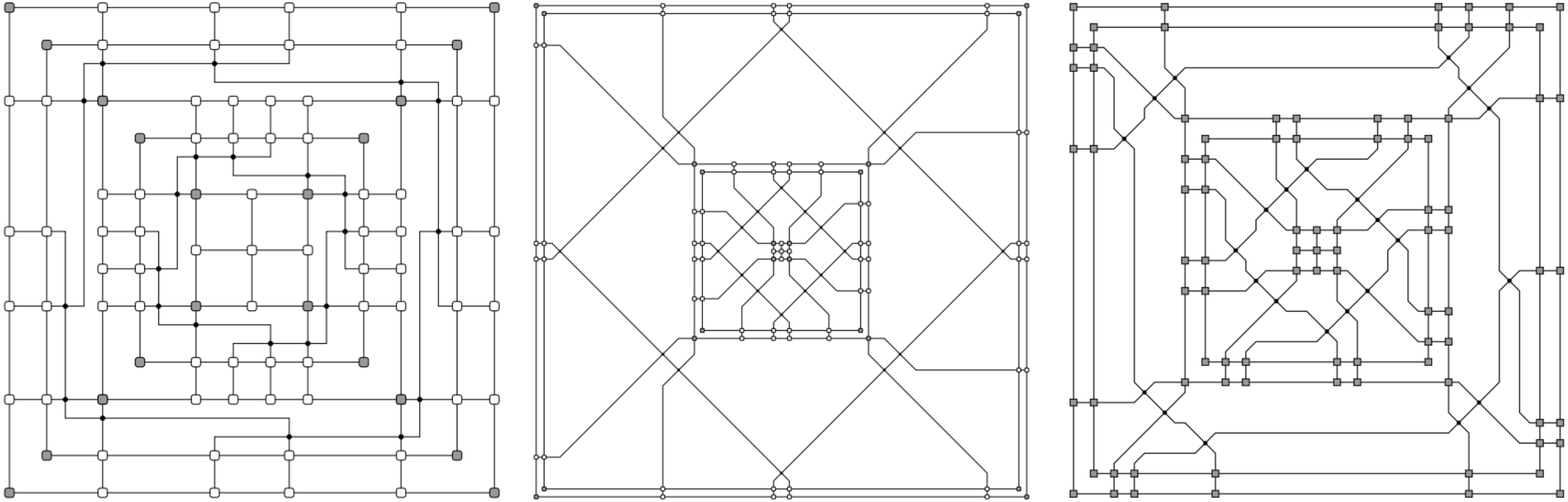






Other variants of orthogonal drawings

- Slanted orthogonal drawings
  - *M. A. Bekos, M. Kaufmann, R. Krug, T. Ludwig, S. Näher, V. Roselli*: J. Graph Algorithms Appl. 18(3) (2014)



Other variants of orthogonal drawings

- Overloaded orthogonal drawings
  - *W. Didimo, E. M. Kornaropoulos, F. Montecchiani, I. G. Tollis: A Visualization Framework and User Studies for Overloaded Orthogonal Drawings. Comput. Graph. Forum 37(1): 288-300 (2018)*