Local and Union Page Numbers

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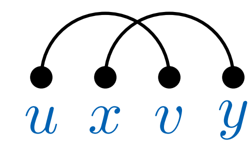
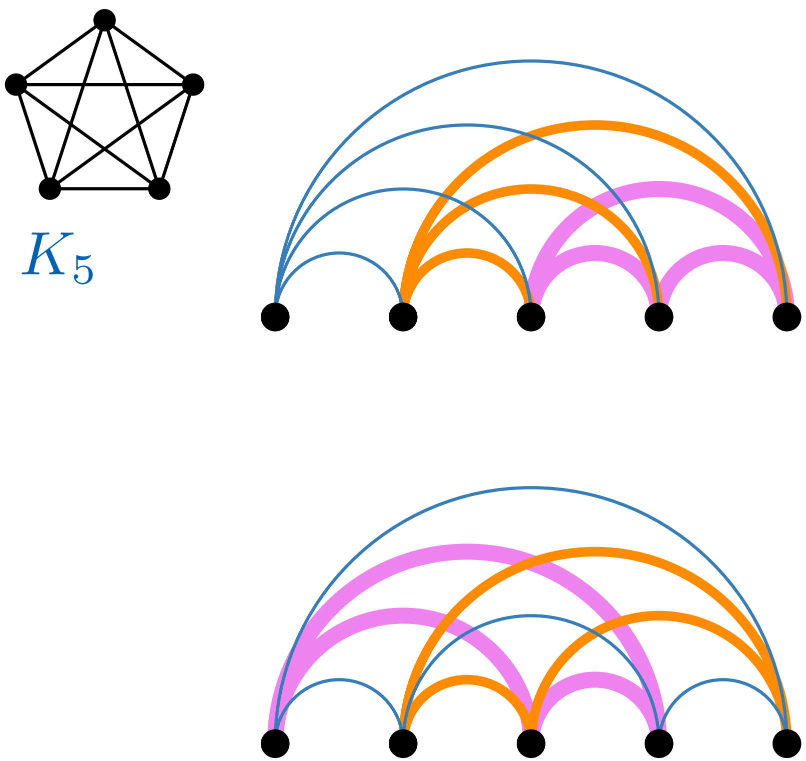
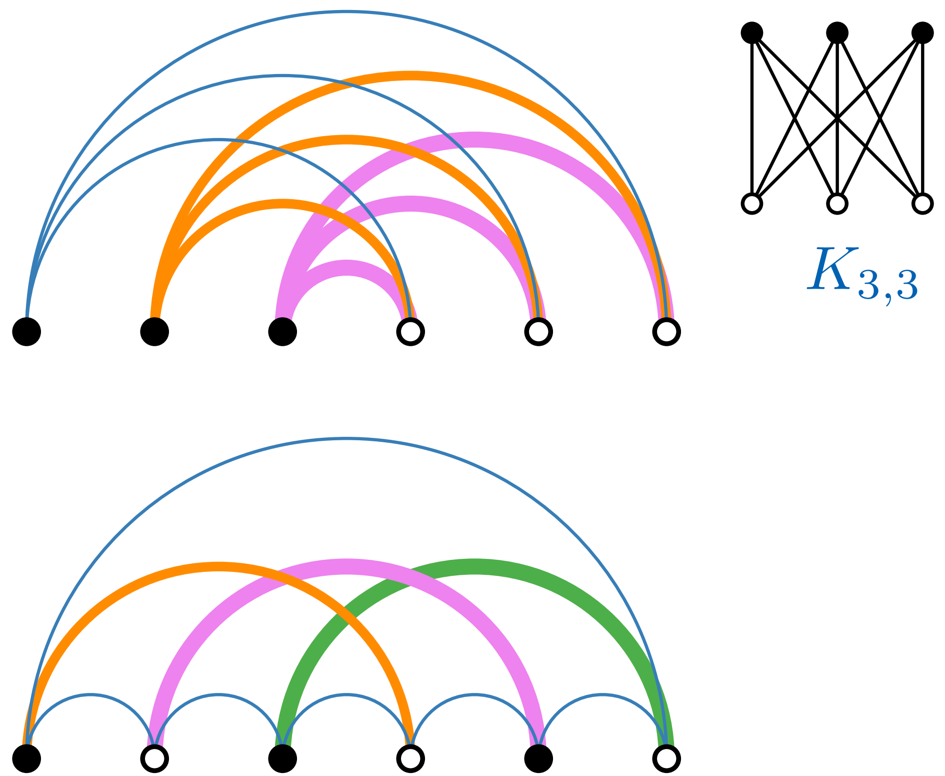
Graph Drawing 2019
September 20, 2019
Pruhonice

book embedding $(\prec, \mathcal{P})$

- ▷ linear vertex ordering $\prec$ ⇝ spine ordering
- ▷ edge partition $\mathcal{P} = \{P_1, \ldots, P_k\}$ ⇝ pages
- ▷ $u \prec x \prec v \prec y, uv \in P_i, xy \in P_j \Rightarrow i \neq j$ ⇝ each page crossing-free







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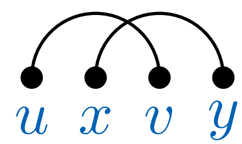

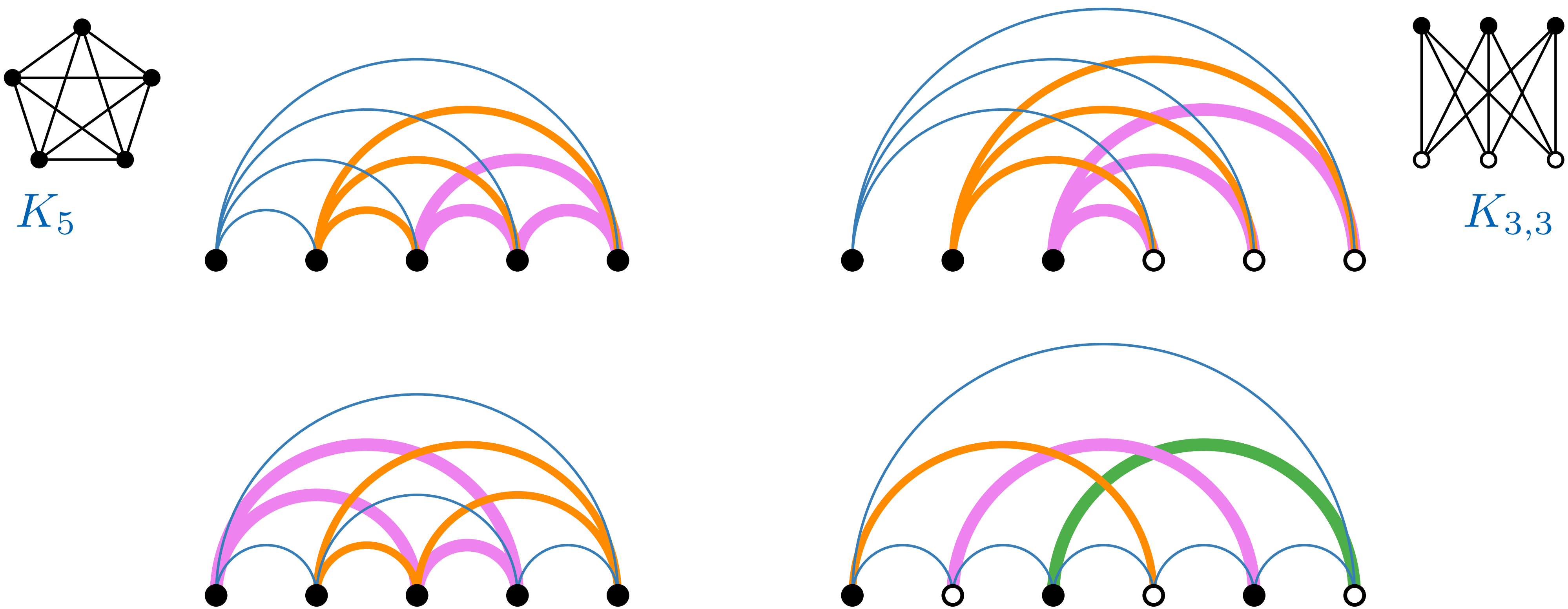

k-**local book embedding:** each vertex on at most k pages

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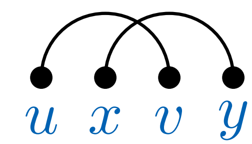
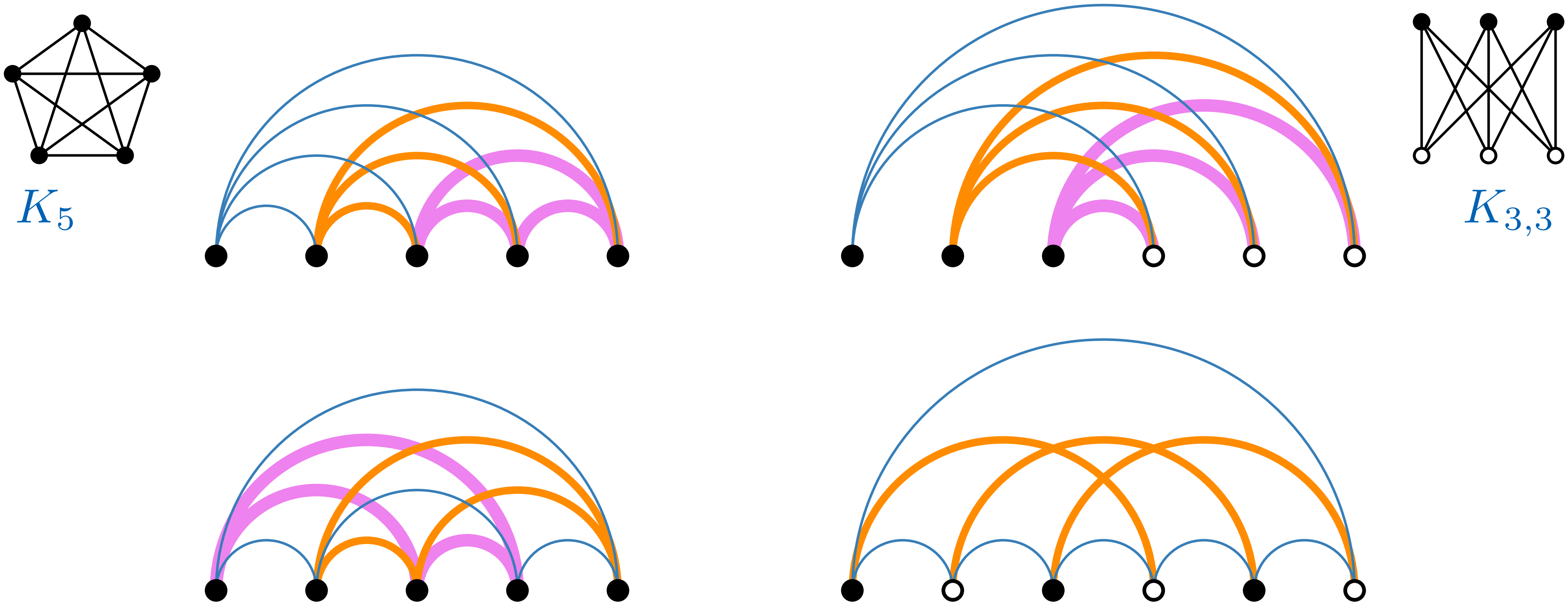




