Julius-Maximilians-
UNIVERSITÄT WÜRZBURG

Chair for
INFORMATICS I
Efficient Algorithms and
Knowledge-Based Systems

Institute for Informatics

Crossing Numbers of Beyond-Planar Graphs

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Universität Würzburg

joint work with

Markus Chimani Fabrizio Montecchiani Pavel Valtr

UNIVERSITÄT OSNABRÜCK

UNIVERSITÀ DEGLI STUDI DI PERUGIA
DAL 1308 COSTRUIAMO IL FUTURO

UNIVERZITA KARLOVA

Crossing ratio

Crossing number $\mathrm{cr}(G)$:

Min. # crossings over all drawings of G

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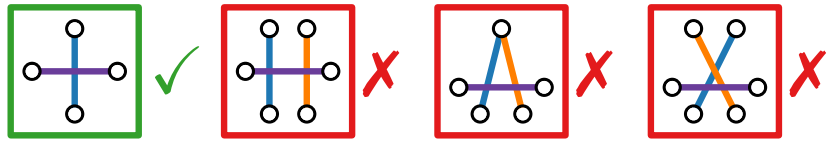

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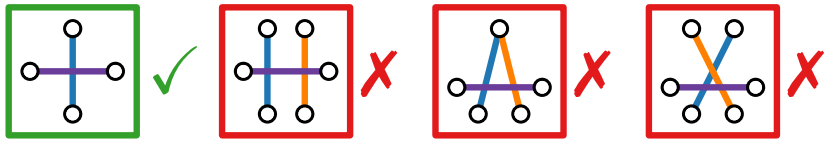

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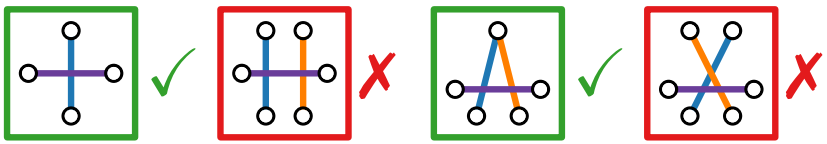

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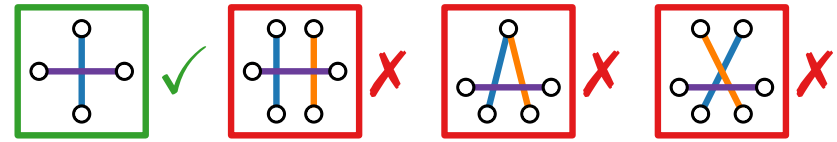

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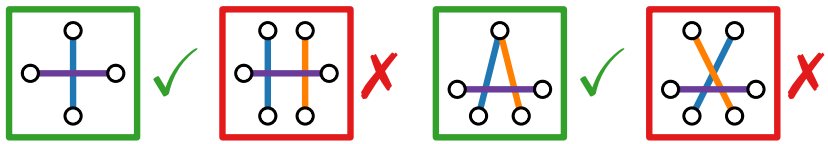

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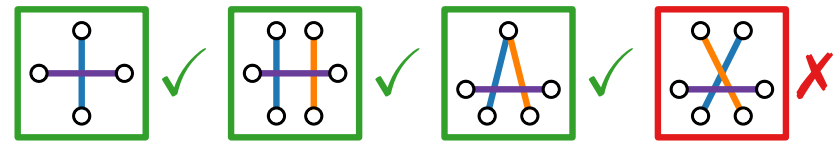

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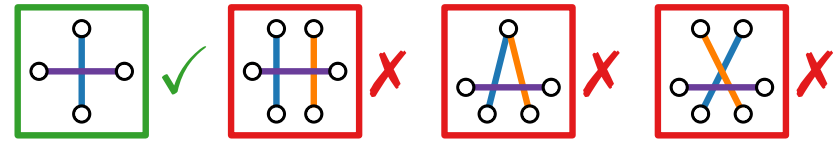

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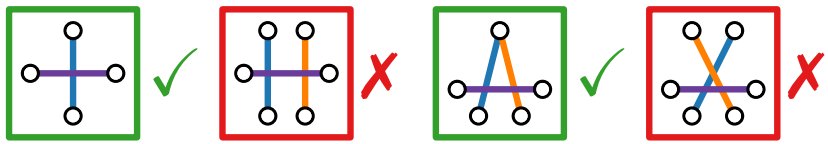

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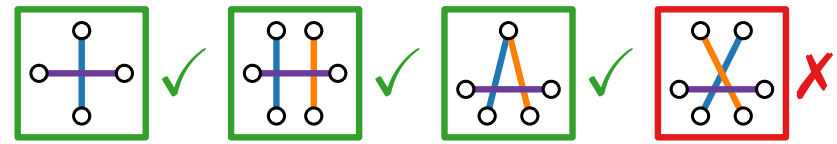

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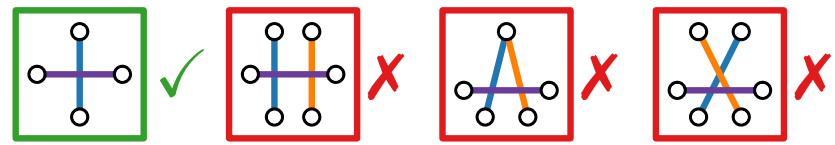

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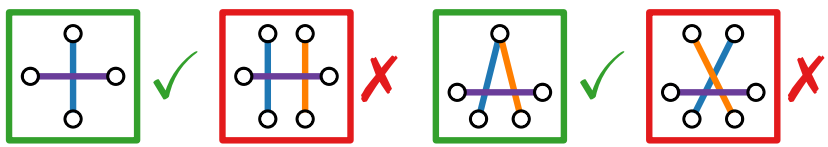

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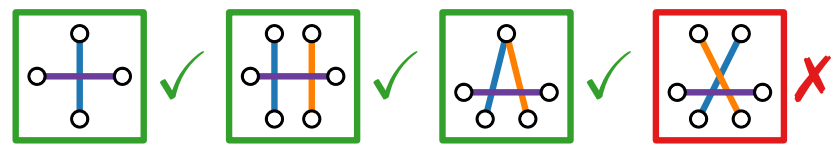

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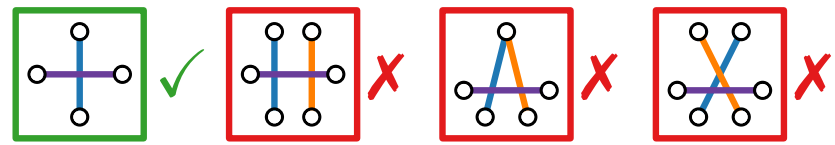

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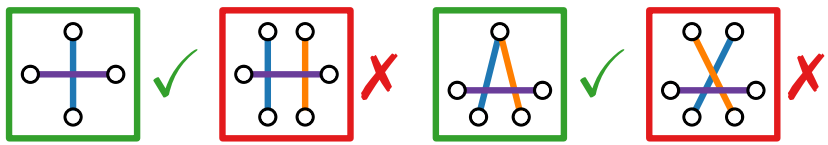

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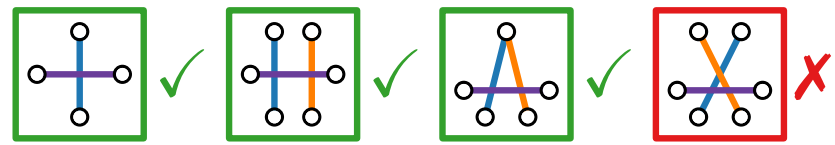

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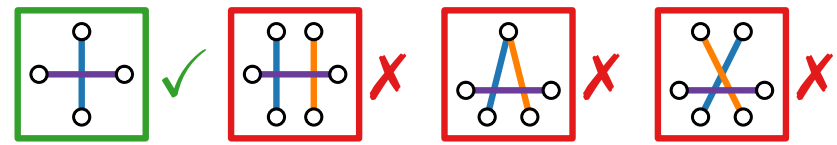

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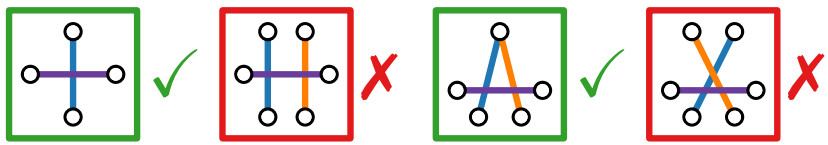

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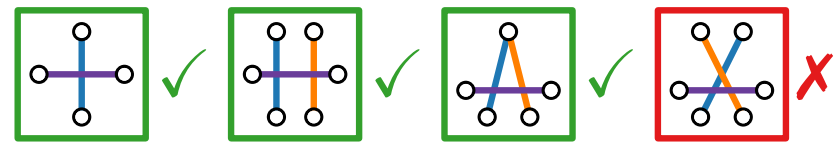

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$\varrho_{\mathrm{fan}}(G)$ $\mathrm{cr}_{\mathrm{fan}}(G)/\mathrm{cr}(G)$ fan-planar

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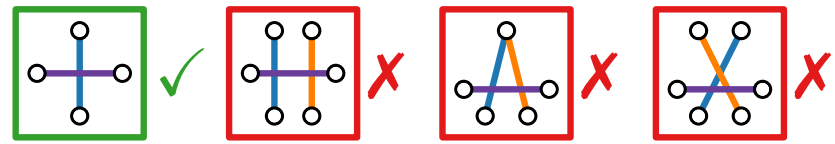

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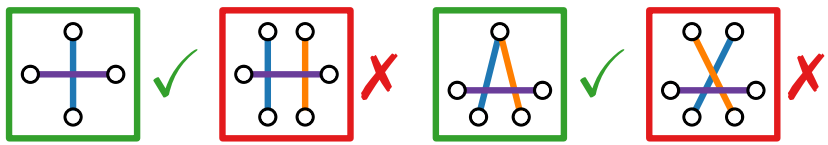

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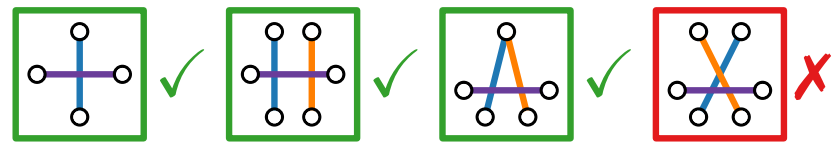

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$\varrho_{\mathrm{fan}}(G)$	$\mathrm{cr}_{\mathrm{fan}}(G)/\mathrm{cr}(G)$	fan-planar
$\varrho_{(k\text{-})\mathrm{qp}}(G)$	$\mathrm{cr}_{(k\text{-})\mathrm{qp}}(G)/\mathrm{cr}(G)$	(k-)quasi-planar

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at most $4n - 8$ edges, $n - 2$ crossings

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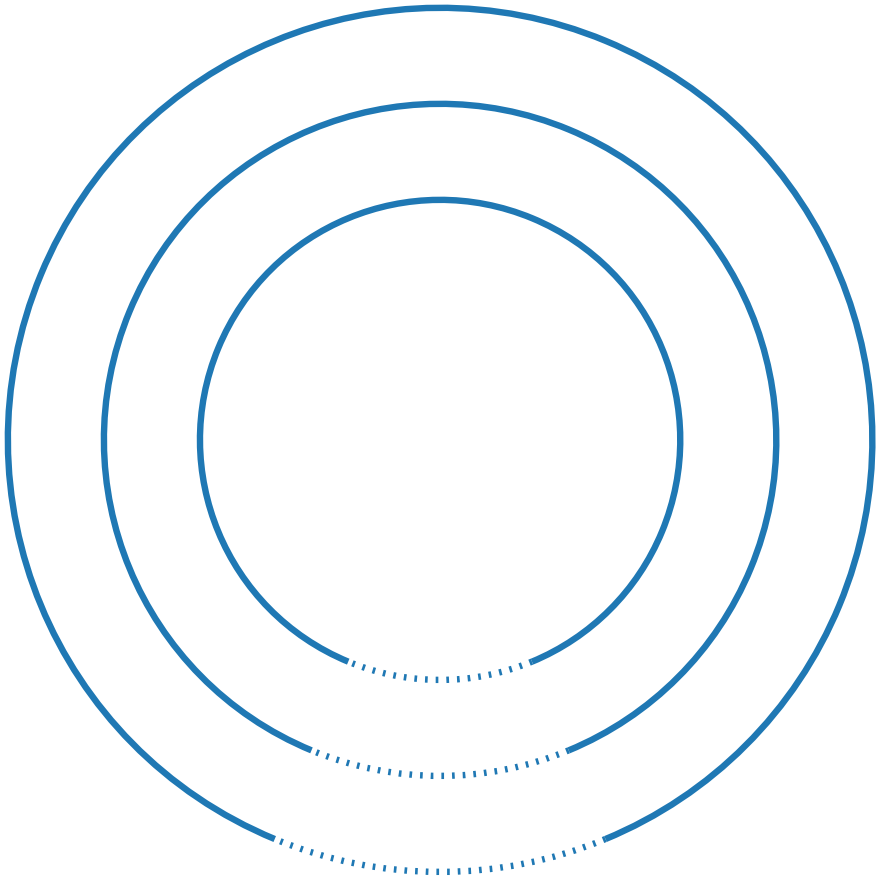

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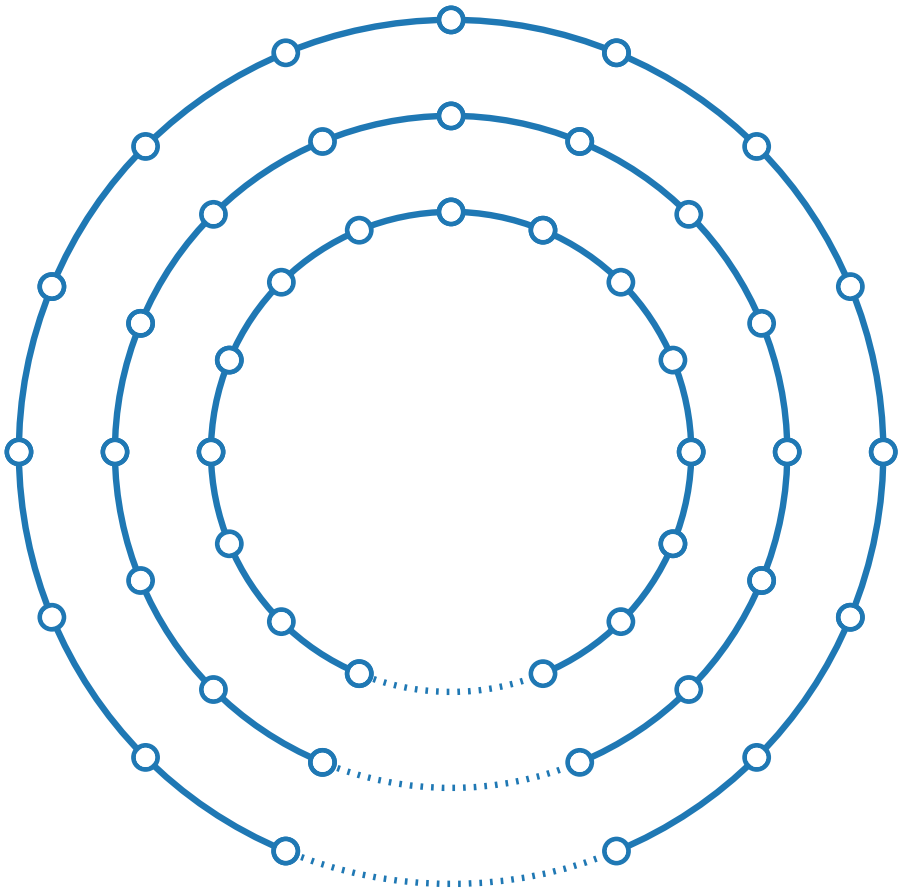

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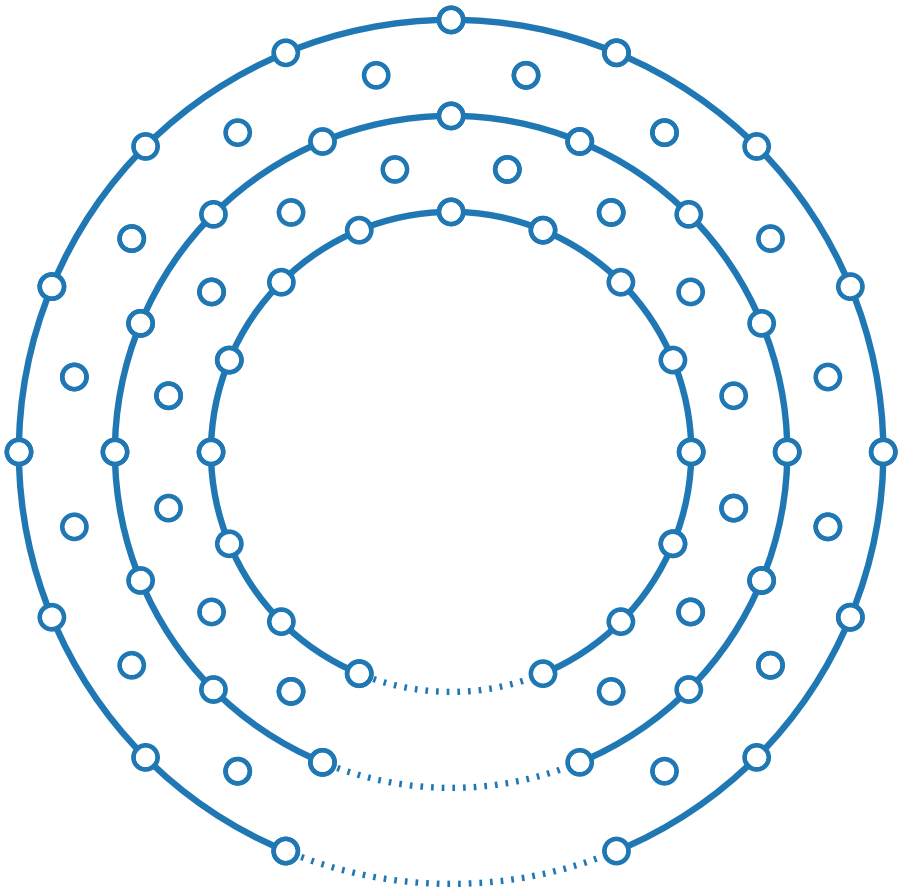