k-**union embedding:** each page crossing-free components

page number $\mathrm{pn}(G) = \min k\colon \exists$ k-page book embedding

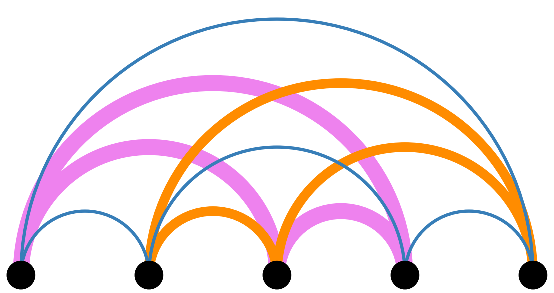
minimize # pages — each page crossing-free

union page number $\mathrm{pn}_u(G) = \min k\colon \exists$ k-union embedding

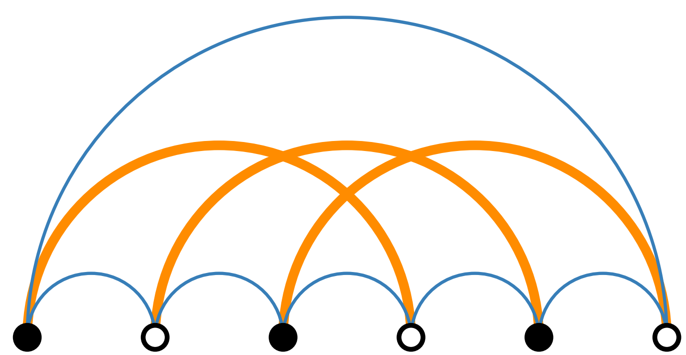
minimize # pages — each page union of crossing-free components

local page number $\mathrm{pn}_\ell(G) = \min k\colon \exists$ k-local book embedding

minimize # pages at any one vertex — each page crossing-free



	pn_ℓ	pn_u	pn
$K_{3,3}$	2	2	3
K_5	2	3	3



Comparison of variants

- For any graph G we have

$$\mathrm{pn}_\ell(G) \le \mathrm{pn}_u(G) \le \mathrm{pn}(G).$$

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A simple lower bound ...

$$|E| = \sum_{P \in \mathcal{P}} \#\{\text{edges in } P\}$$

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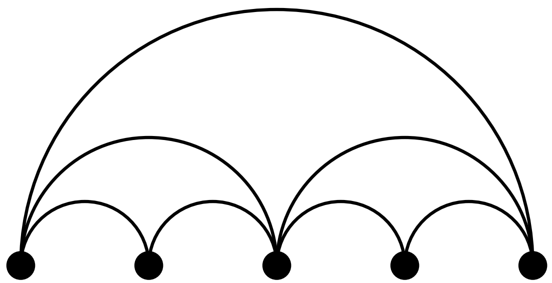

as each page is outerplanar

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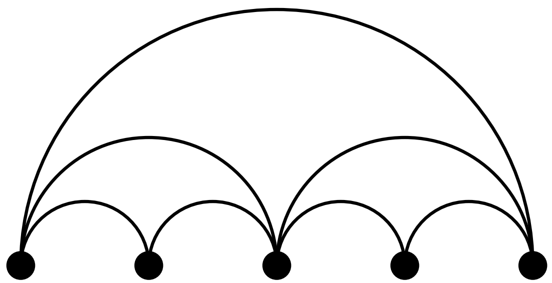
$$\leq 2 \cdot \mathrm{pn}_\ell(G)|V|$$



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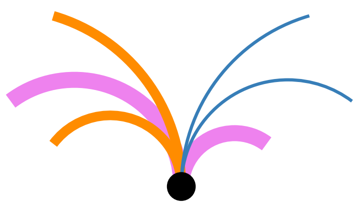

as each vertex is on at most $\mathrm{pn}_\ell(G)$ pages

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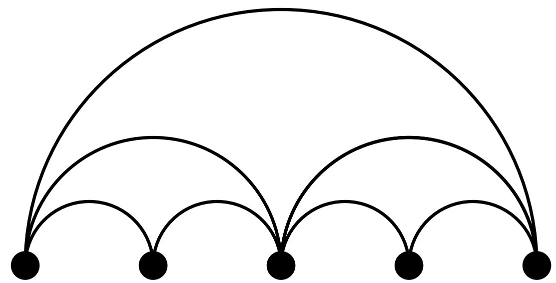

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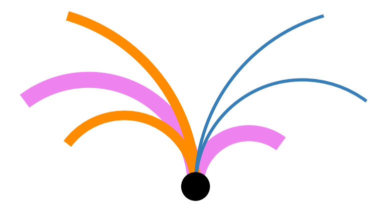

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Hence

$$\mathrm{pn}_\ell(G) \geq \frac{|E|}{2|V|} = \frac{1}{4} \cdot \mathrm{avd}(G)$$

$$\mathrm{avd}(G) = \frac{\sum_v \deg(v)}{|V|} = \frac{2|E|}{|V|}$$

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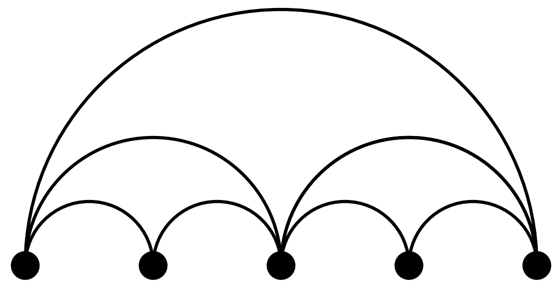

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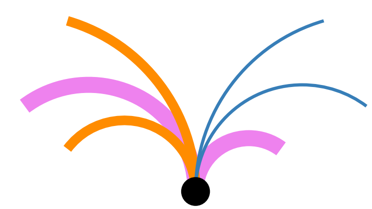

as each vertex is on at most $\mathrm{pn}_\ell(G)$ pages

Hence

$$\mathrm{pn}_\ell(G) \geq \frac{|E|}{2|V|} = \frac{1}{4} \cdot \mathrm{avd}(G)$$

$$\implies \quad \mathrm{pn}_\ell(G) \geq \frac{1}{4}\,\mathrm{mad}(G)$$

$$\mathrm{avd}(G) = \frac{\sum_v \deg(v)}{|V|} = \frac{2|E|}{|V|}$$

Comparison of variants

▷ For any graph G we have

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A simple lower bound ...

$$\mathrm{pn}_\ell(G) \ge \frac{1}{4}\,\mathrm{mad}(G)$$

... gives also an upper bound

$\mathrm{mad}(G) = k$

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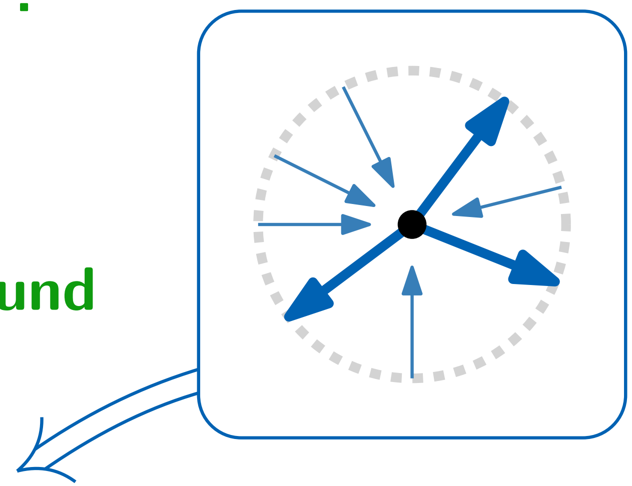

$\mathrm{mad}(G) = k$

$\Longrightarrow$ orientation with $\mathrm{outdeg}(v) \le k/2 + 1$

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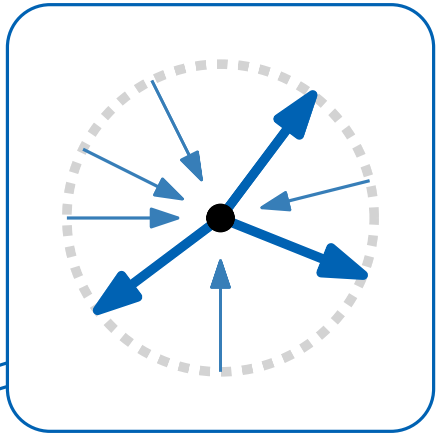

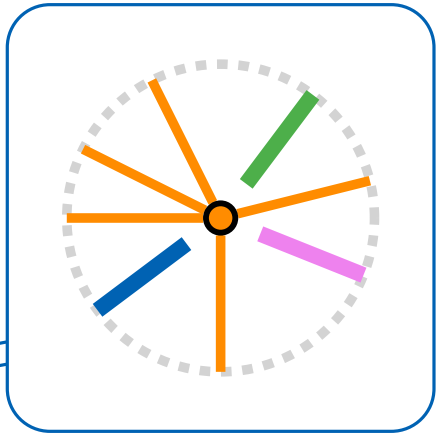

$\mathrm{mad}(G) = k$

$\implies$ orientation with $\mathrm{outdeg}(v) \leq k/2+1$

$\implies$ $(k/2+2)$-local star partition

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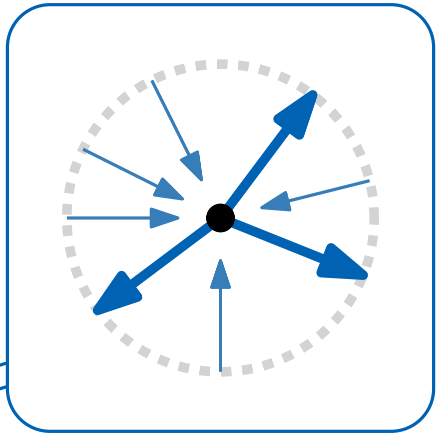