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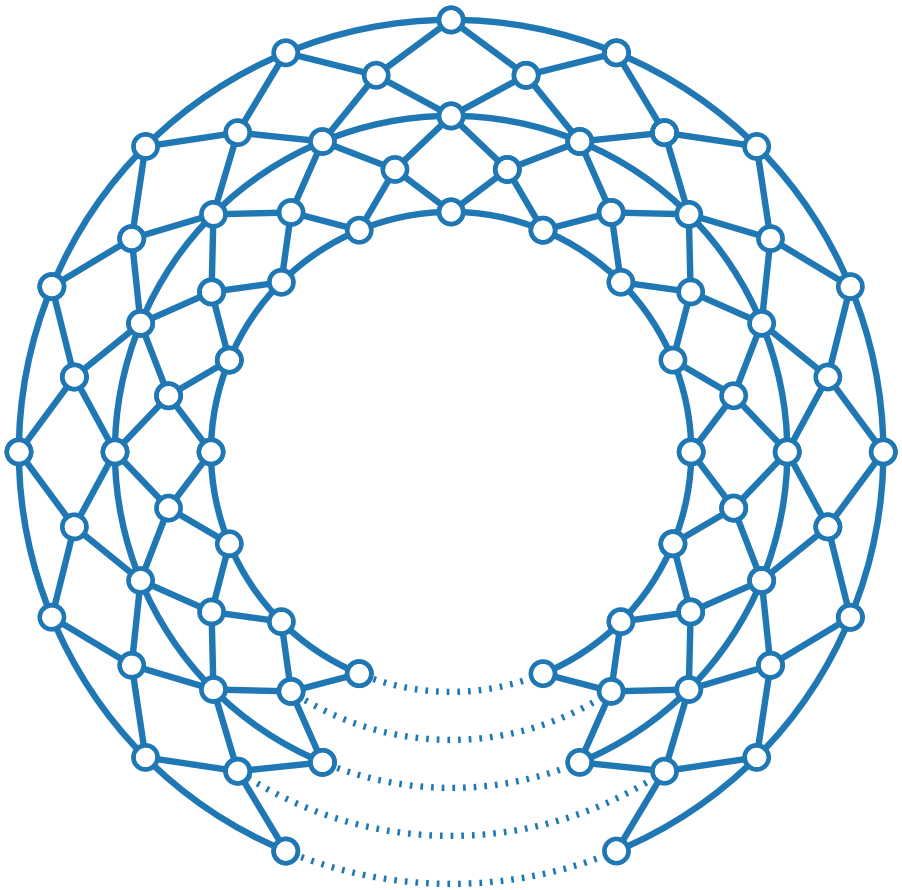

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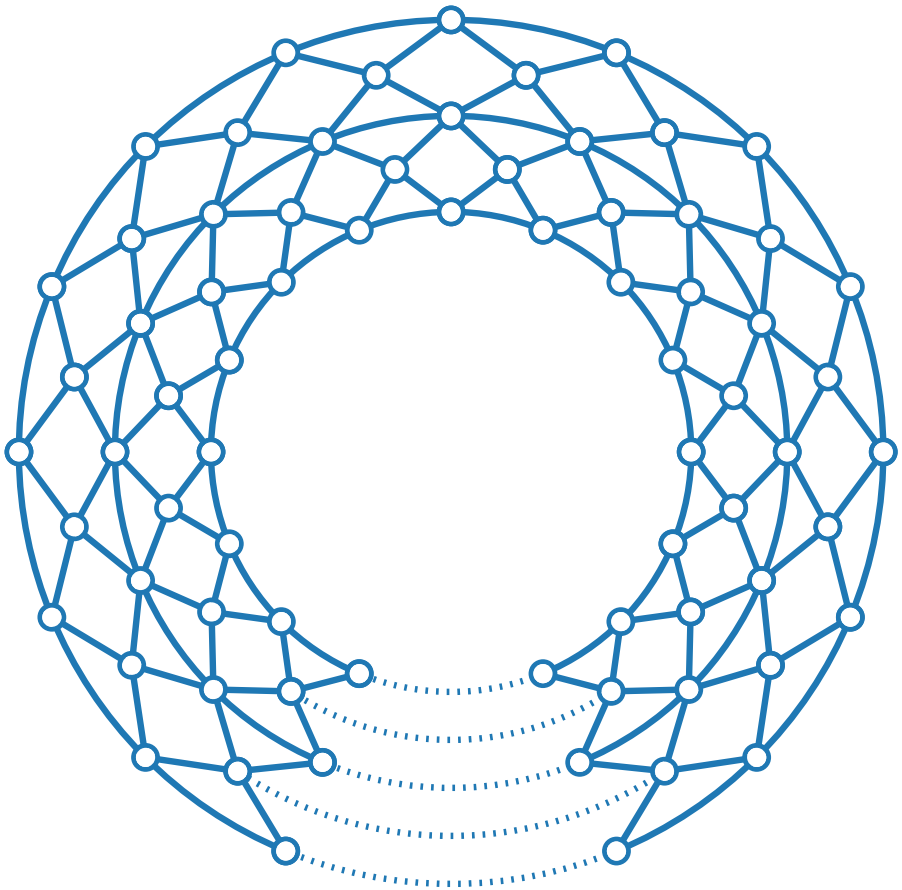

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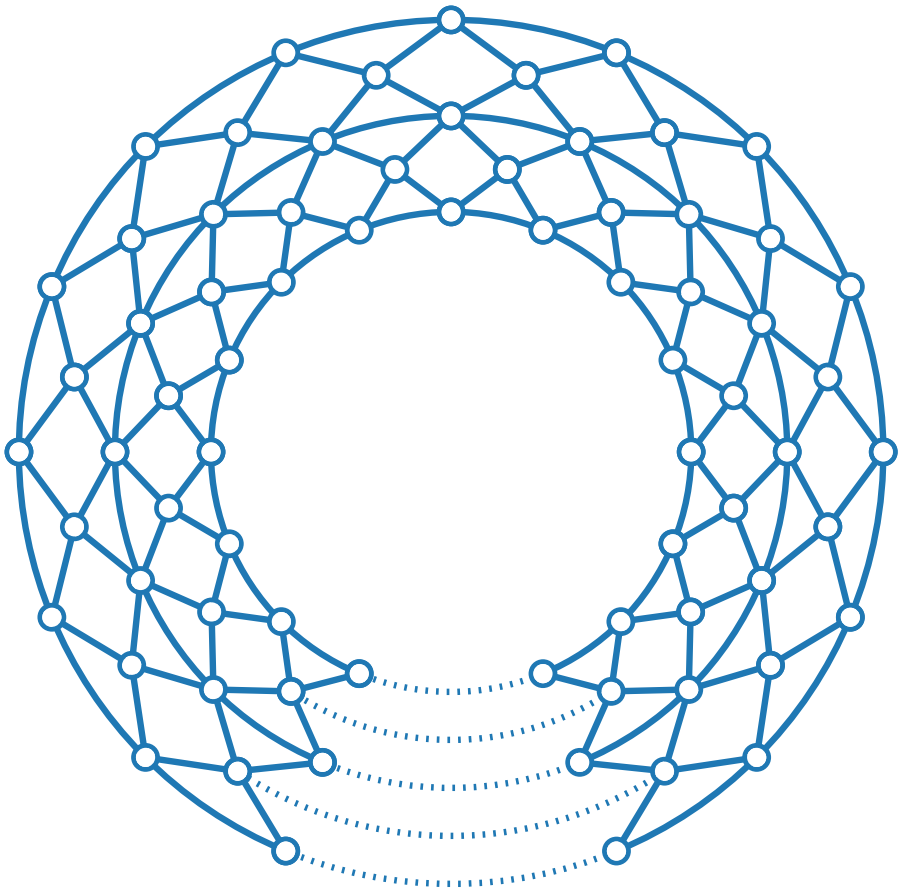

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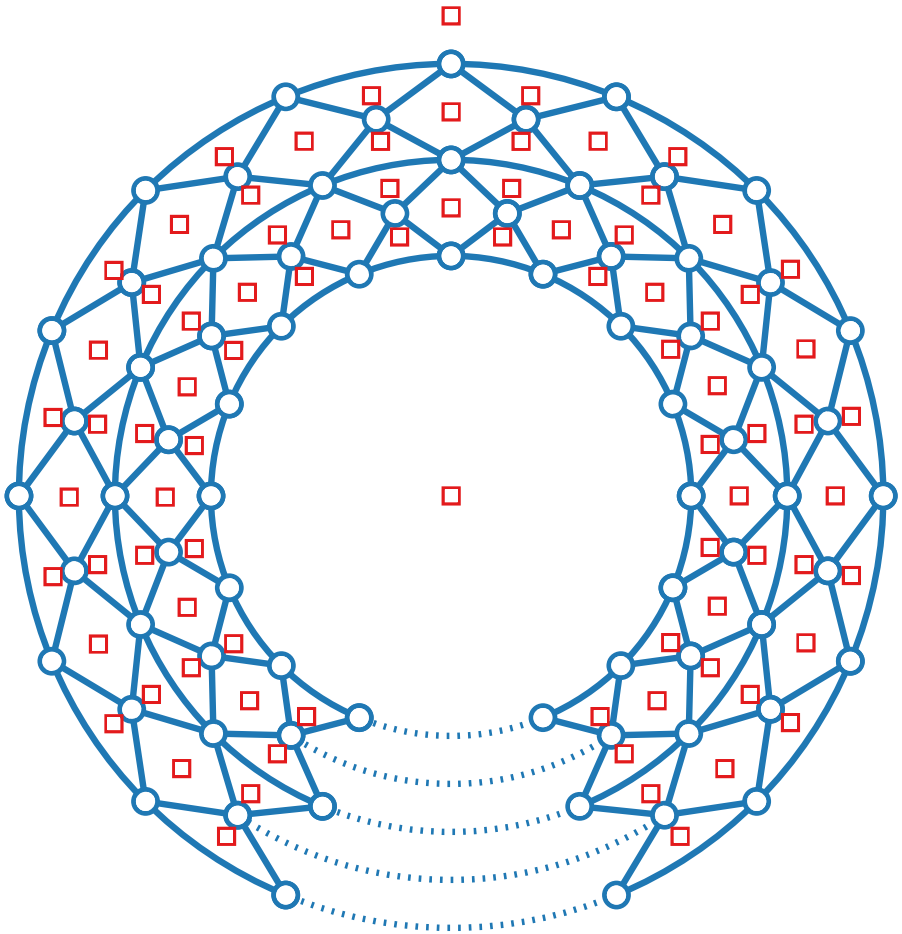

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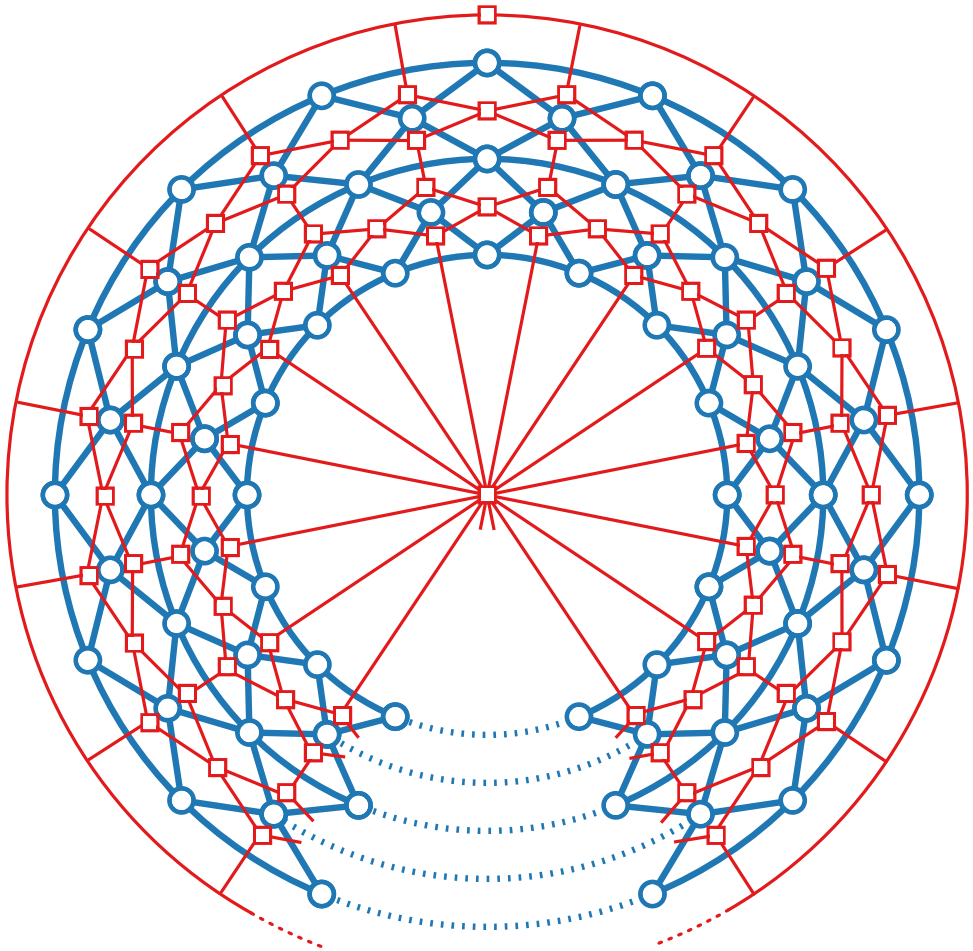

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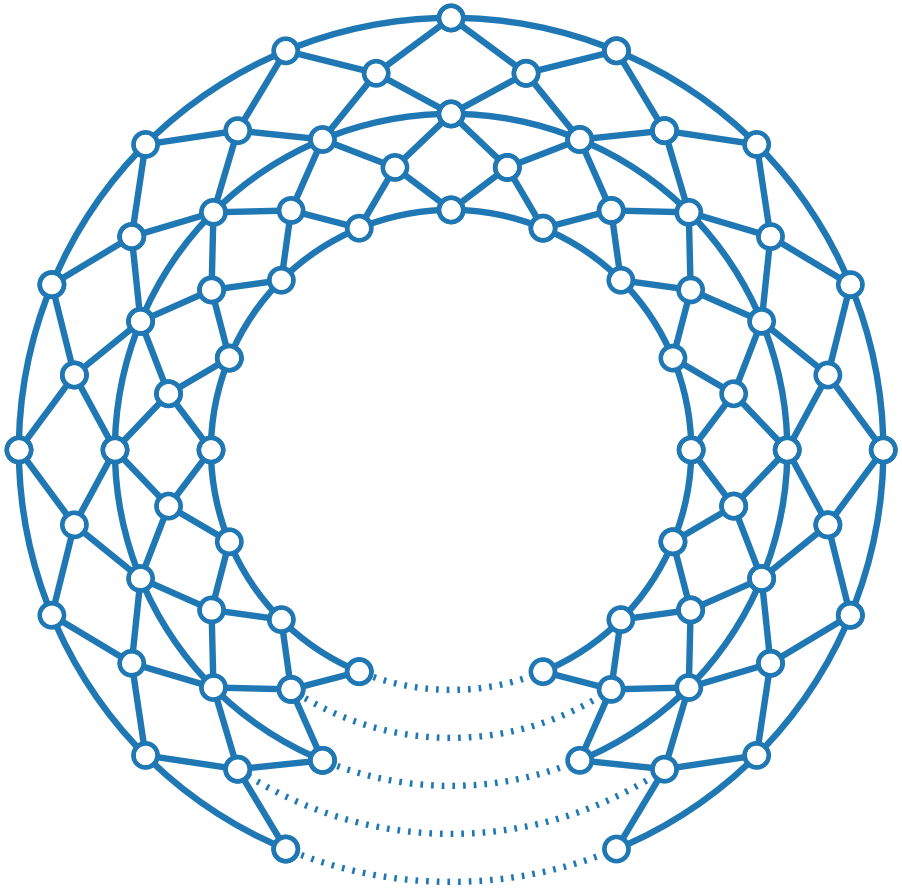

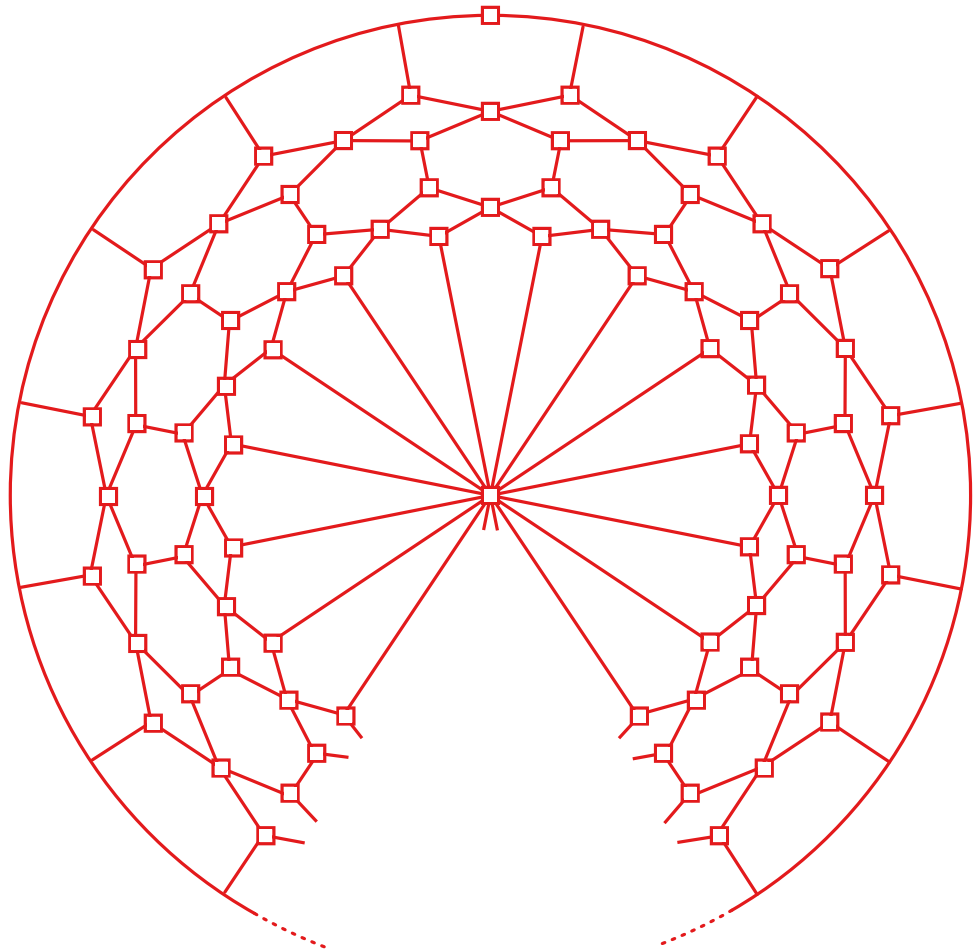

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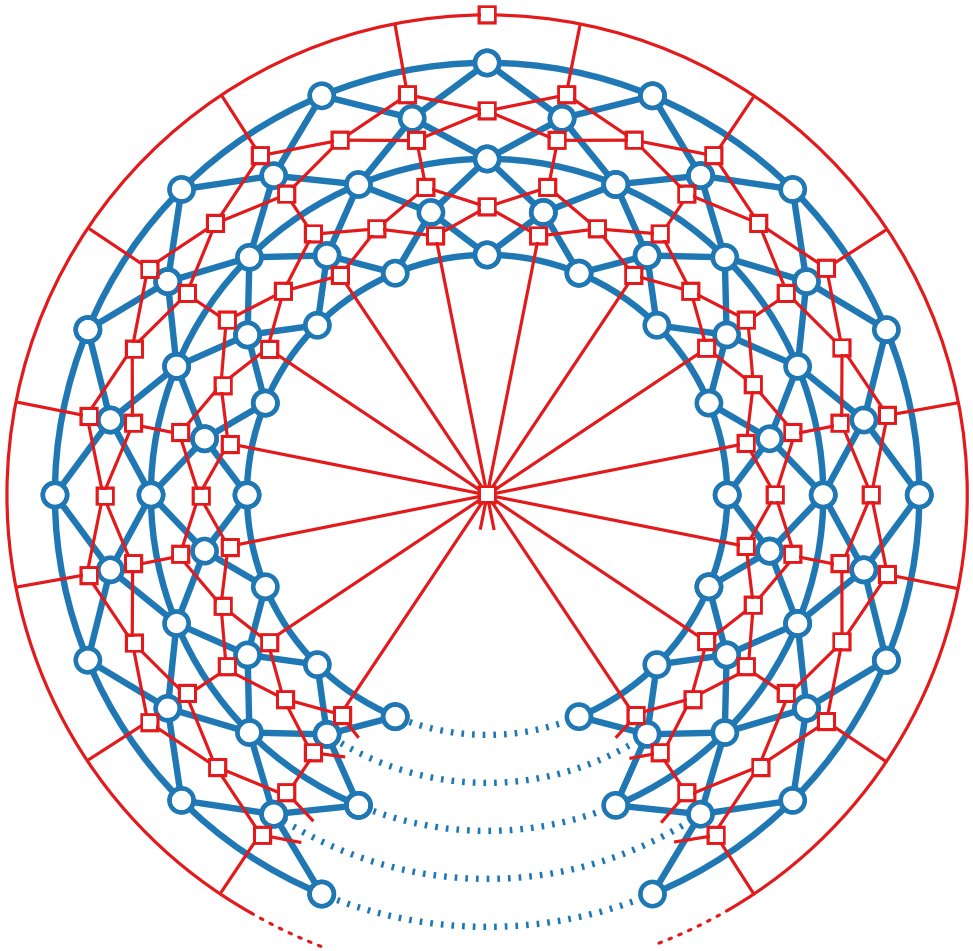

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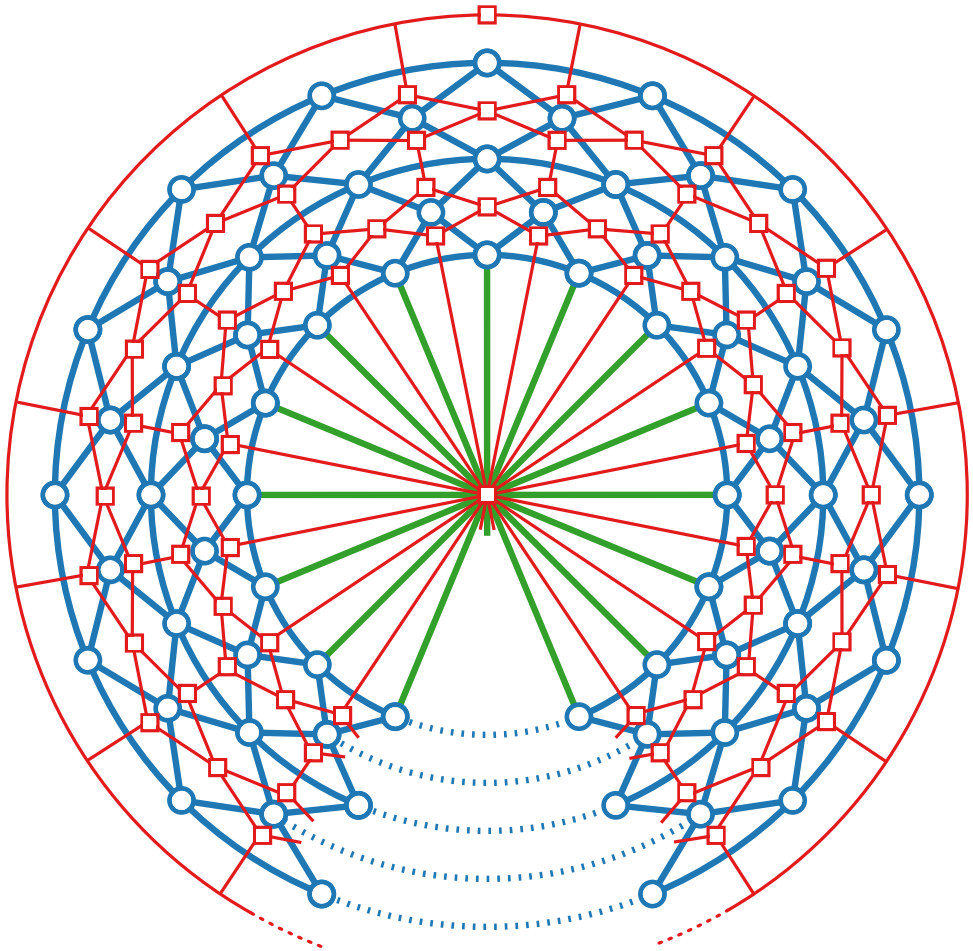

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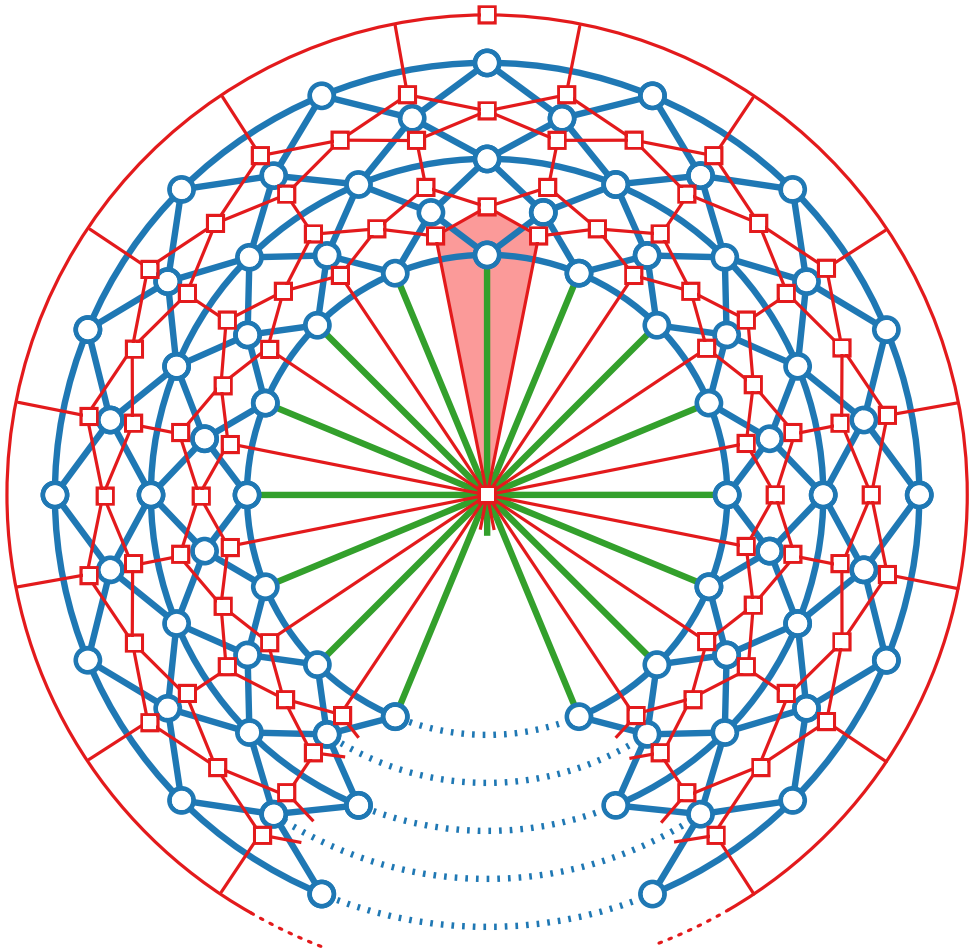

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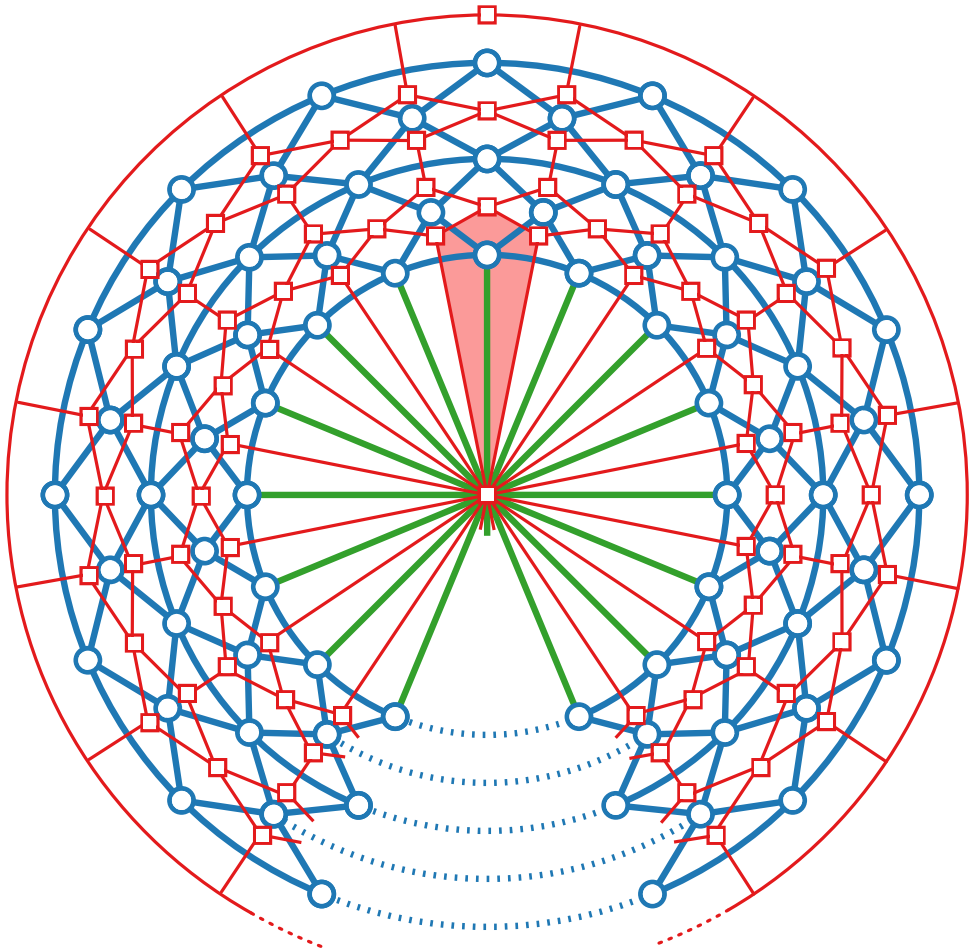

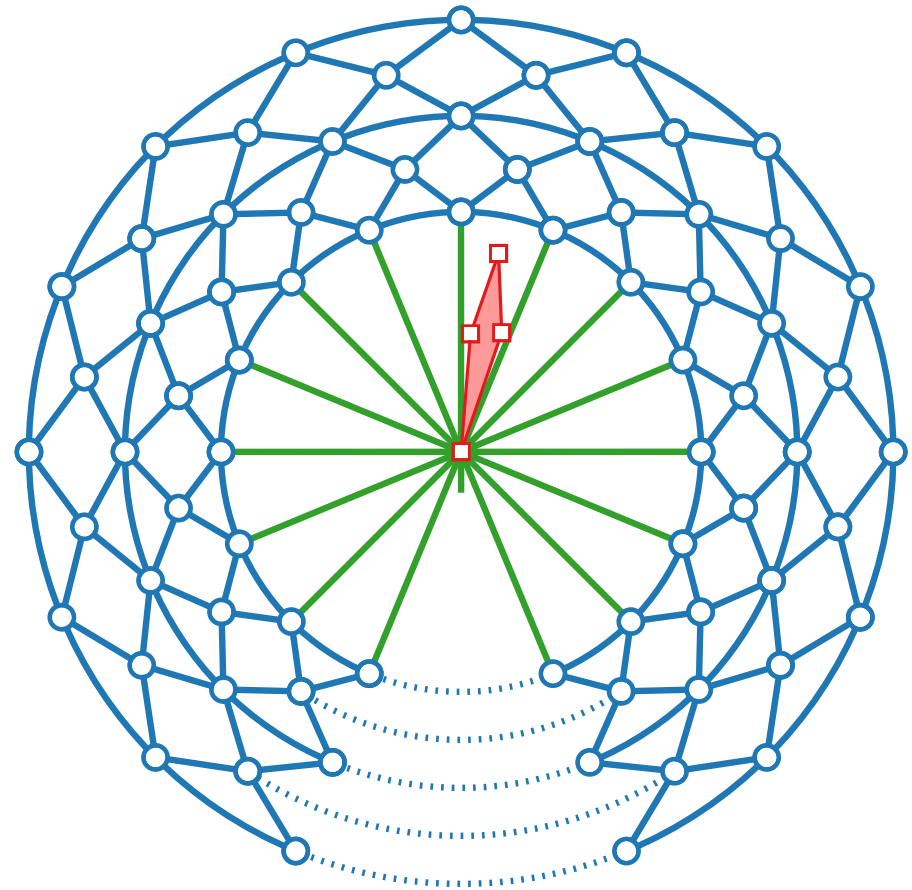

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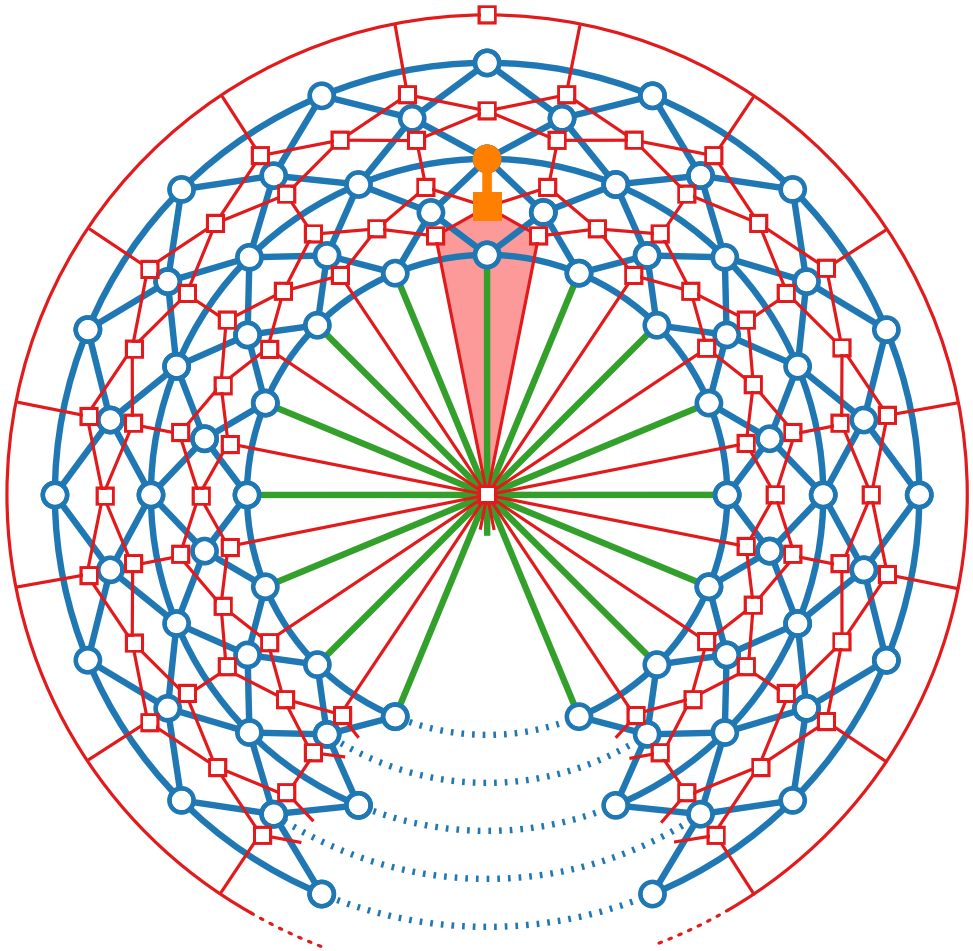