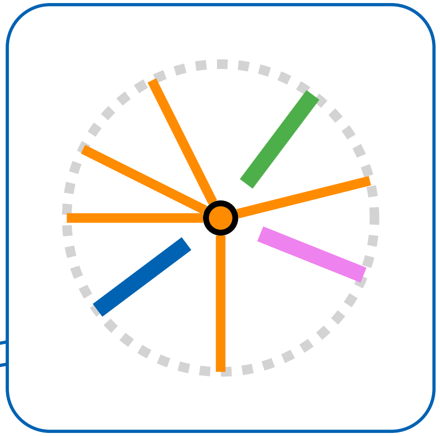

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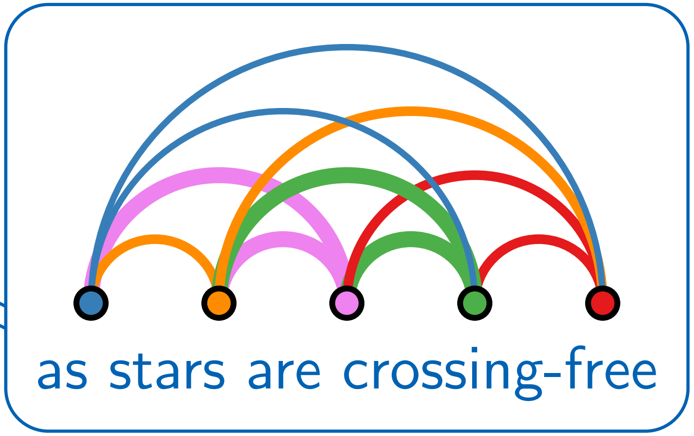
$\Longrightarrow$ $(k/2 + 2)$-local star partition



$$\Longrightarrow \mathrm{pn}_\ell(G) \leq \frac{1}{2}\,\mathrm{mad}(G) + 2$$

Comparison of variants

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A simple lower bound . . .

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$$\mathrm{pn}_\ell(G) \leq \frac{1}{2}\,\mathrm{mad}(G) + 2$$

... also for union page number

$\mathrm{mad}(G) = k$

$\implies$ $k+2$ star forests partition

$\implies$ $\mathrm{pn}_u(G) \leq \mathrm{mad}(G) + 2$

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$\mathrm{mad}(G) = k$

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$\implies$ $\mathrm{pn}_u(G) \leq \mathrm{mad}(G) + 2$

Corollary.

$$\mathrm{pn}_u(G) \leq 4\,\mathrm{pn}_\ell(G) + 2$$

but there are n-vertex k-regular graphs with

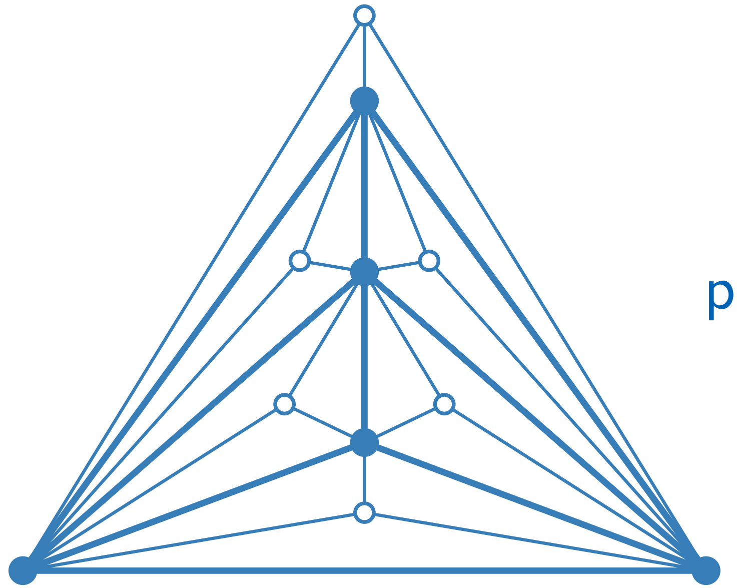
$$\mathrm{pn}_u(G) \leq k + 2$$

and

$$\mathrm{pn}(G) = \Omega\left(\sqrt{k}n^{\frac{1}{2}-\frac{1}{k}}\right)$$

"local and union page numbers are tied to density, classical page number is tied to structure"

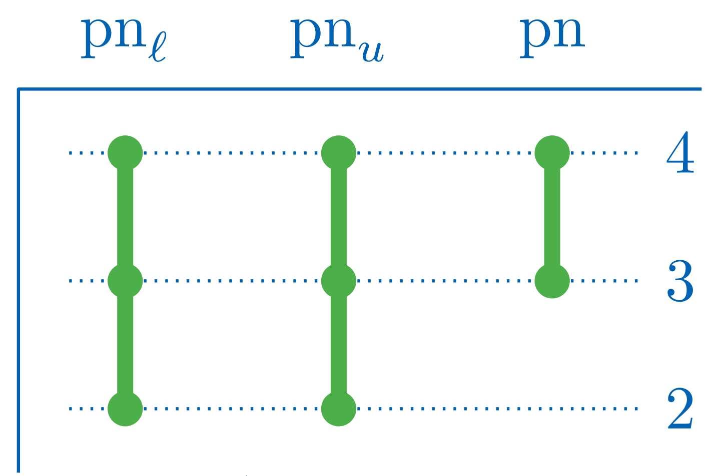
Planar graphs



$\mathrm{pn}(G) = 3$

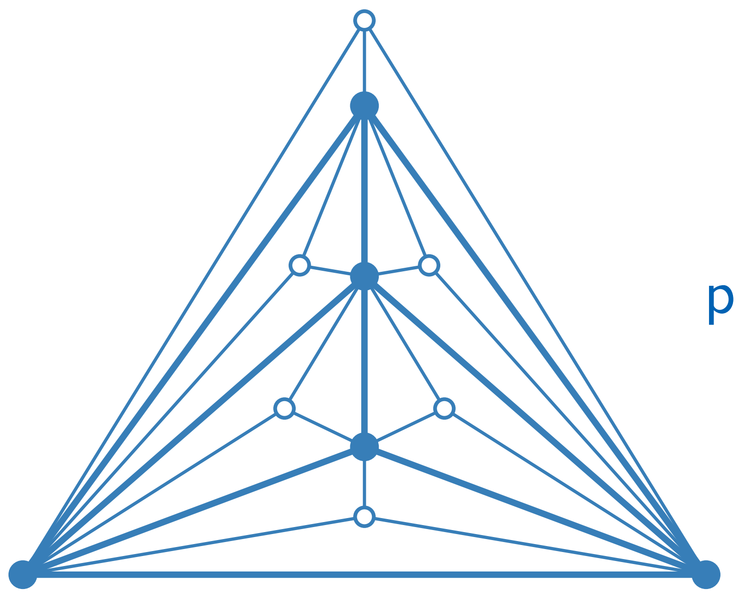
non-hamiltonian triangulation

max. within planar graphs



$\frac{1}{4}\,\mathrm{mad}(G) \le \mathrm{pn}_\ell(G) \le \mathrm{pn}_u(G)$

Planar graphs



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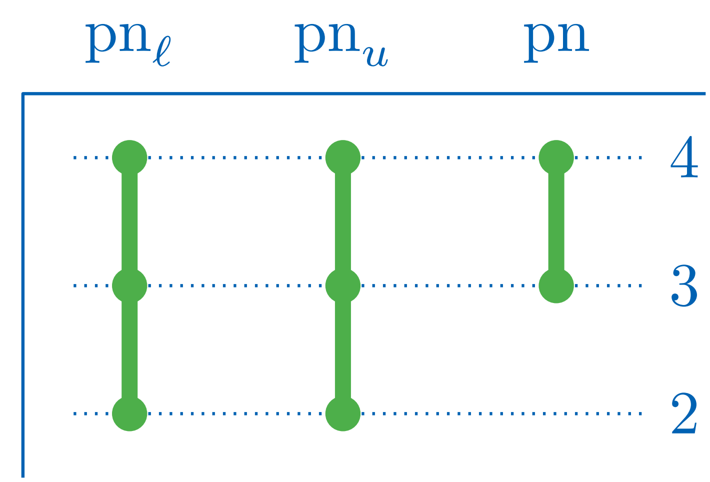
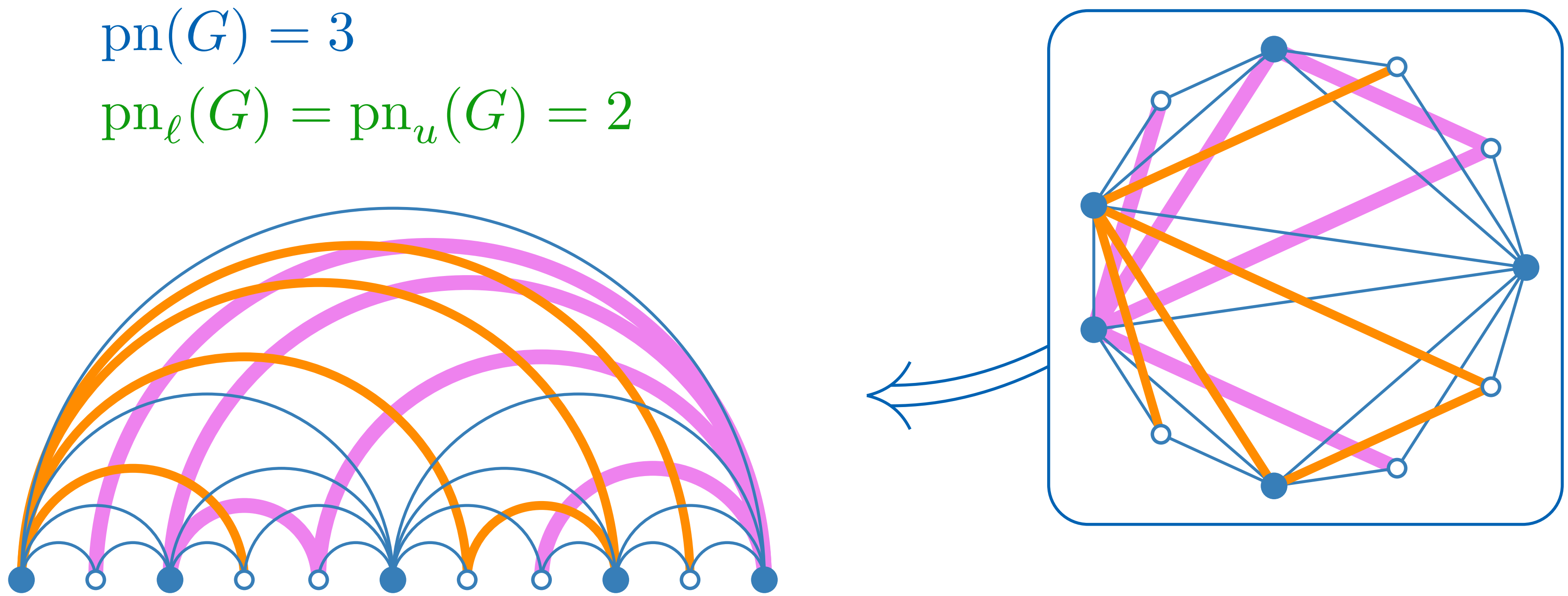