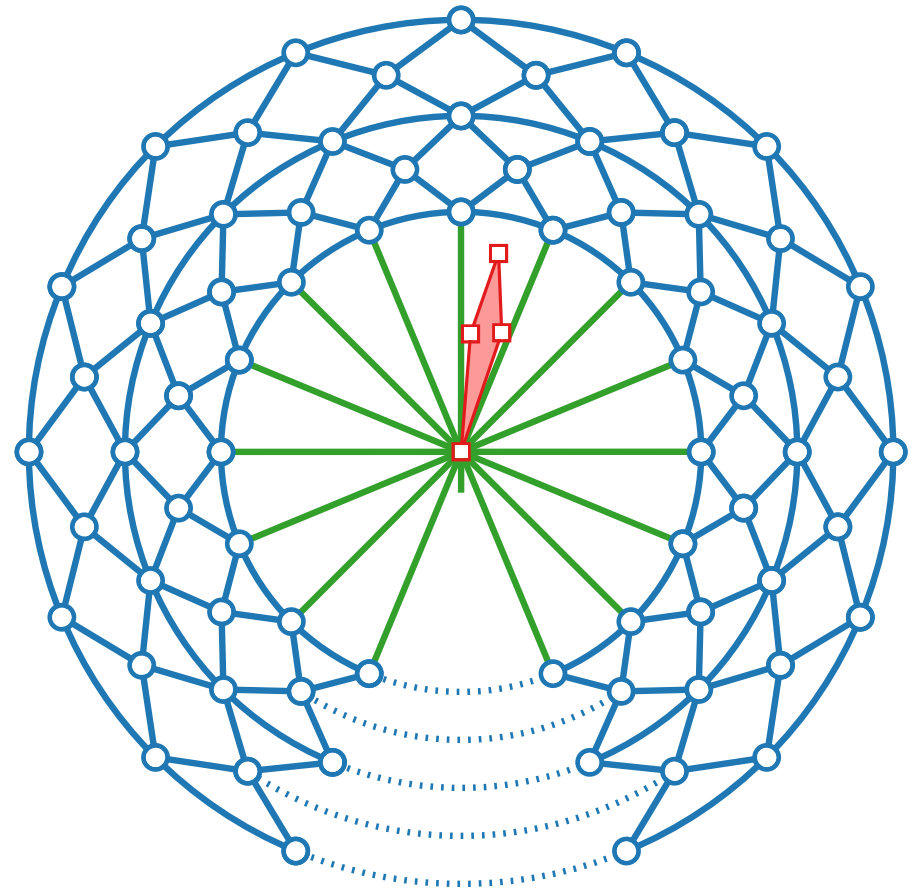

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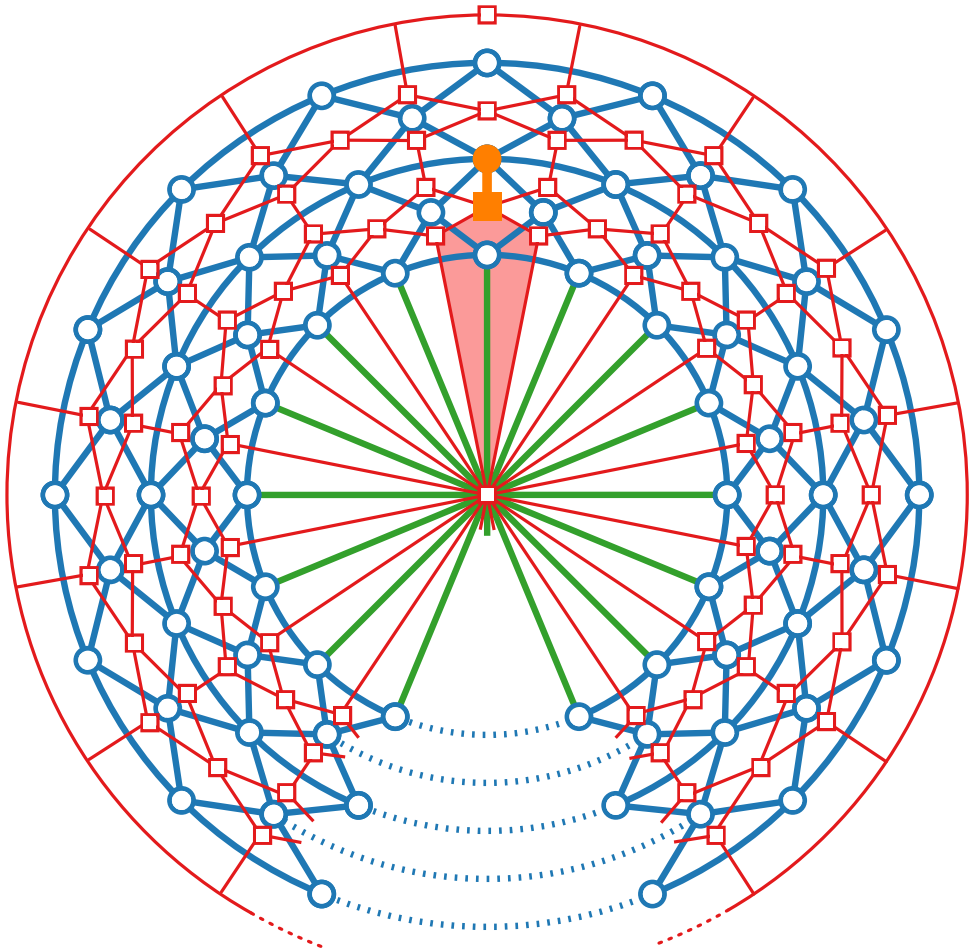

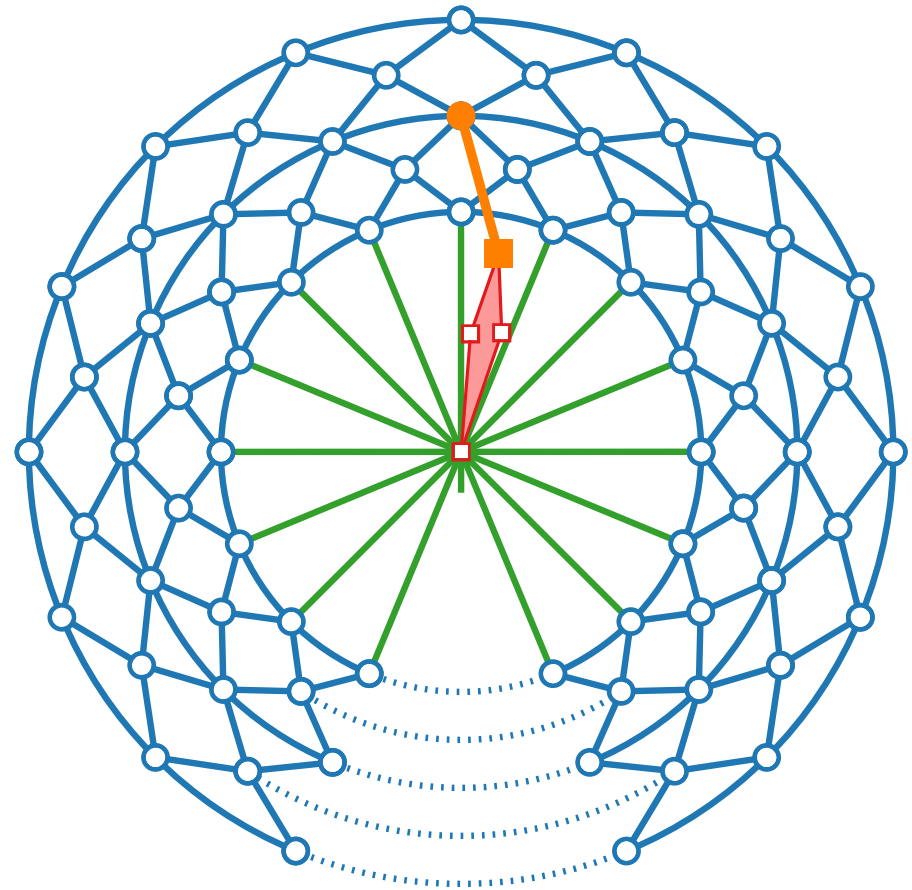

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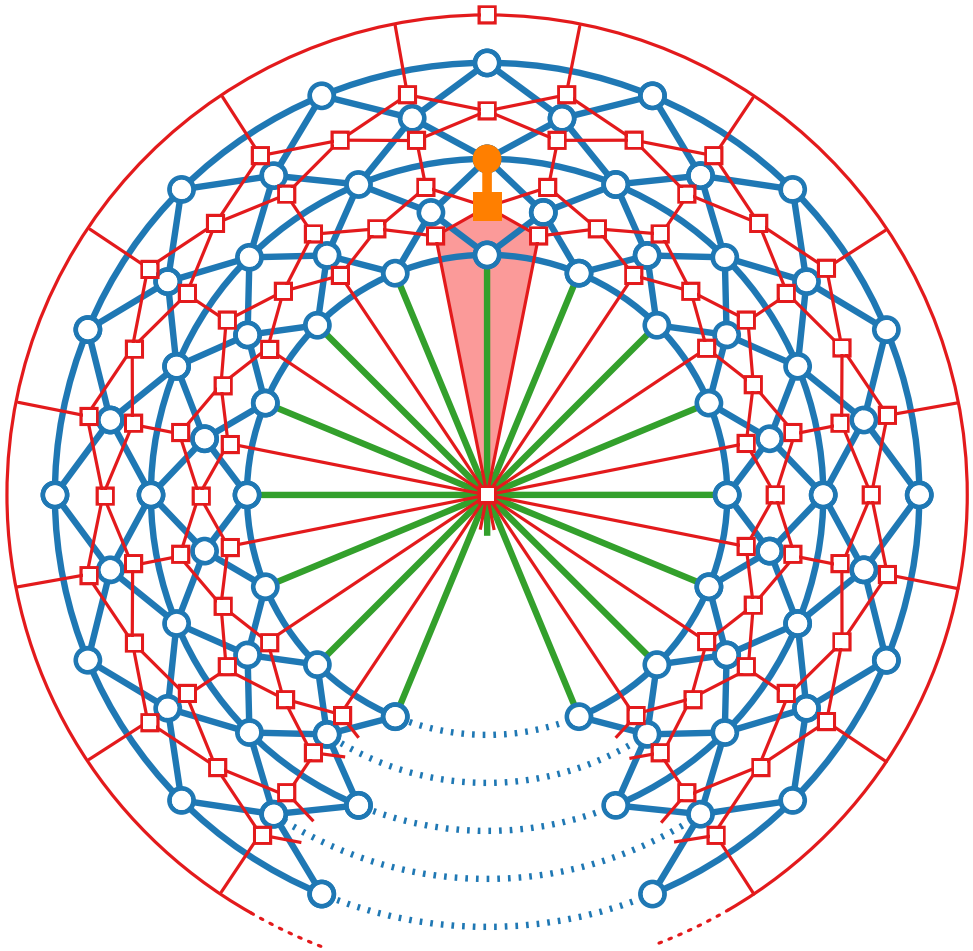

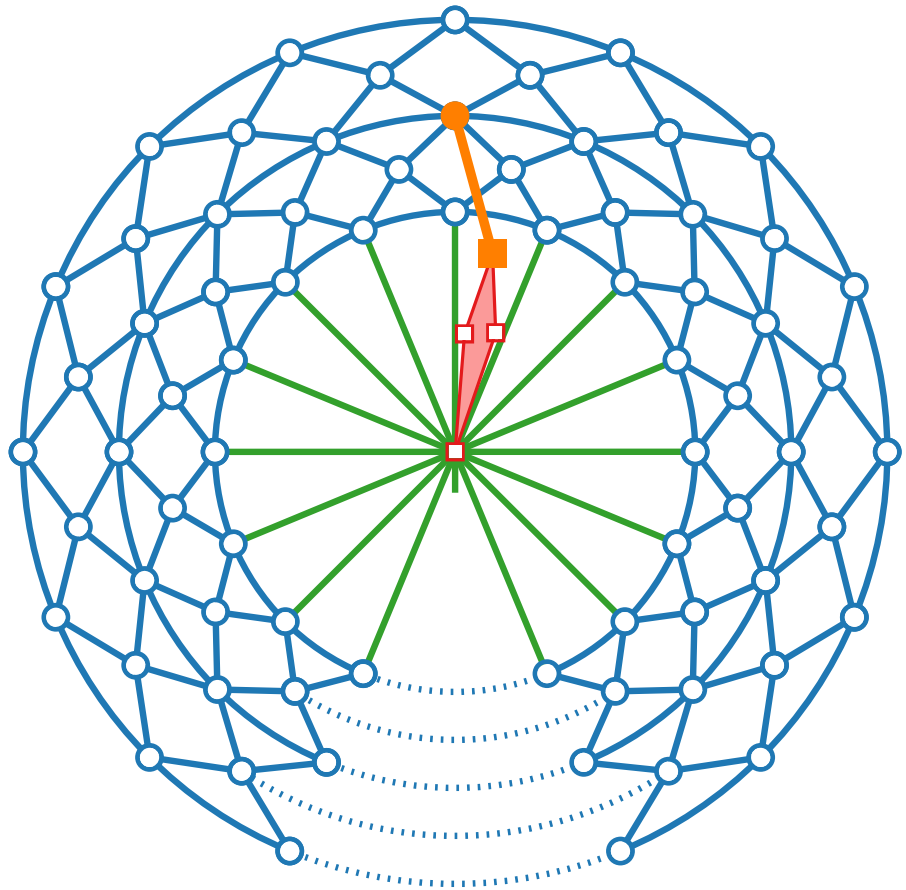

not 1-planar, 2 crossings

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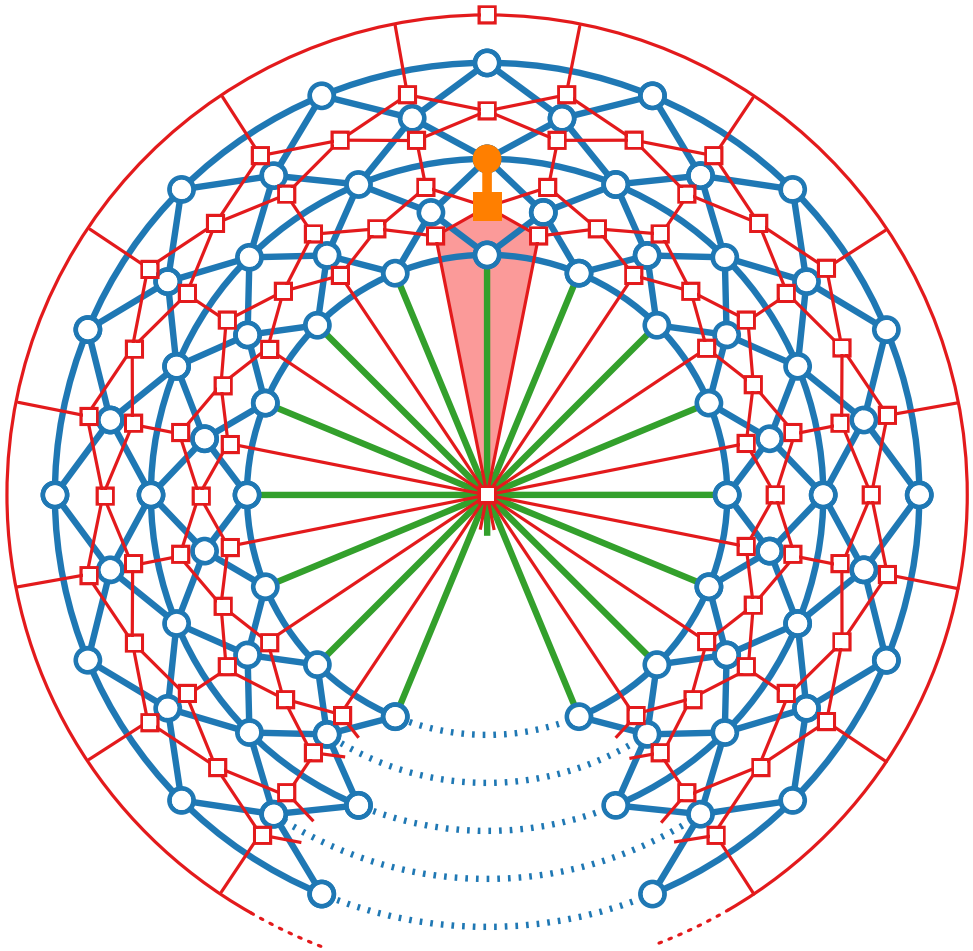
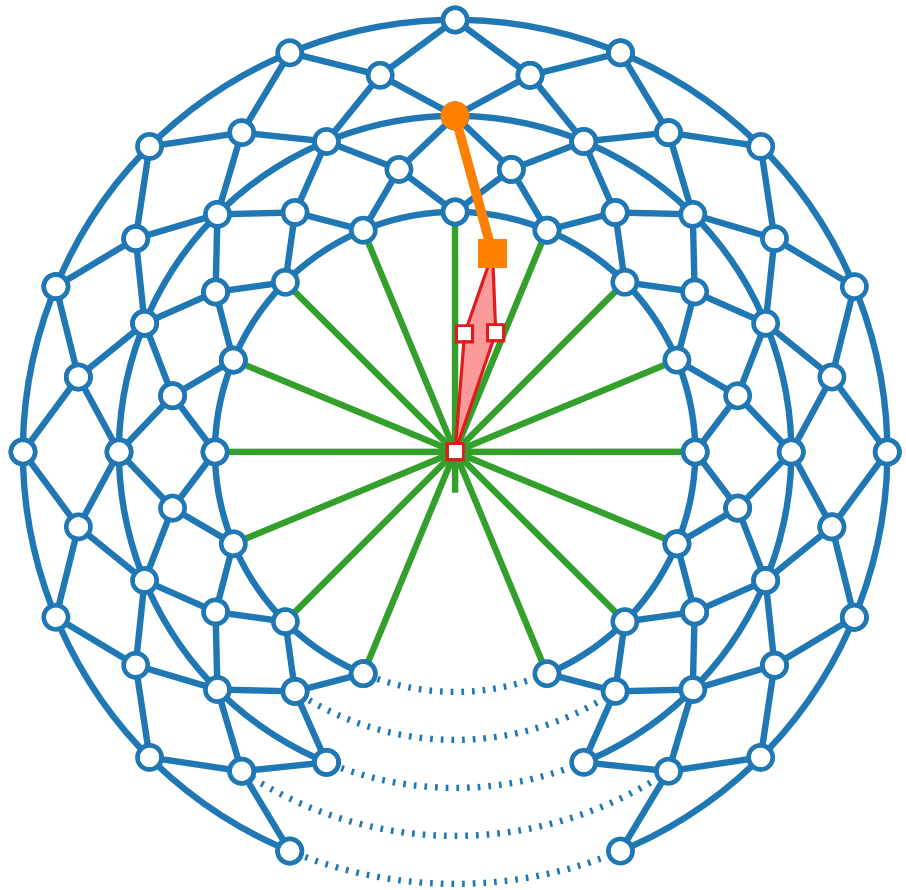
at most $4n - 8$ edges, $n - 2$ crossings $\Rightarrow \varrho_{1\text{-pl}} \leq n/2 - 1$

Graph G Dual G^* Fixing edges Special edge



1-planar, $n - 2$ crossings



not 1-planar, 2 crossings

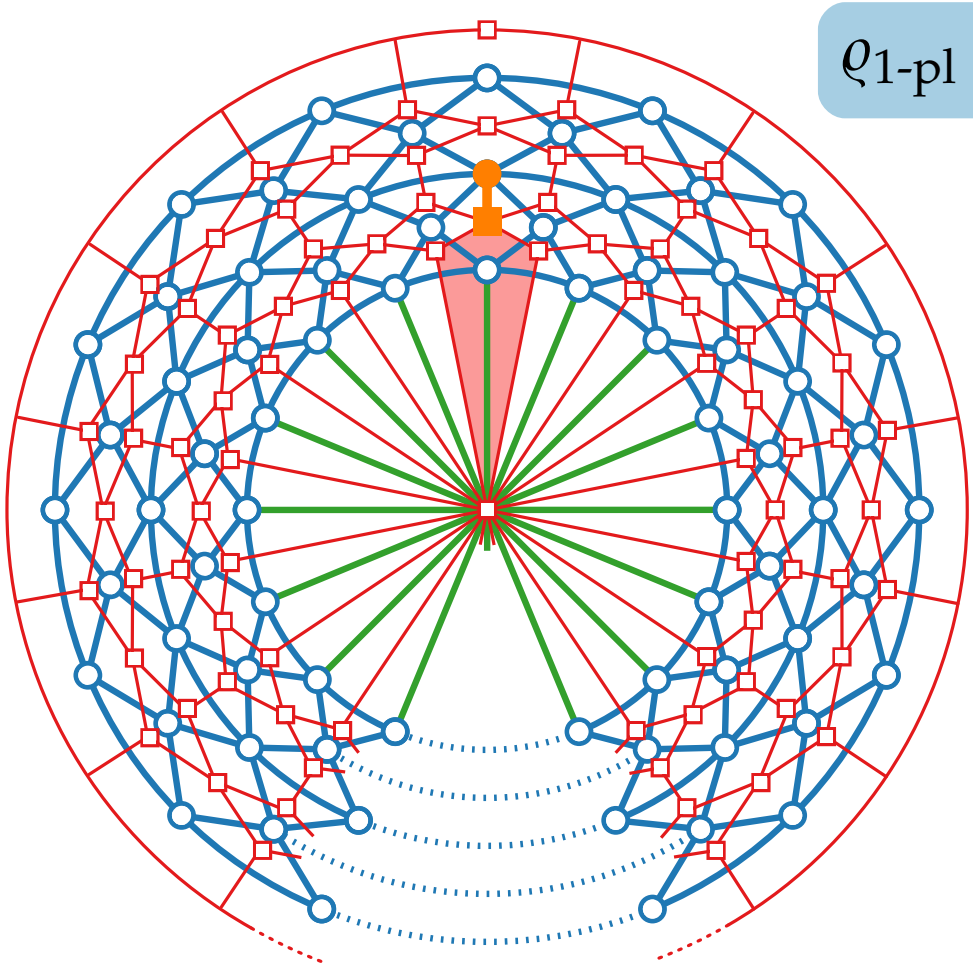
1-Planar Graphs

[Korzhik & Mohar '13]
This is the only 1-planar embedding of G

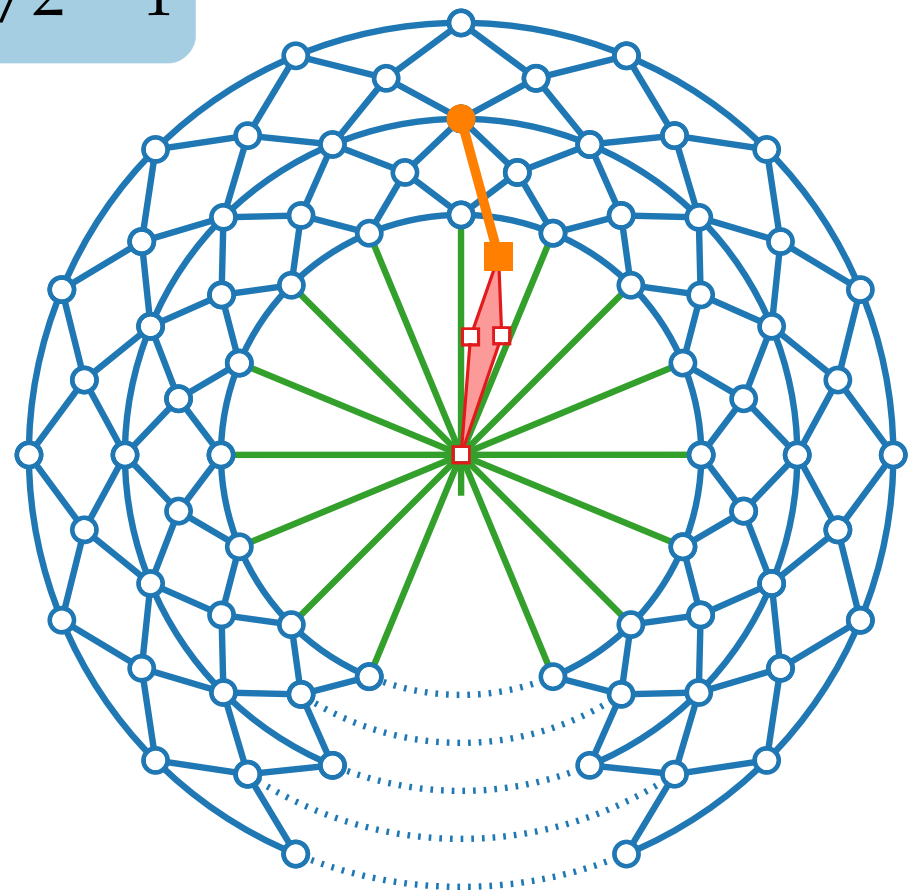
at most $4n - 8$ edges, $n - 2$ crossings $\Rightarrow \varrho_{1\text{-pl}} \leq n/2 - 1$

Graph G Dual G^* Fixing edges Special edge

$\varrho_{1\text{-pl}} = n/2 - 1$



1-planar, $n - 2$ crossings



not 1-planar, 2 crossings

Fan-Planar Graphs

at most $5n - 10$ edges

Fan-Planar Graphs

at most $5n - 10$ edges $\Rightarrow O(n^2)$ crossings

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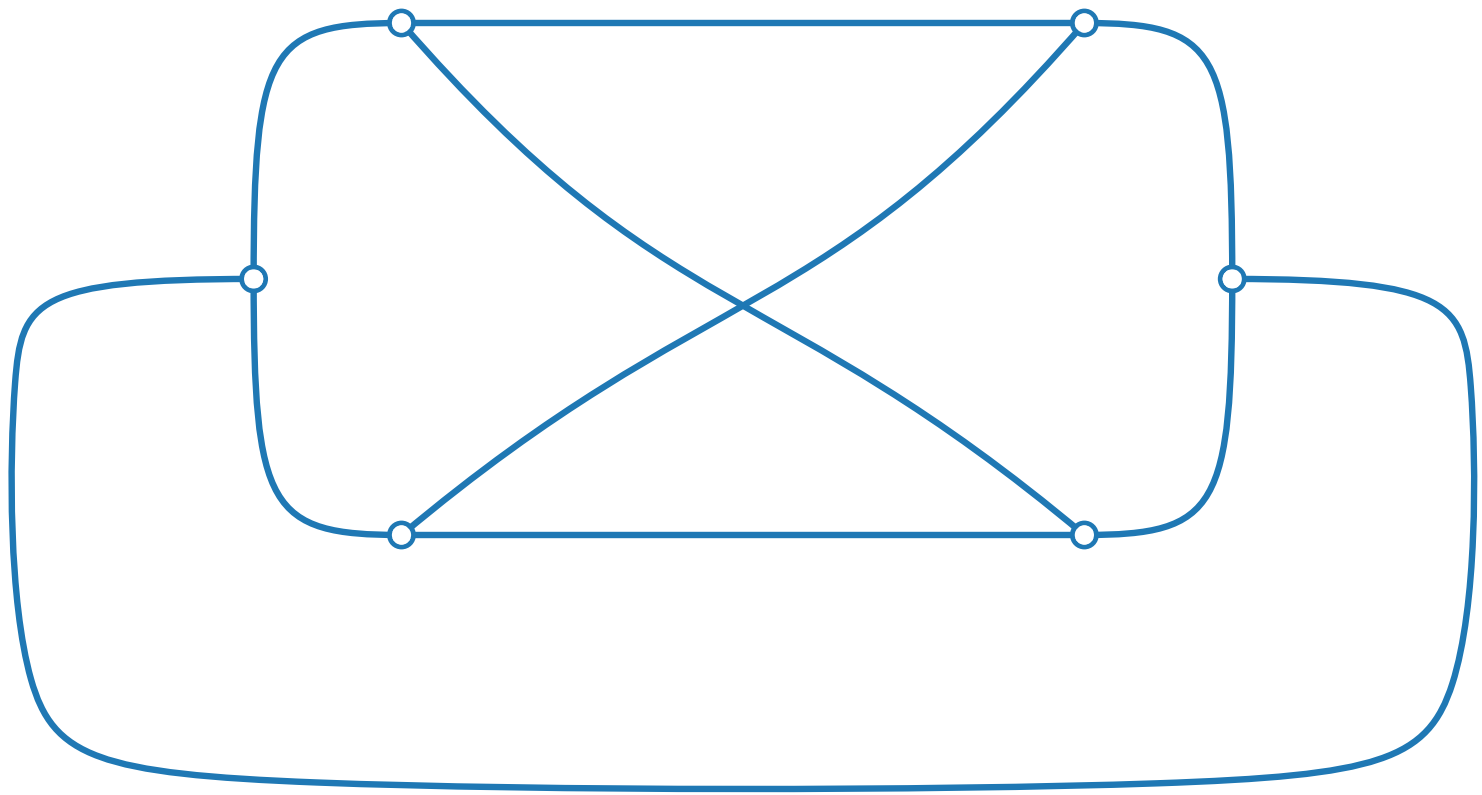
$K_{3,3}$



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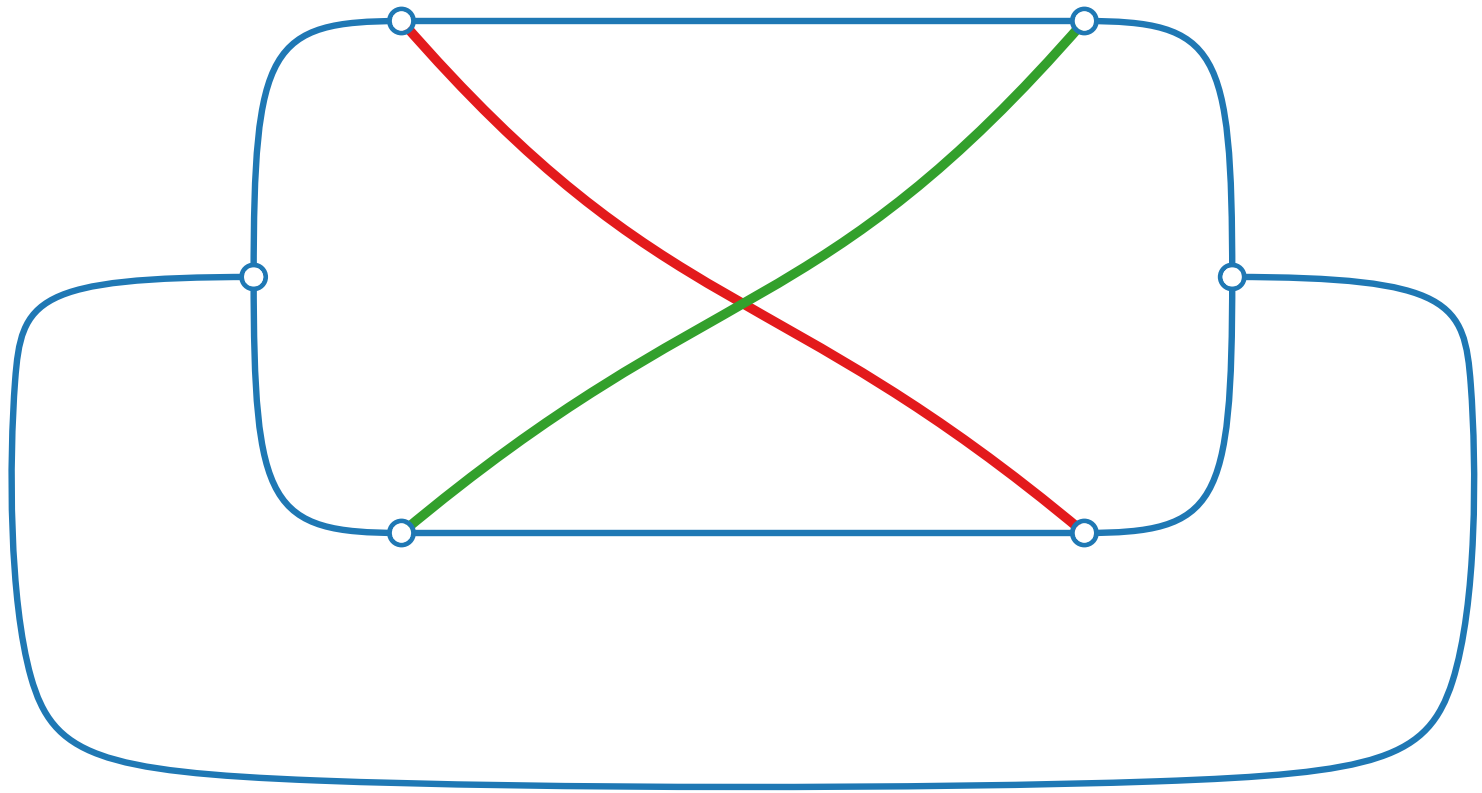