$\mathrm{pn}(G) = 3$

$\mathrm{pn}_\ell(G) = \mathrm{pn}_u(G) = 2$



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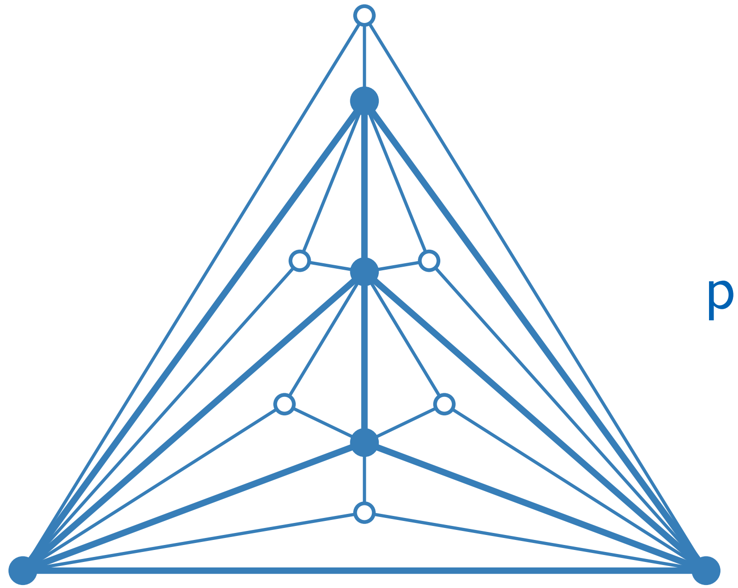

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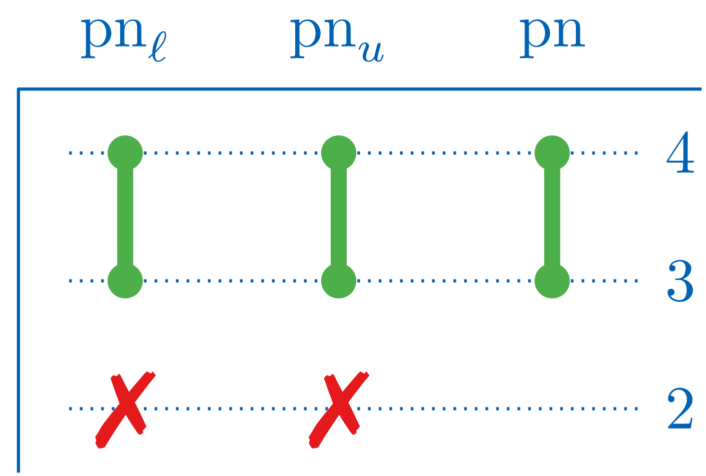

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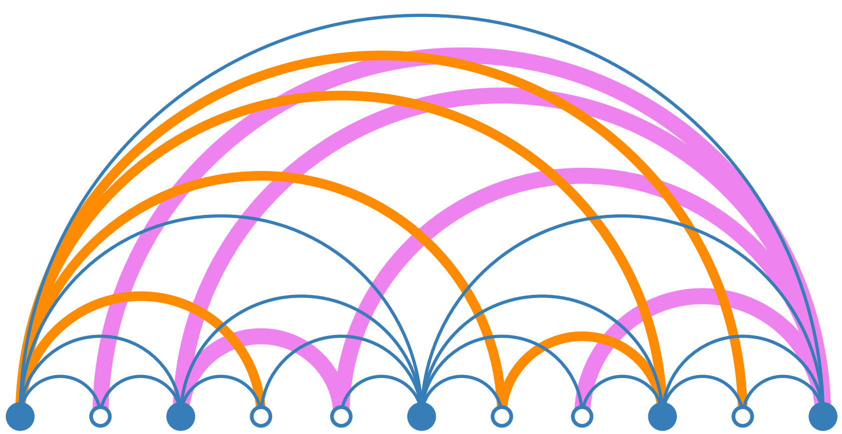
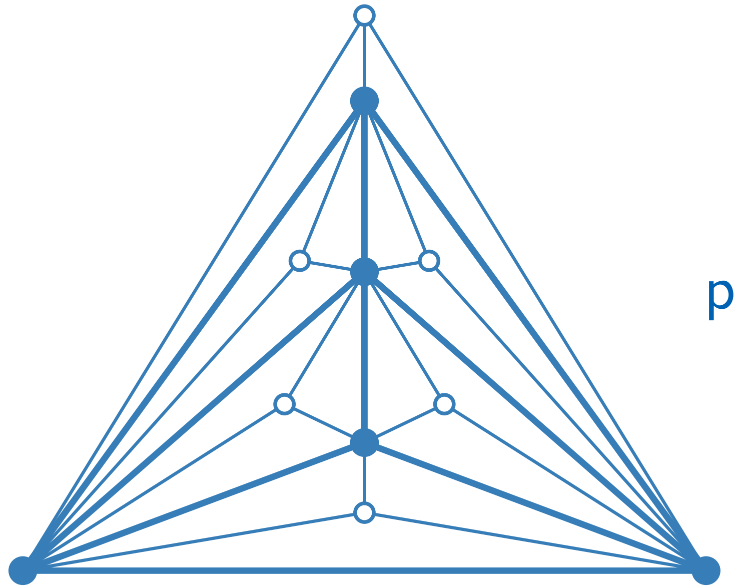


Theorem.

there is a planar graph G with

$\mathrm{pn}_u(G) \geq \mathrm{pn}_\ell(G) \geq 3$

Planar graphs



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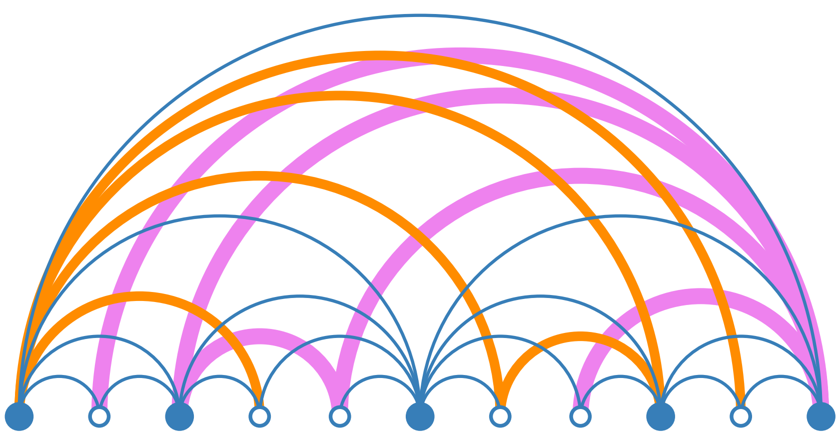
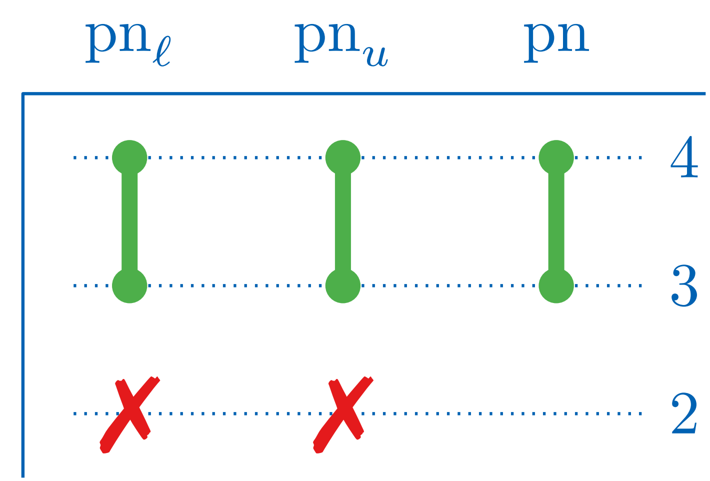


max. within planar graphs



G planar

$\implies$ orientation with $\mathrm{outdeg}(v) \leq 3$

$\implies$ 4-local star partition

$\implies \mathrm{pn}_\ell(G) \leq 4$

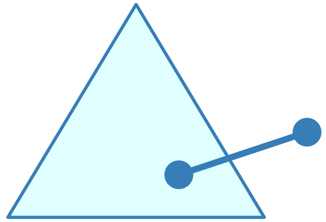
G planar

$\implies$ 5 star forest partition

$\implies \mathrm{pn}_u(G) \leq 5$

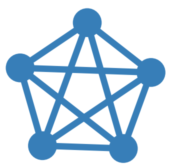
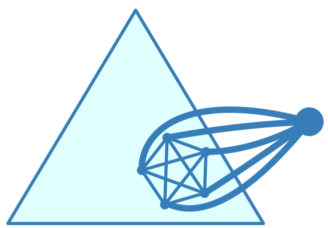
k-**Trees** (graphs of treewidth k)