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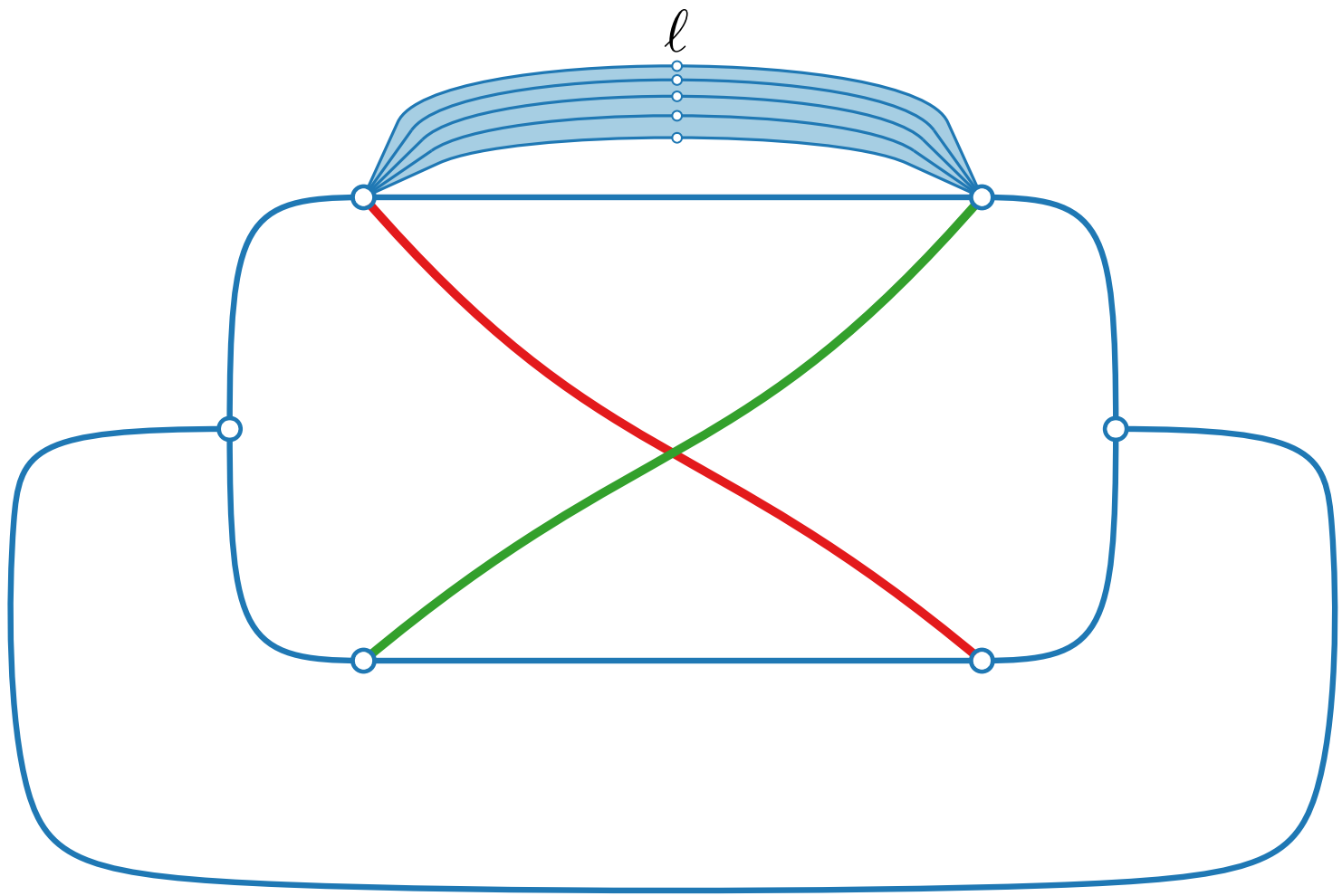

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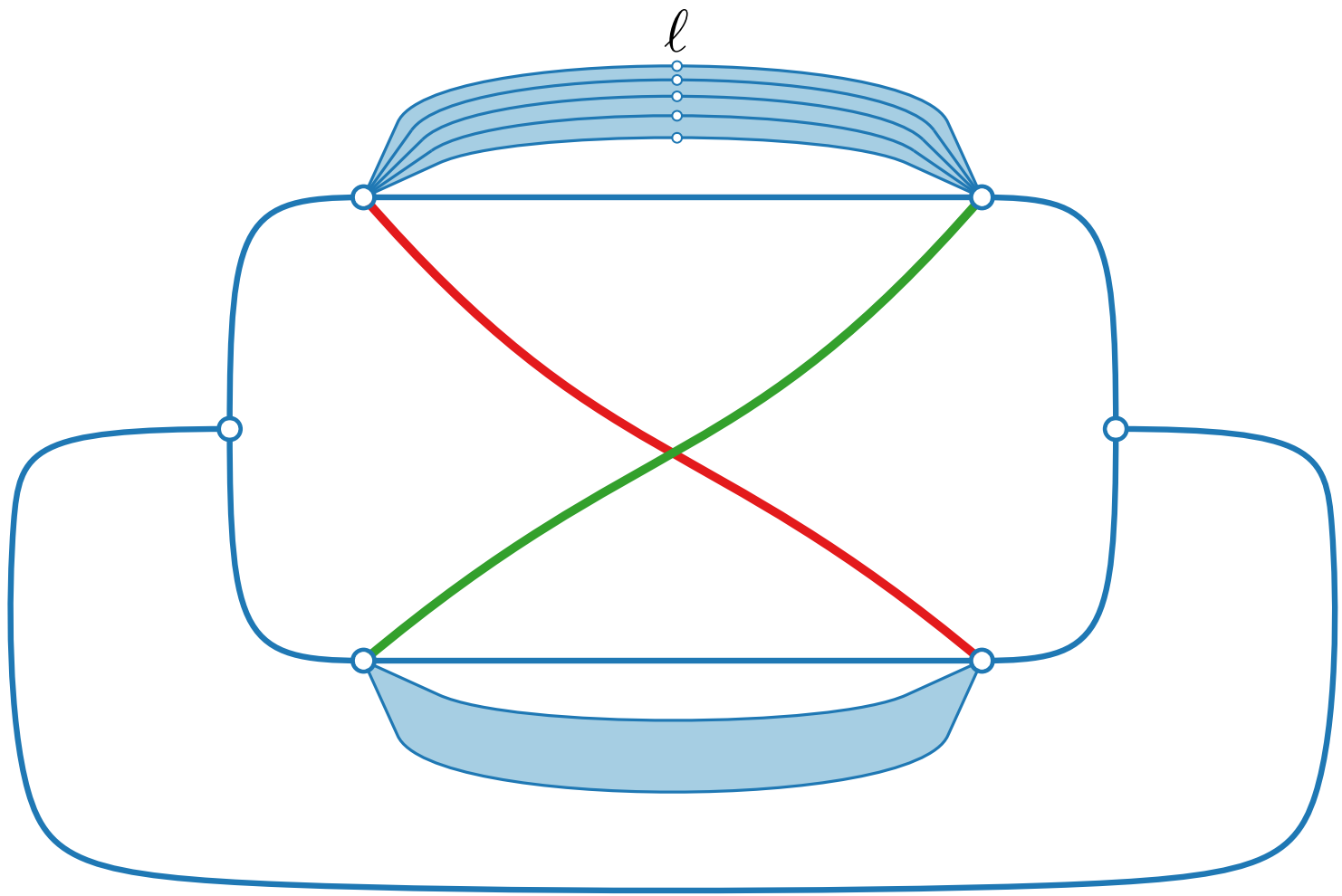

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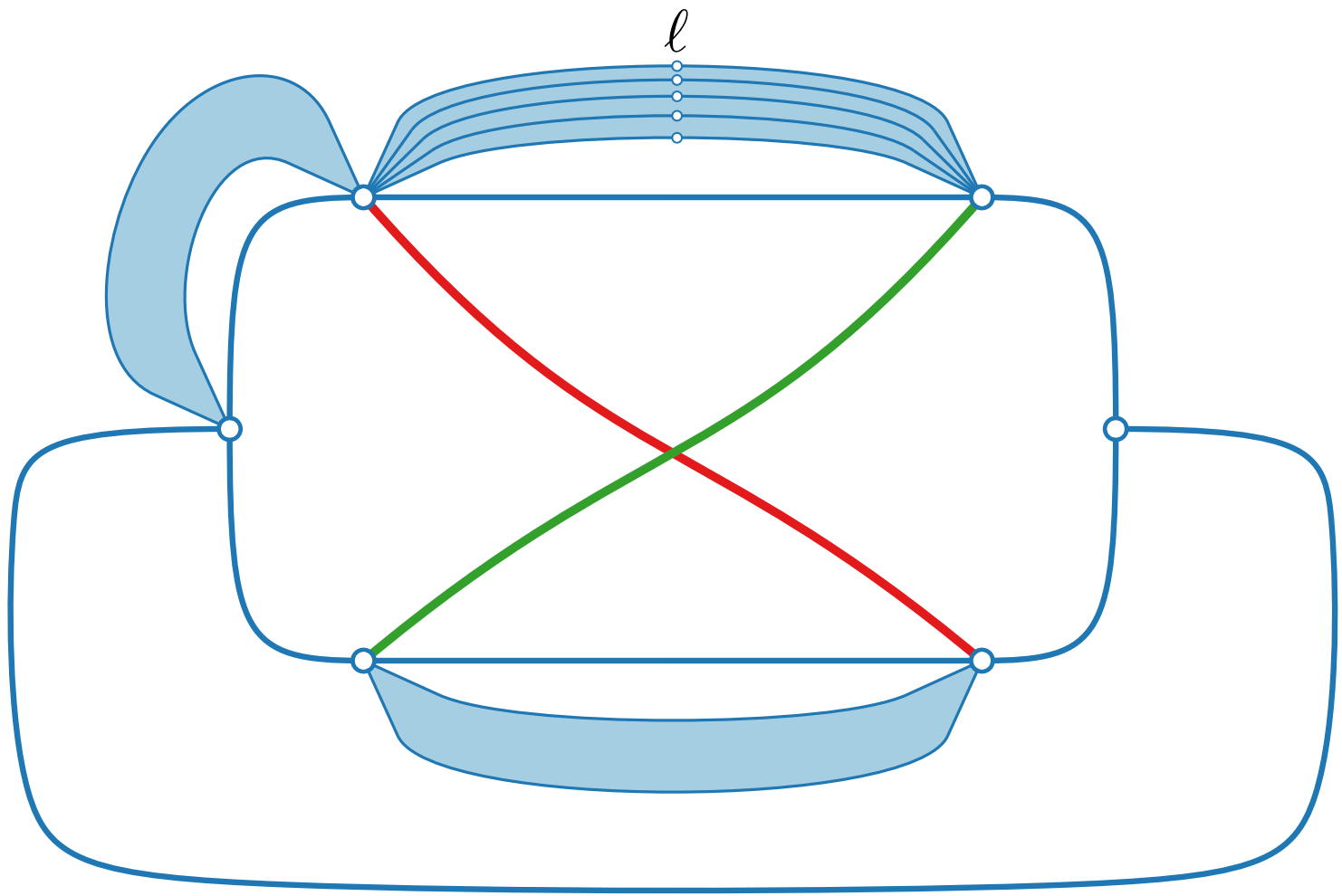

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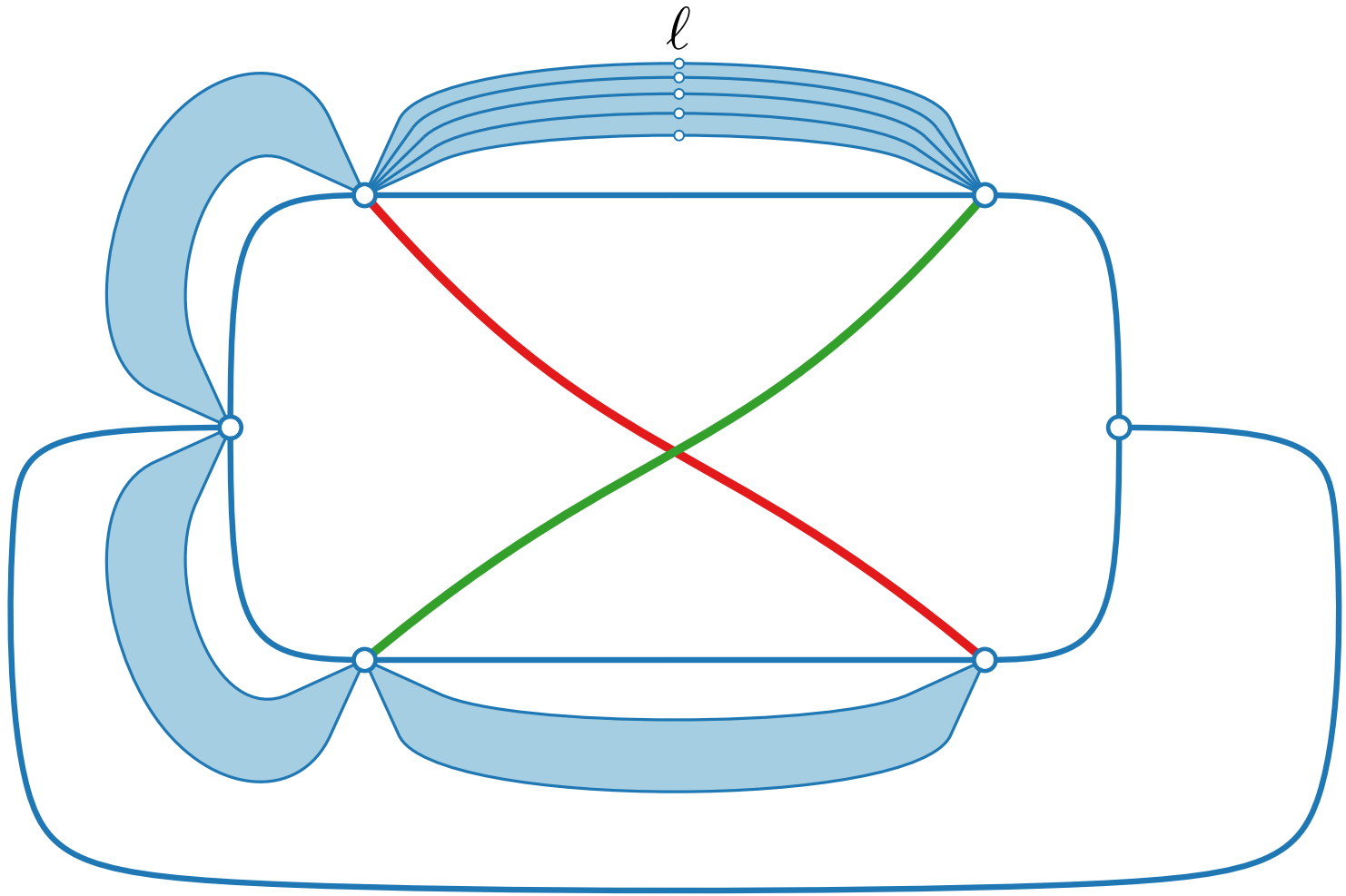

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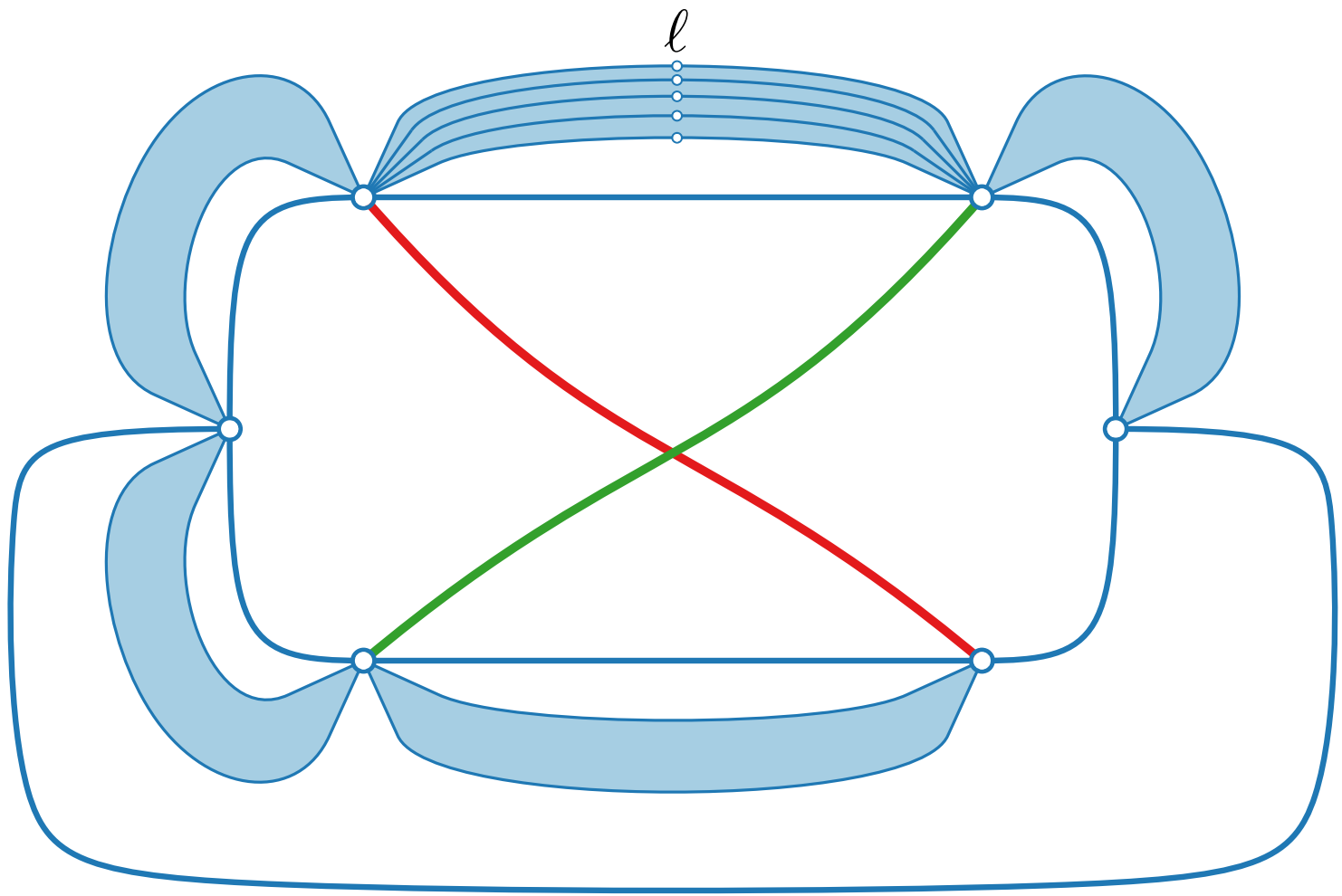