1-tree:

 or 

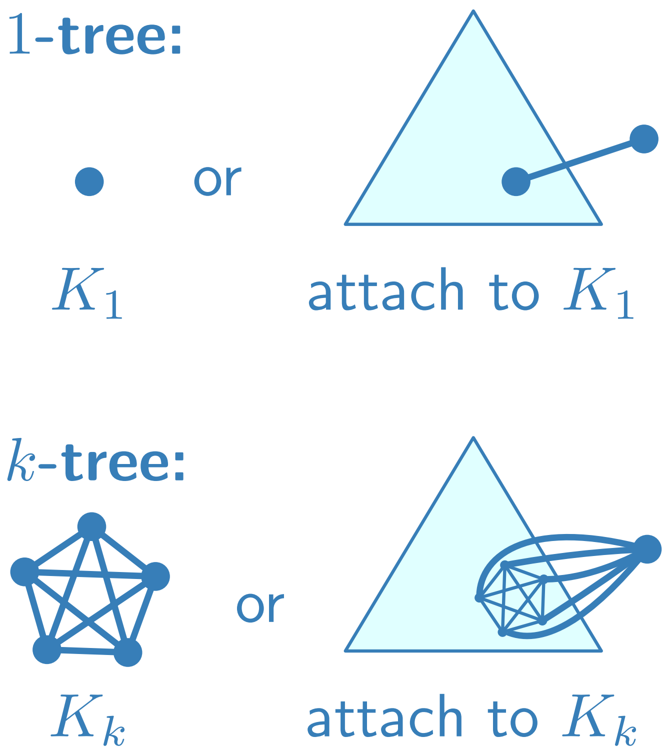
K_1 attach to K_1

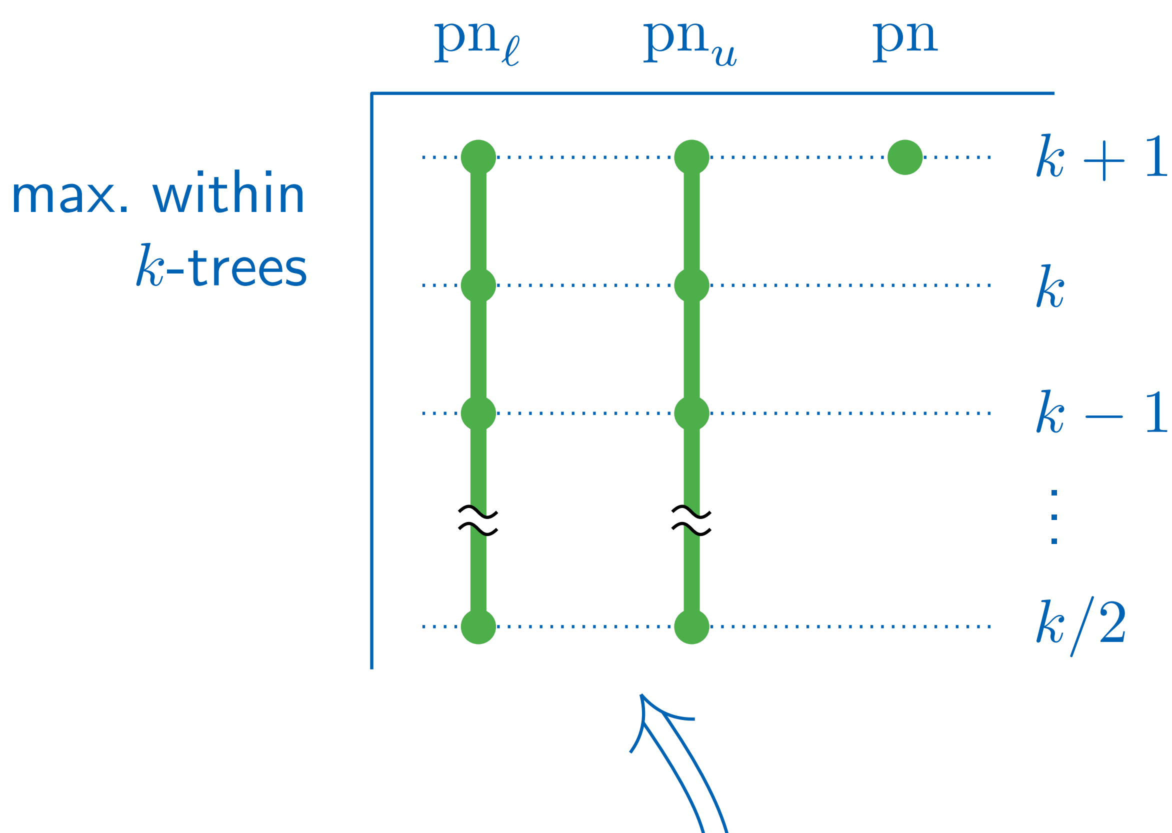
k**-tree:**

 or 

K_k attach to K_k

k-Trees (graphs of treewidth k)



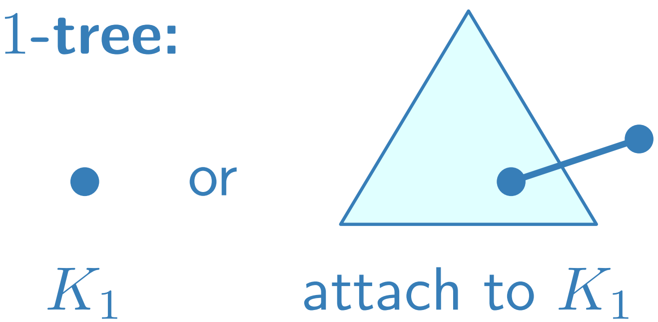


$$\tfrac{1}{4}\,\mathrm{mad}(G) \le \mathrm{pn}_\ell(G) \le \mathrm{pn}_u(G)$$

$$|E| \approx k|V| \Longrightarrow \mathrm{mad}(G) \approx 2k$$

k-**Trees** (graphs of treewidth k)

1-tree:



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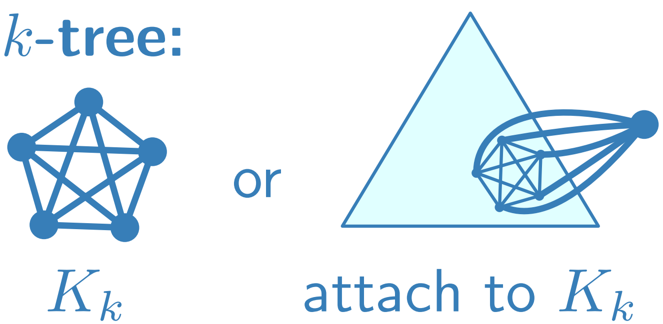
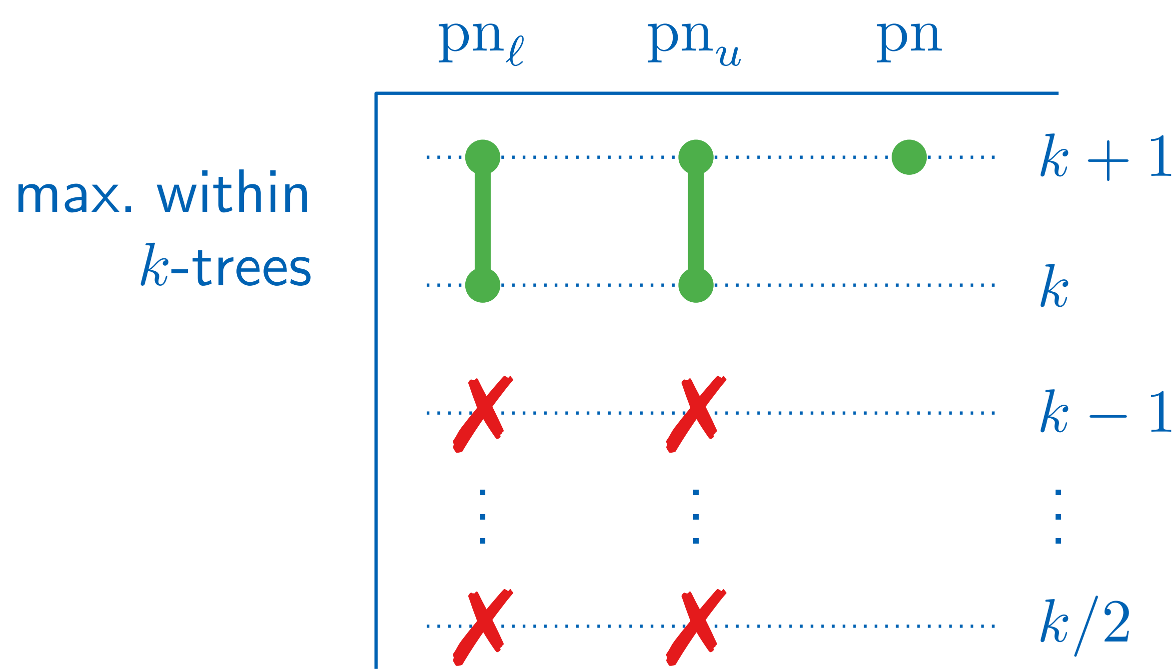
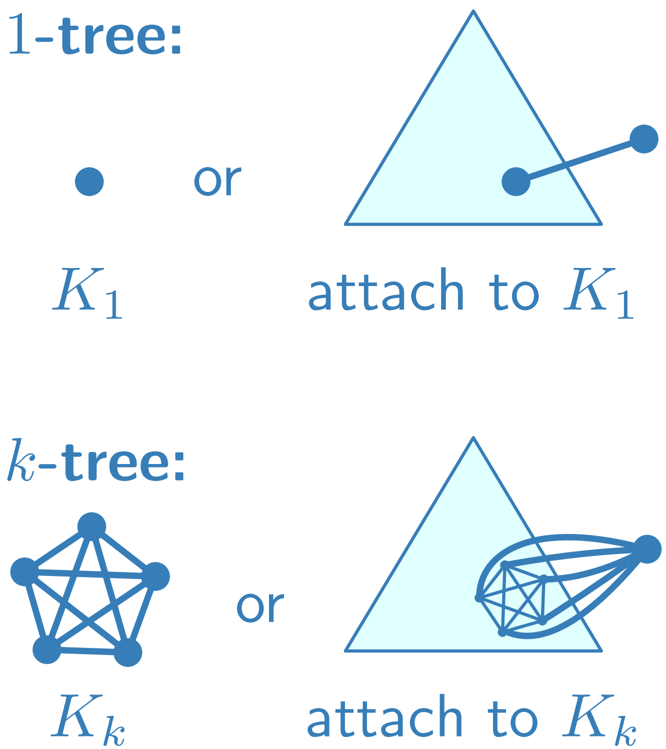


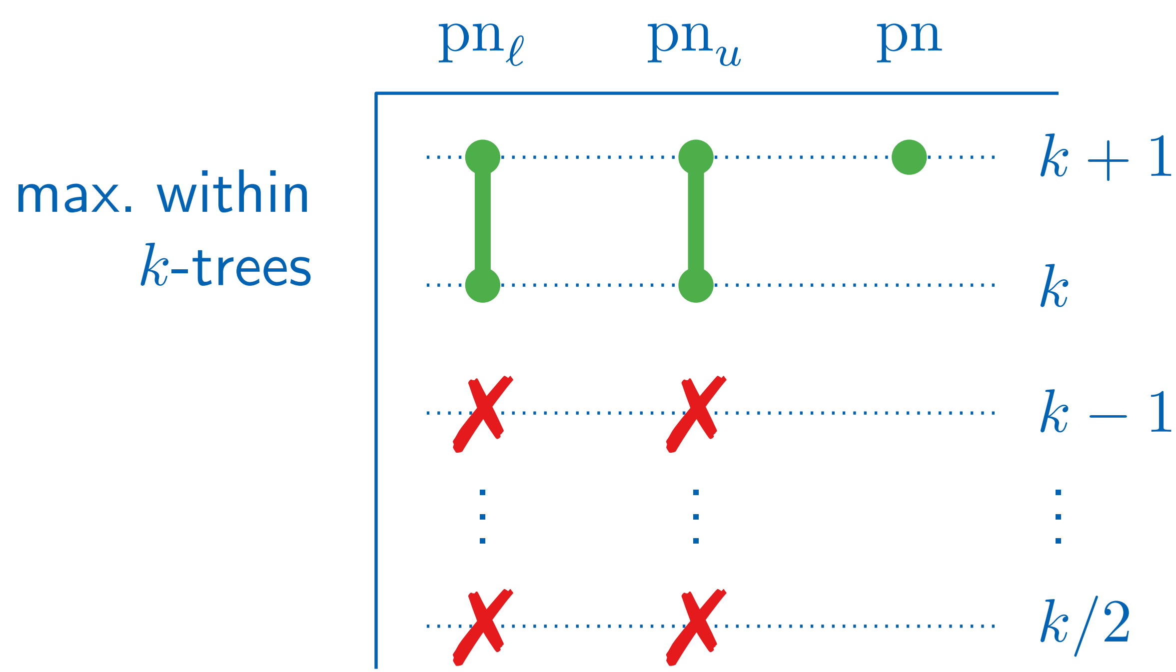


Theorem.

ℓ-local book embedding for every k-tree $\Longrightarrow$ ℓ-local book embedding for every k-tree with a forest on each page

k-**Trees** (graphs of treewidth k)





G k-tree

$\implies$ orientation with $\mathrm{outdeg}(v) \leq k$

$\implies$ $(k+1)$-local star partition

$\implies$ $\mathrm{pn}_\ell(G) \leq k+1$

G k-tree

$\implies$ $k+1$ star forest partition

$\implies$ $\mathrm{pn}_u(G) \leq k+1$

k-**Trees** (graphs of treewidth k)

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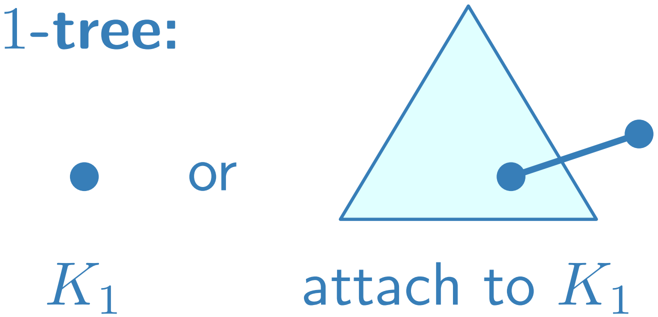
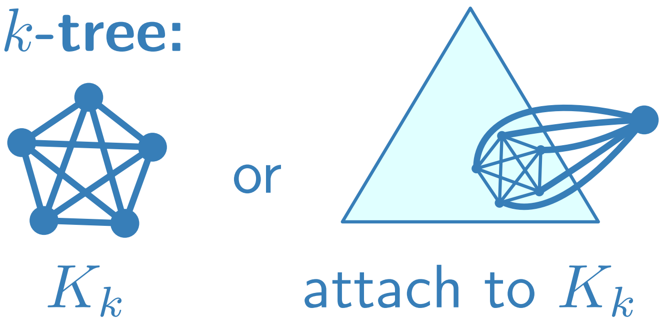
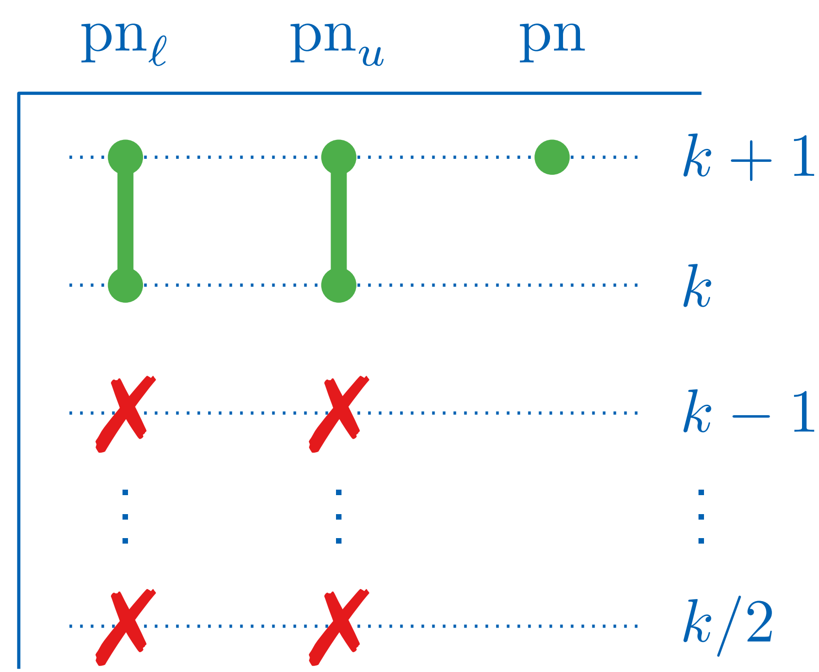


k-**tree:**



max. within k-trees

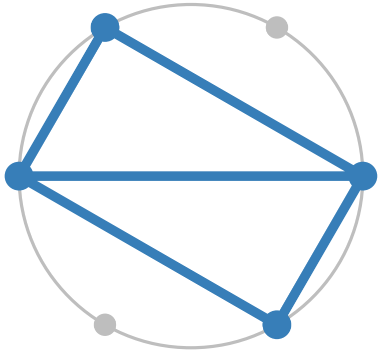


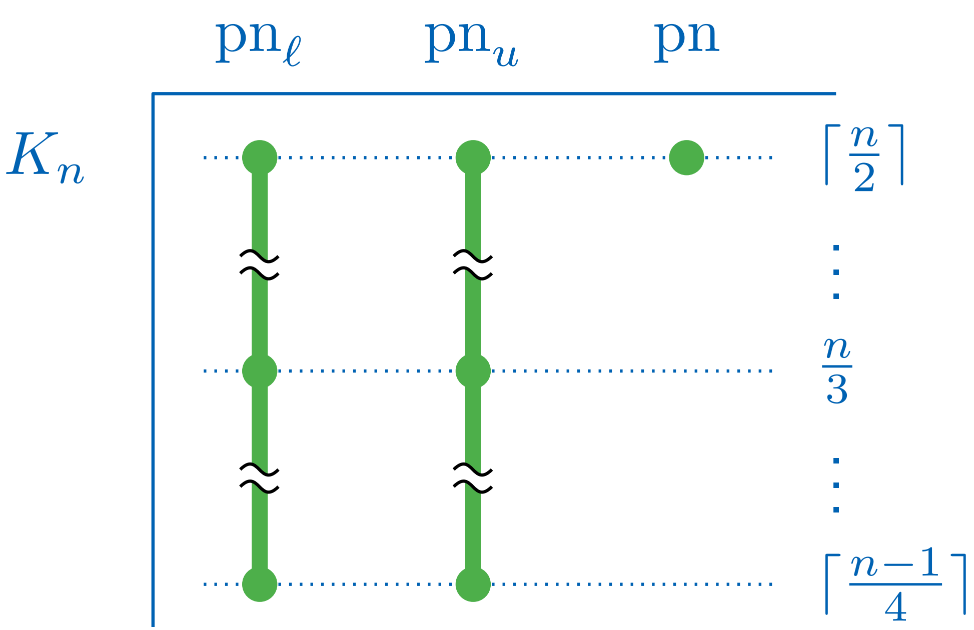
A possible approach?

- ▷ consider the unique $(k+1)$-coloring of G
- ▷ then any two color classes induce a tree
  - ⇝ k trees at each vertex
  - ⇝ can be combined to k or $k+1$ forests

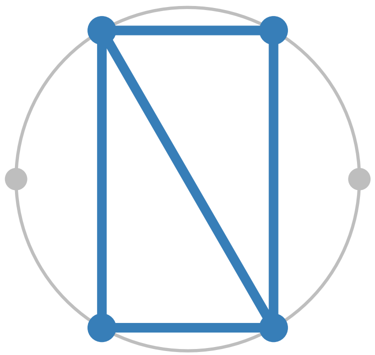
Still open:
Find the spine ordering!