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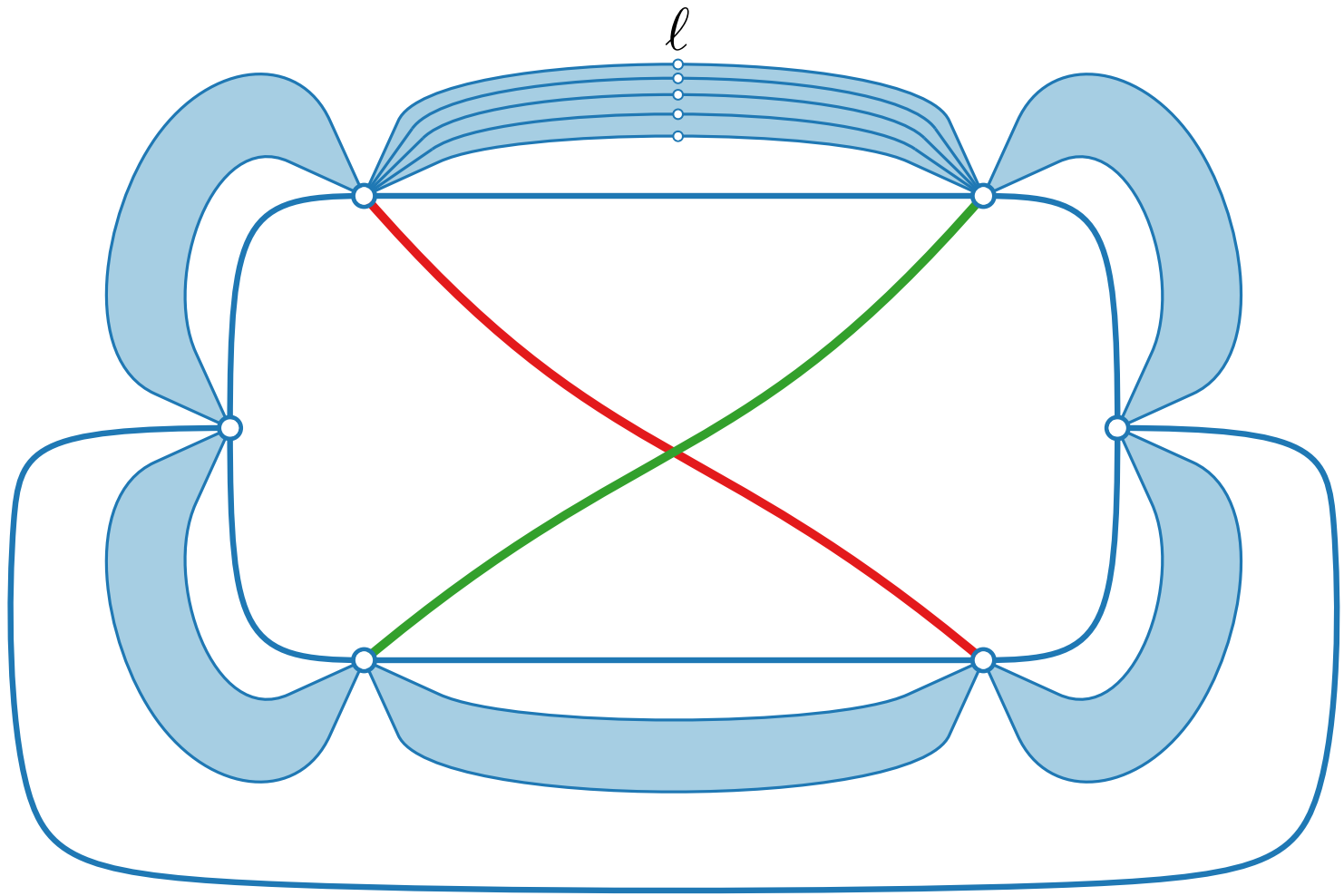

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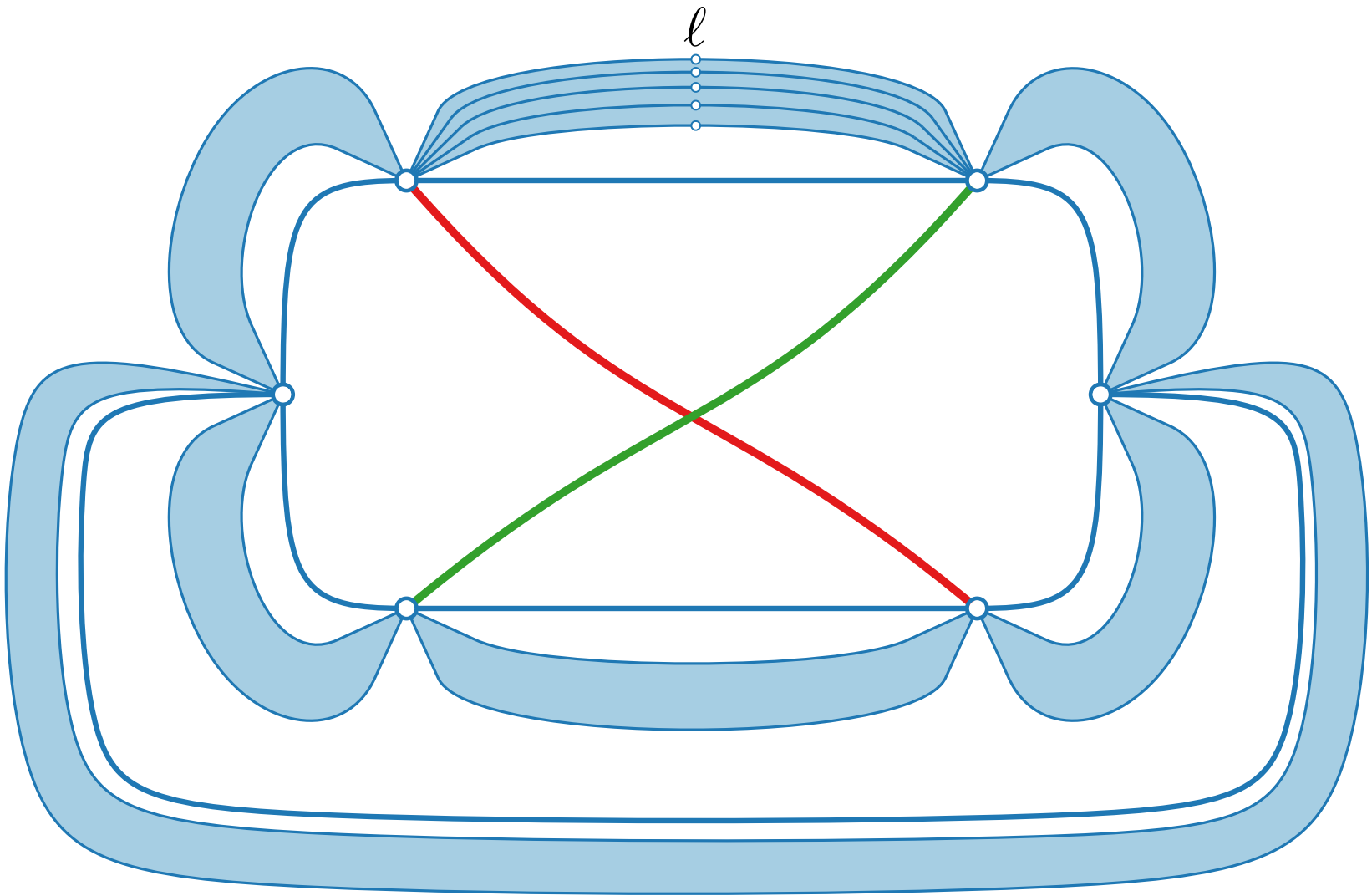

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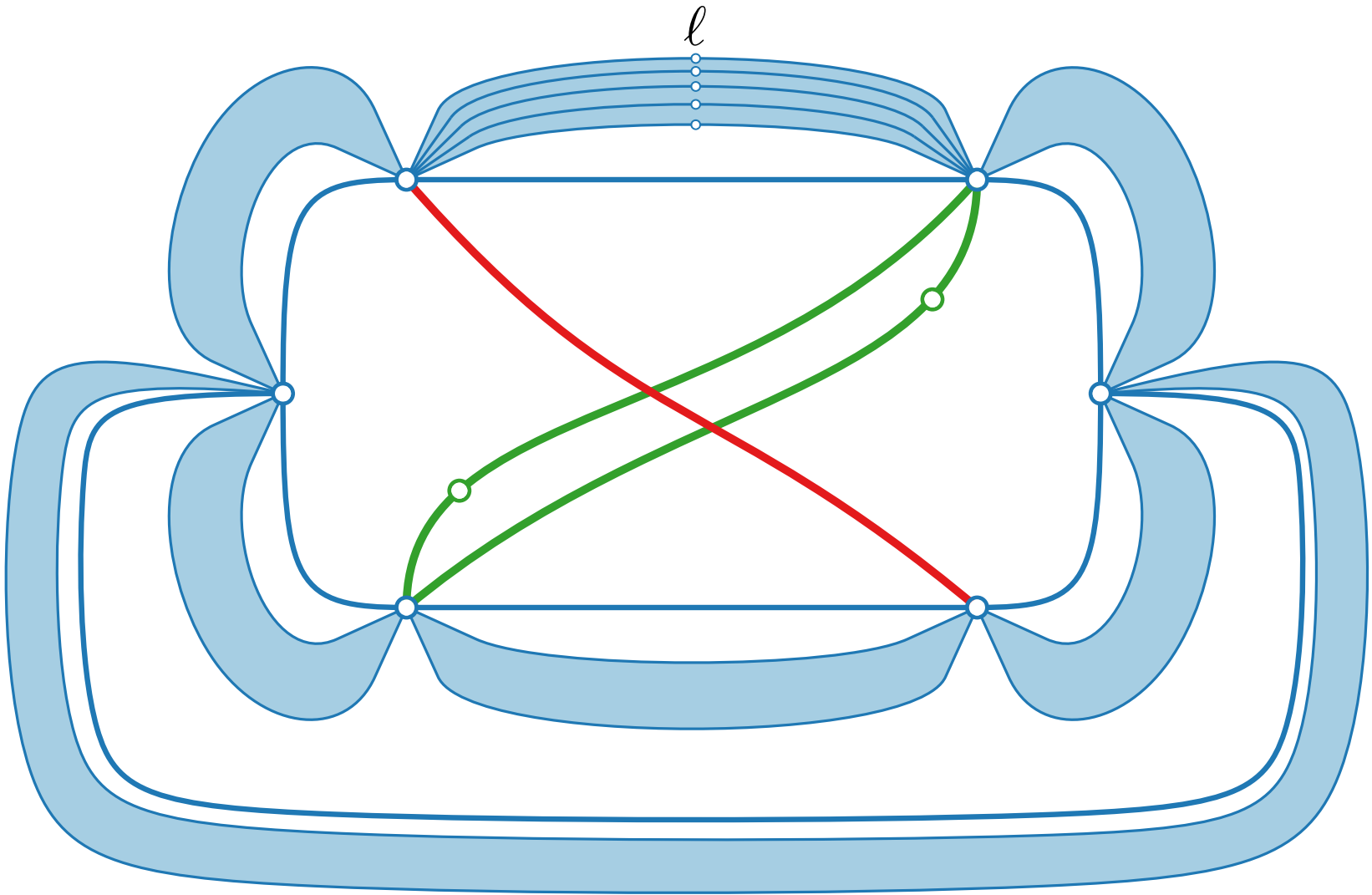

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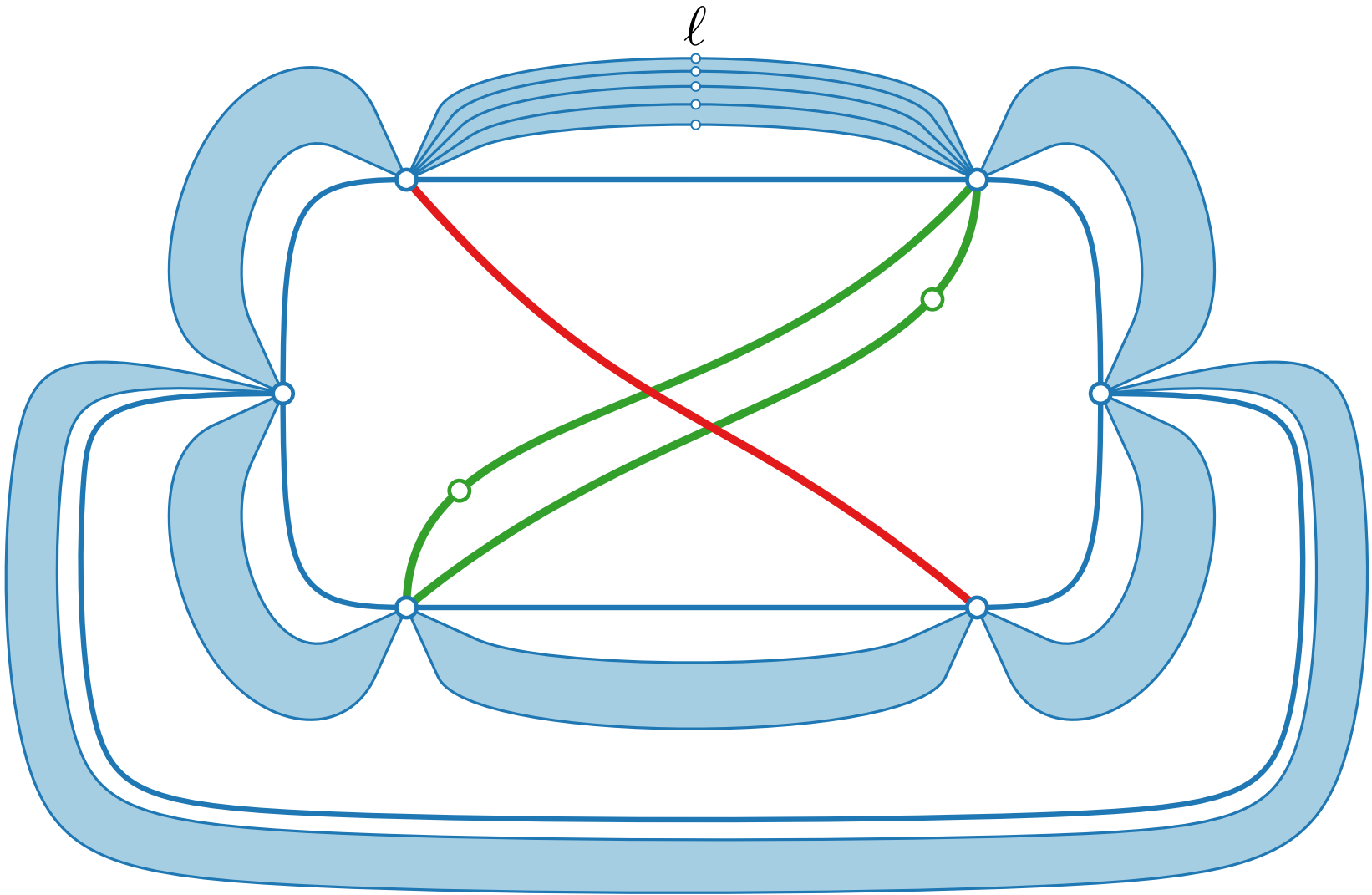
not fan-planar, 2 crossings