Complete graphs

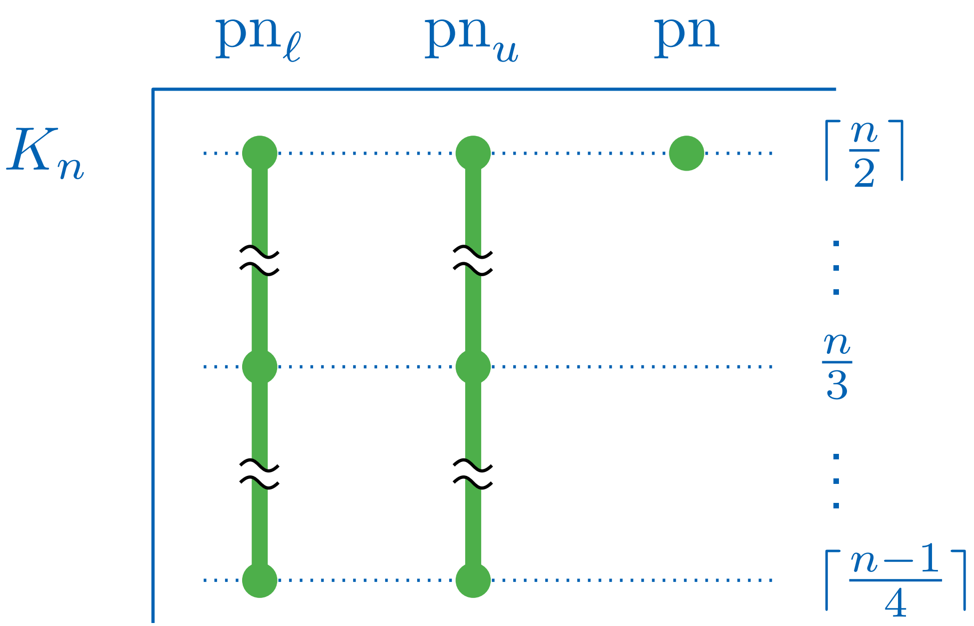




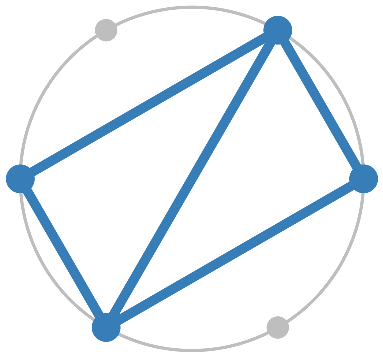
Complete graphs



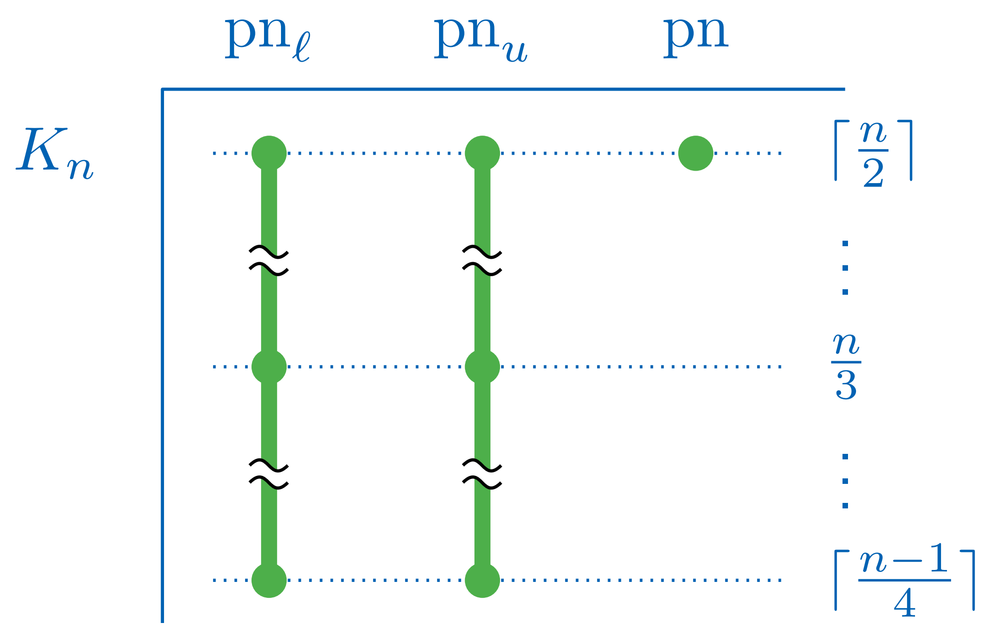
$\mathrm{pn}_\ell(K_6) = 2$



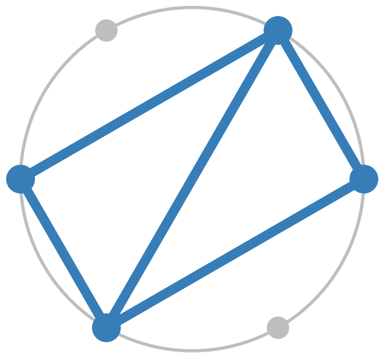
Complete graphs



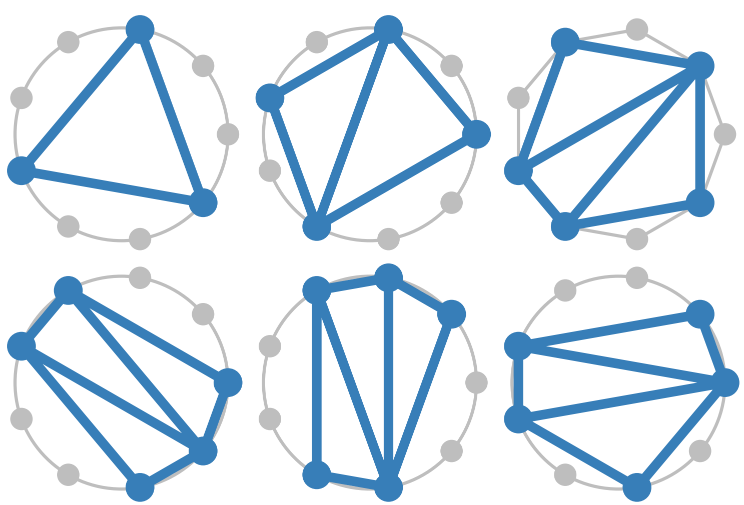
$\mathrm{pn}_\ell(K_6) = 2$



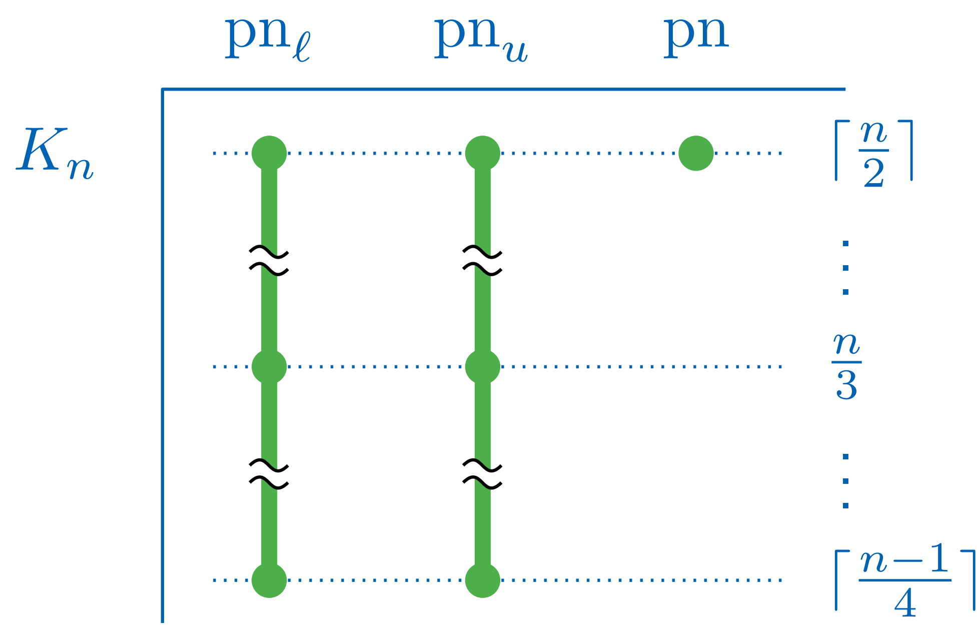
Complete graphs



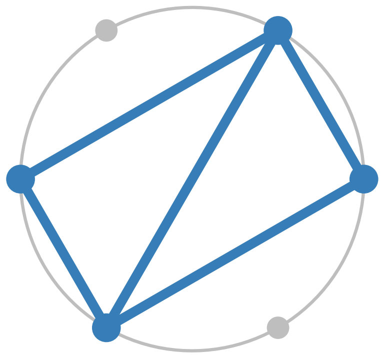
$\mathrm{pn}_\ell(K_6) = 2$



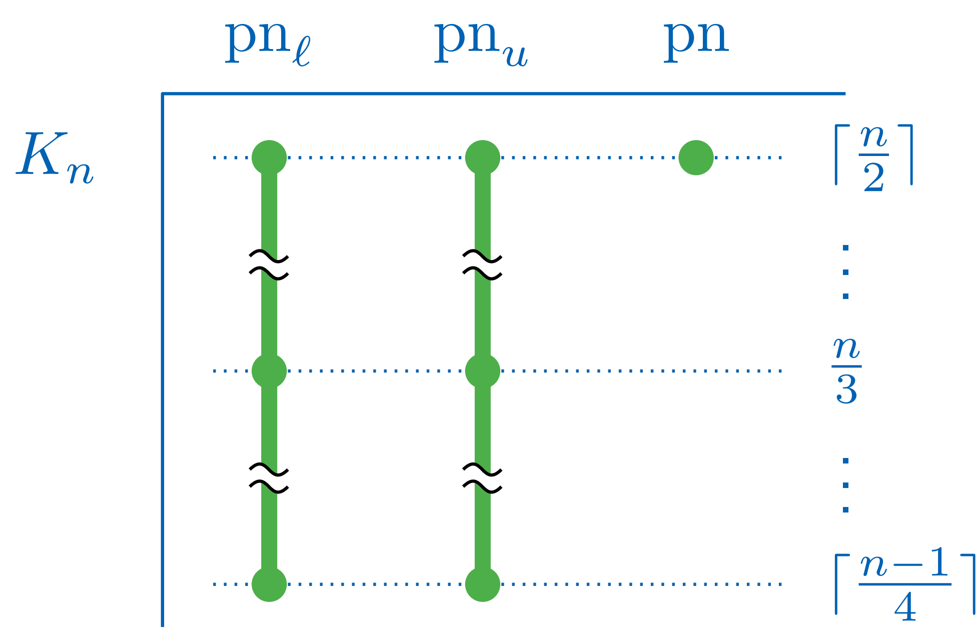
$\mathrm{pn}_\ell(K_9) = 3$

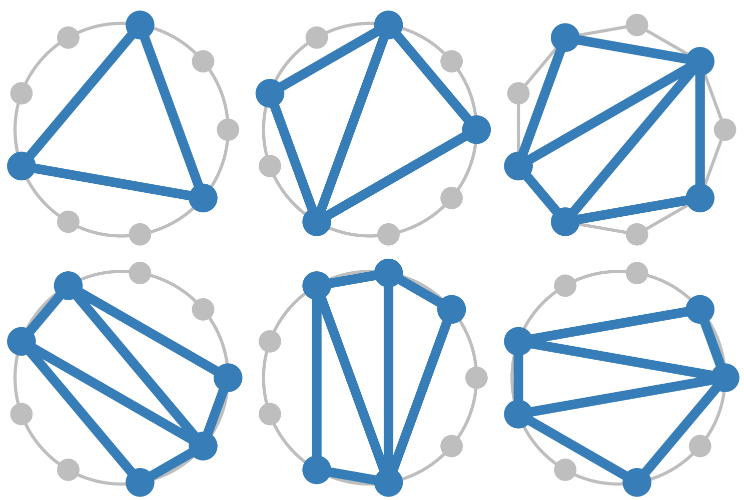


Complete graphs

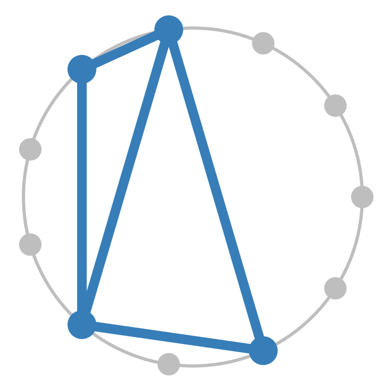


$\mathrm{pn}_\ell(K_6) = 2$



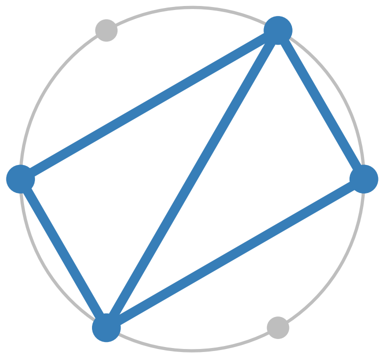


$\mathrm{pn}_\ell(K_9) = 3$

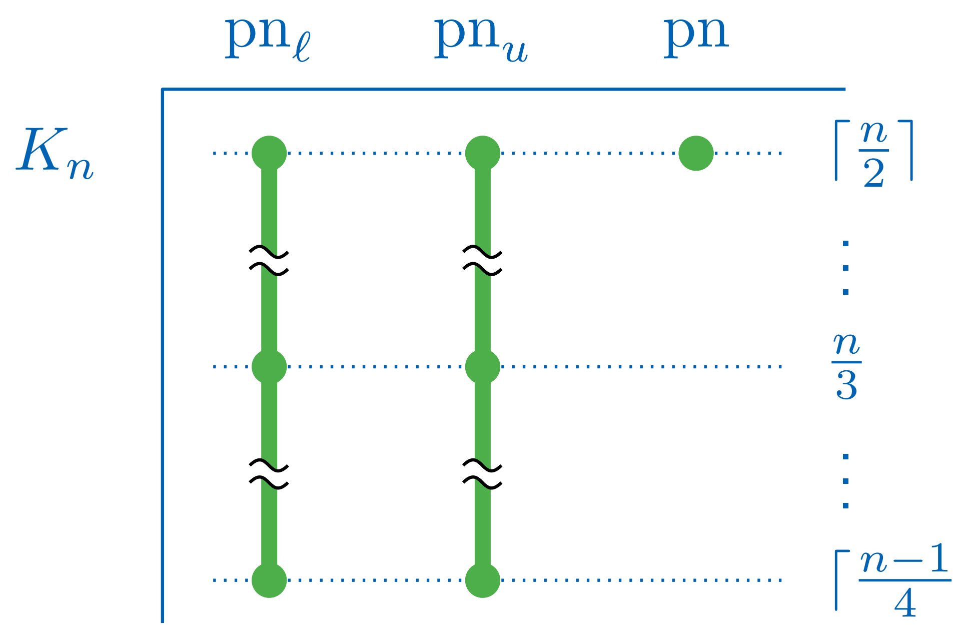


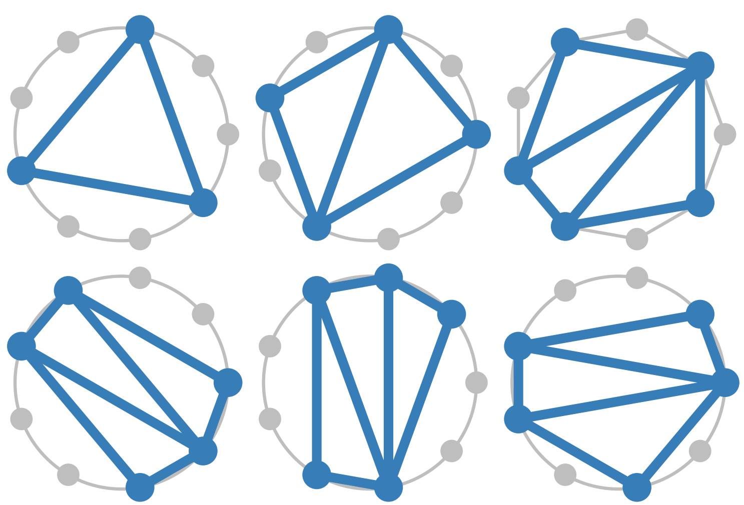
$\mathrm{pn}_\ell(K_{11}) = 4$

Complete graphs

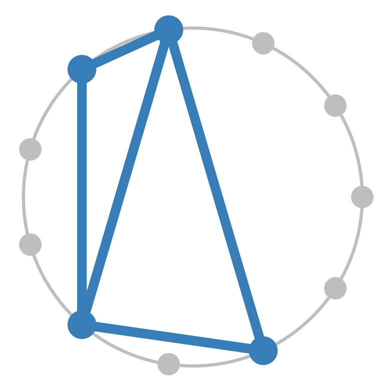


$\mathrm{pn}_\ell(K_6) = 2$

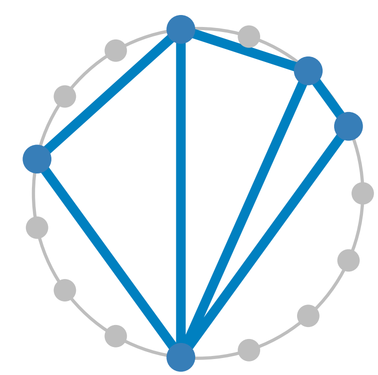




$\mathrm{pn}_\ell(K_9) = 3$

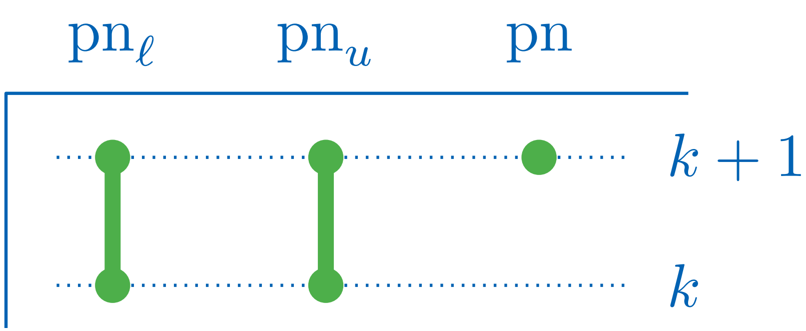


$\mathrm{pn}_\ell(K_{11}) = 4$

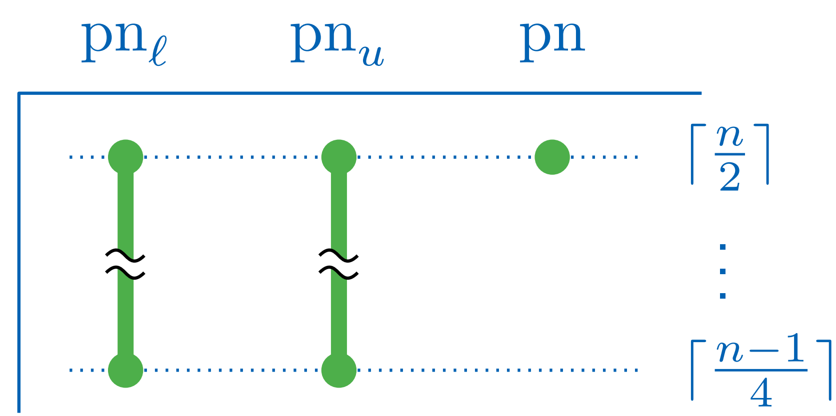


$\mathrm{pn}_\ell(K_{15}) \leq 5$

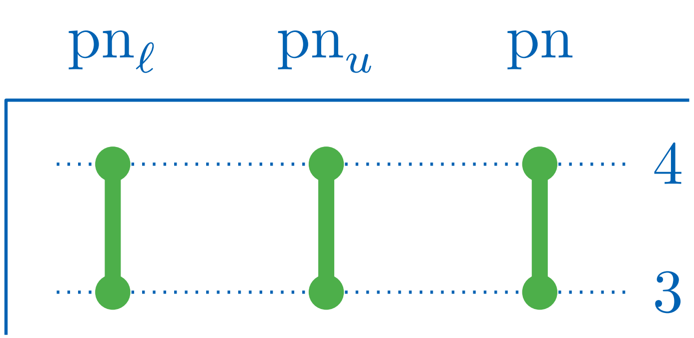
Open problems



k-trees, treewidth k



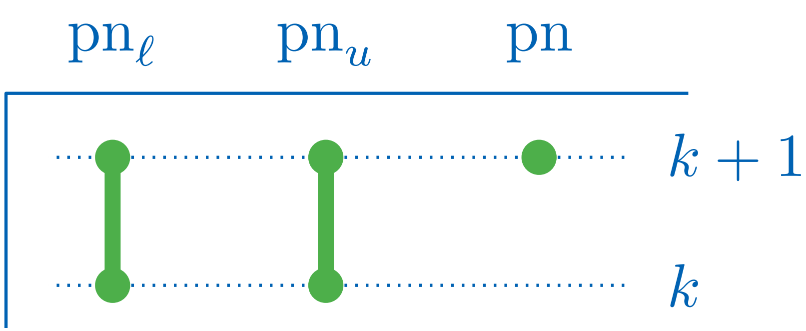
complete graphs, K_n



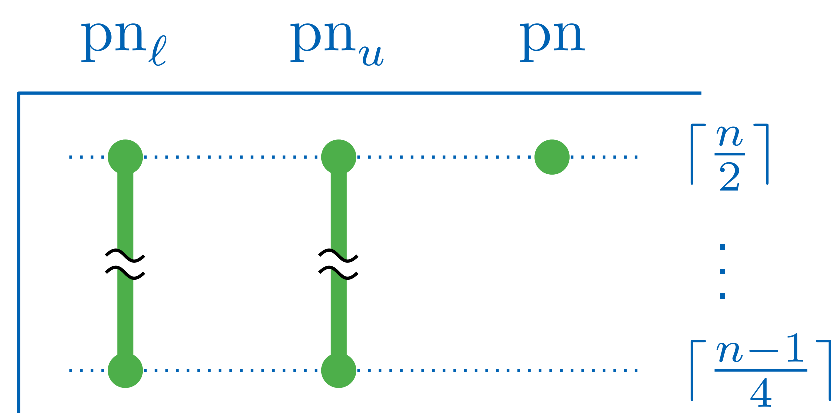
planar graphs

- ▷ computational complexity ?
- ▷ maximum $\mathrm{pn}_u(G)/\,\mathrm{pn}_\ell(G)$?
- ▷ $K_{m,n}$?
- ▷ local and union queue numbers ?

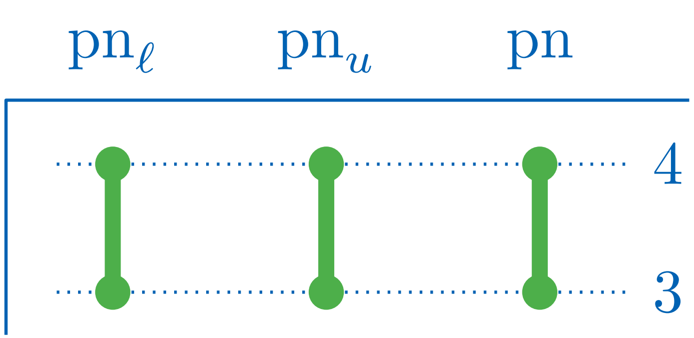
Open problems



k-trees, treewidth k



complete graphs, K_n



planar graphs

Thank you

- ▷ computational complexity ?
- ▷ maximum $\mathrm{pn}_u(G)/\,\mathrm{pn}_\ell(G)$?
- ▷ $K_{m,n}$?
- ▷ local and union queue numbers ?