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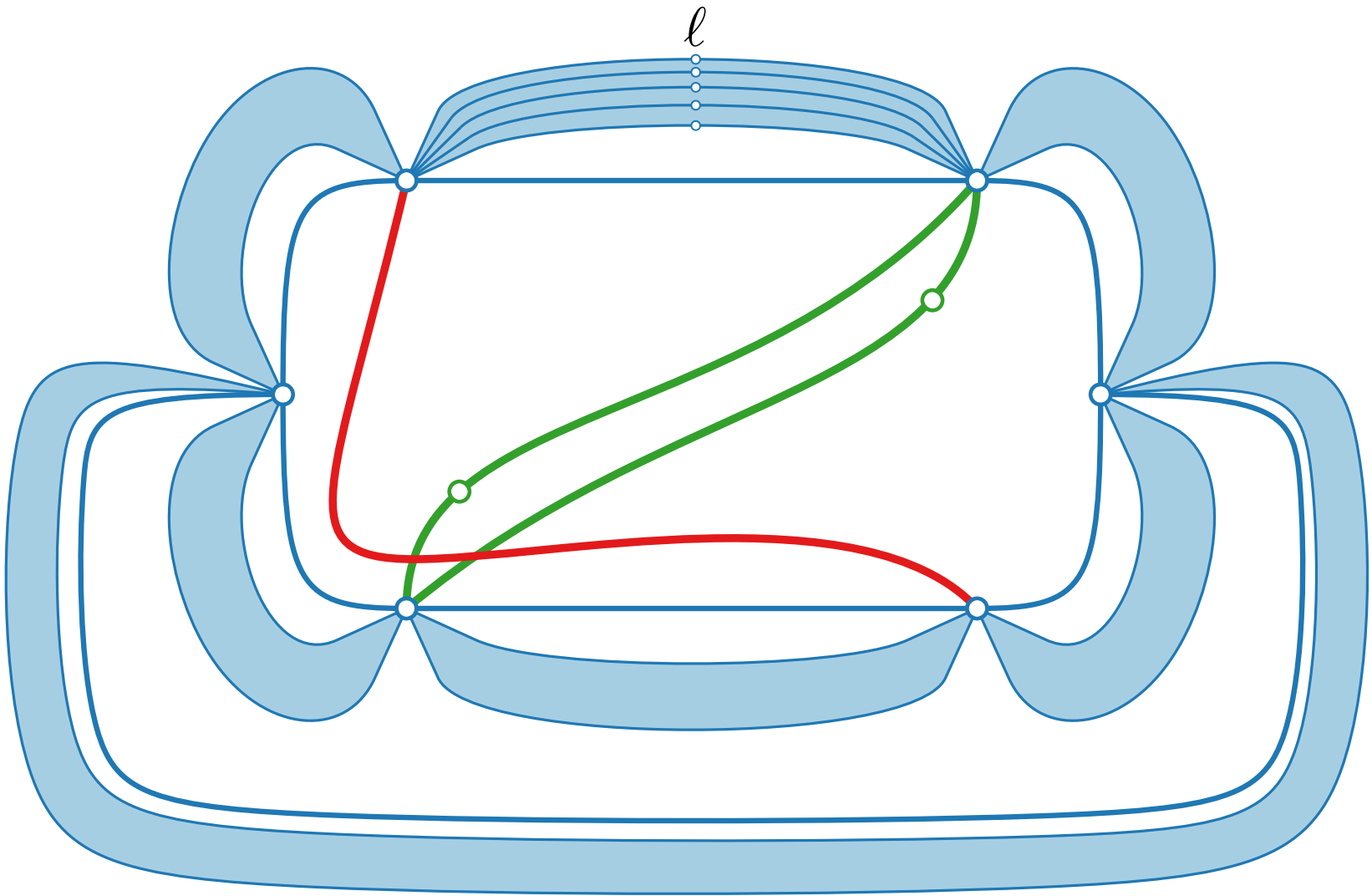

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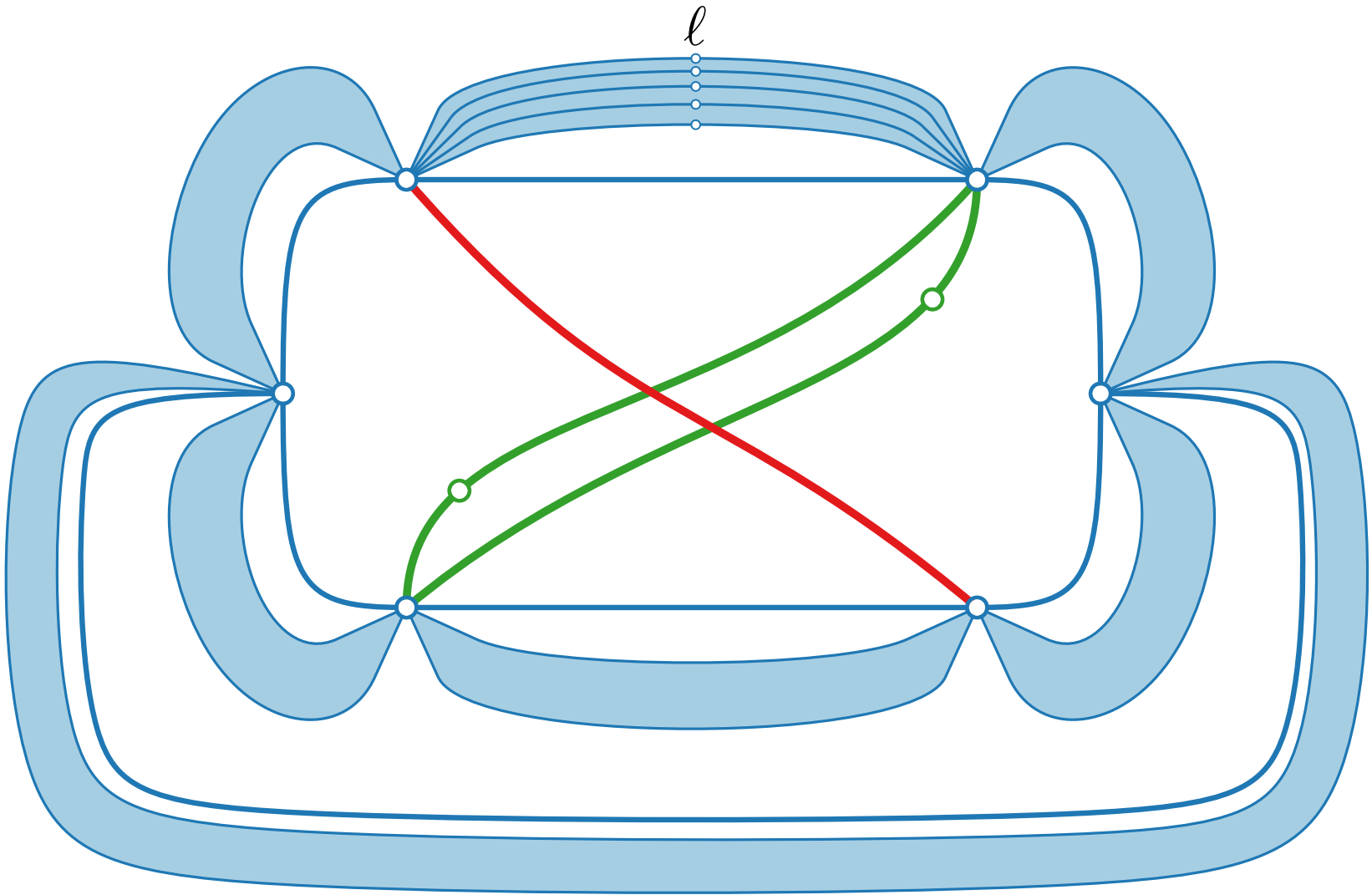

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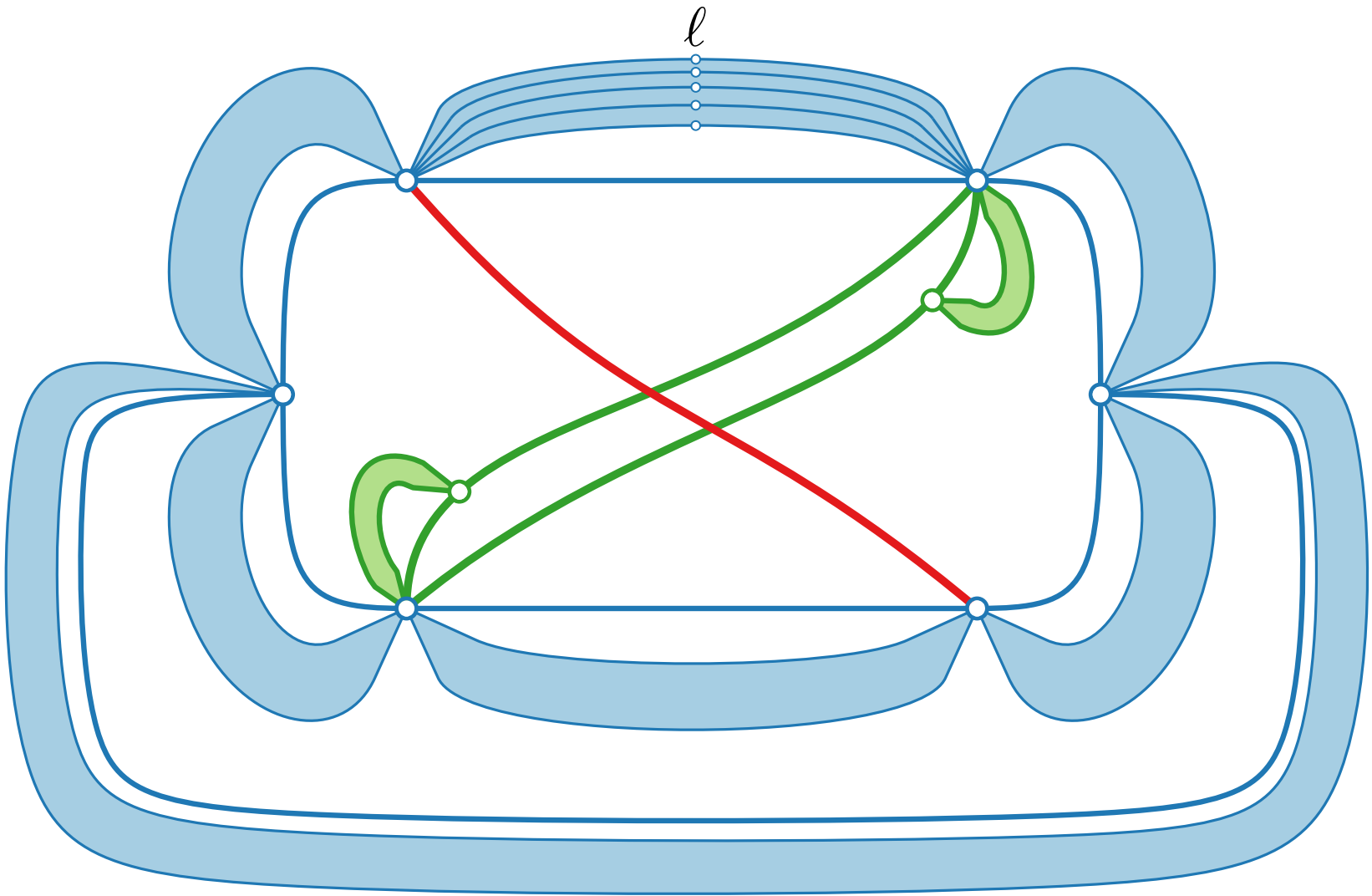

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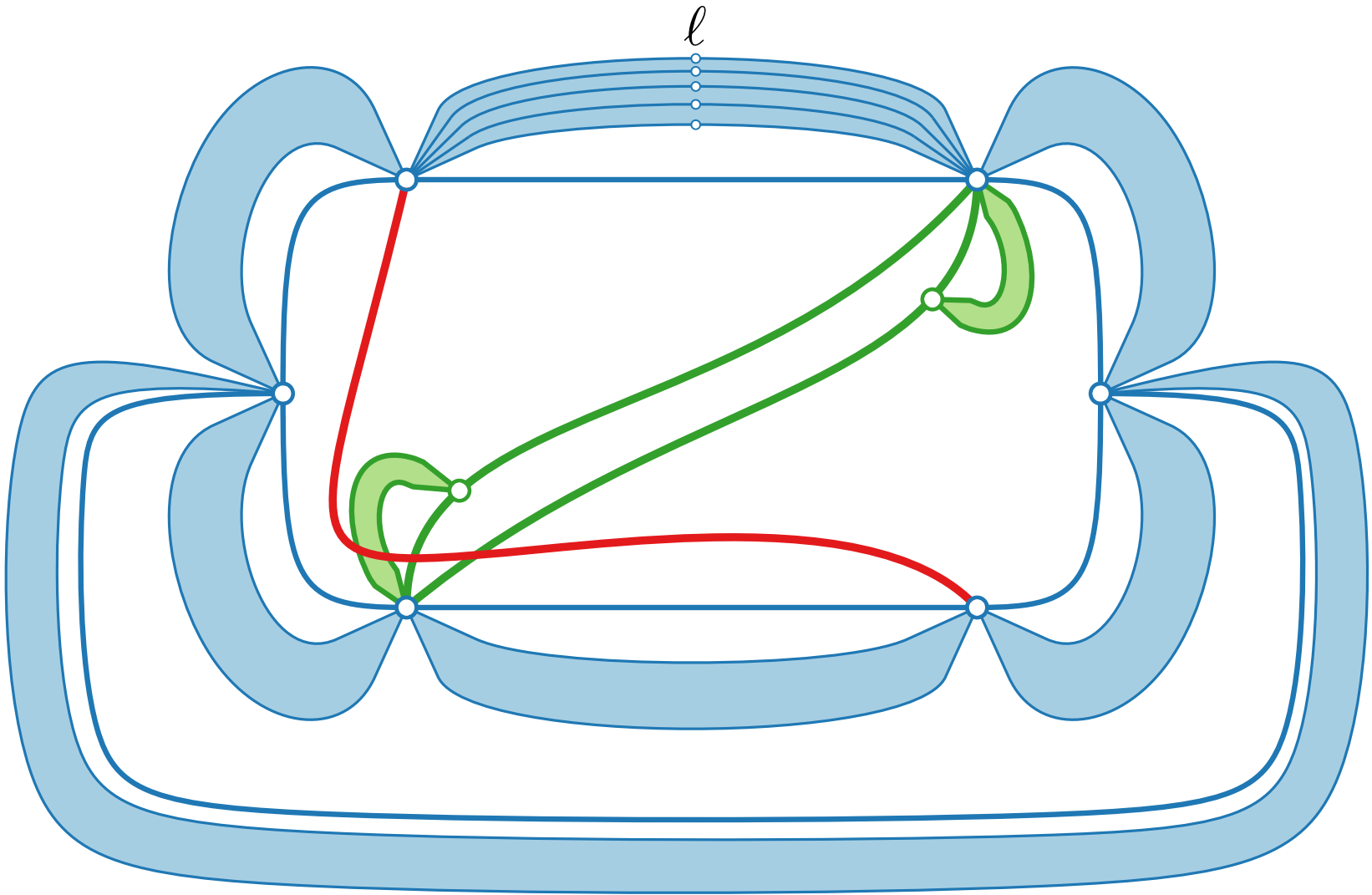
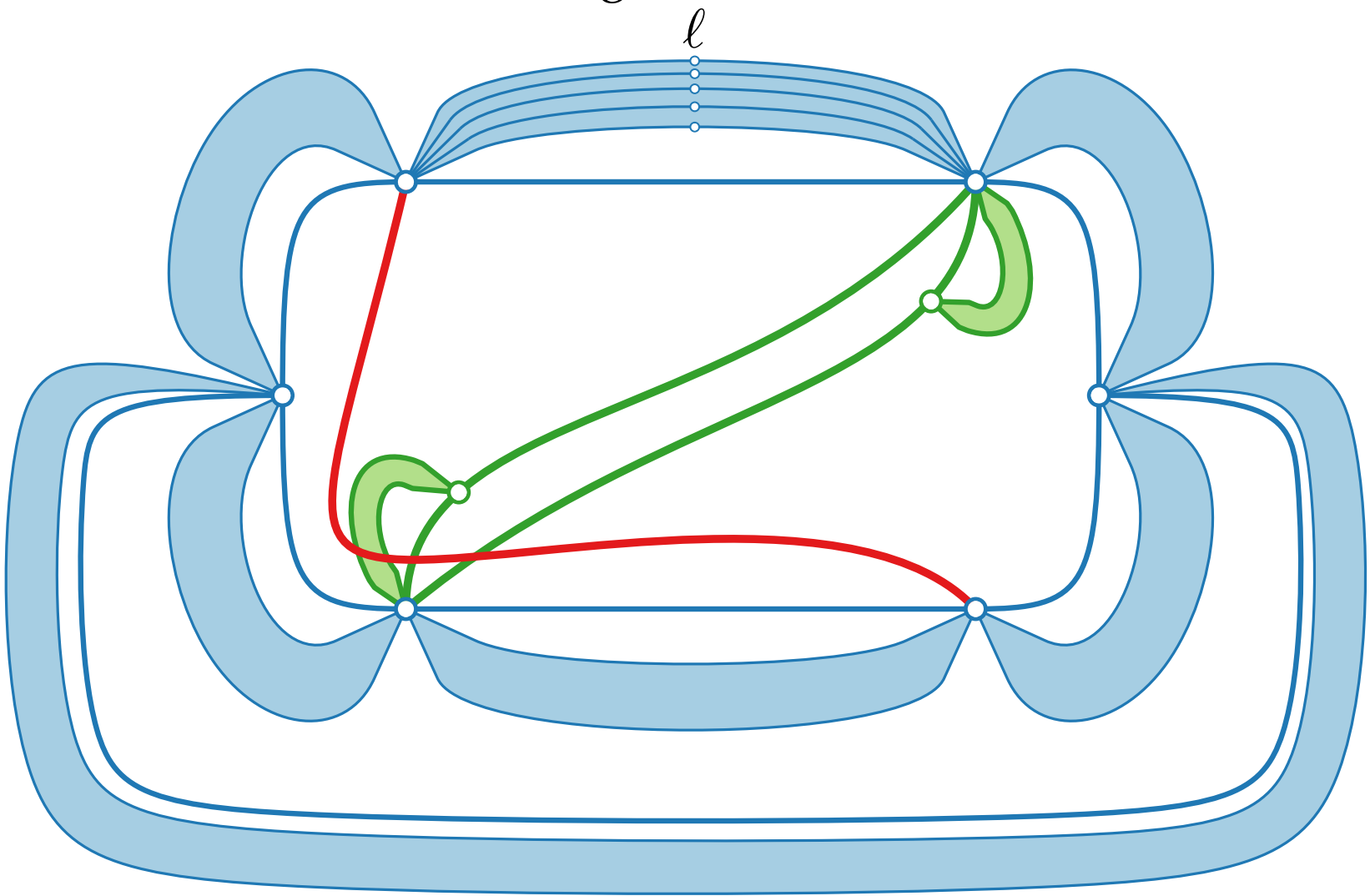


Fan-Planar Graphs

at most $5n - 10$ edges $\Rightarrow O(n^2)$ crossings $\Rightarrow \varrho_{\text{fan}} \in O(n^2)$

not fan-planar, 2 crossings

fan-planar, $\ell + 1 \approx n/12$ crossings



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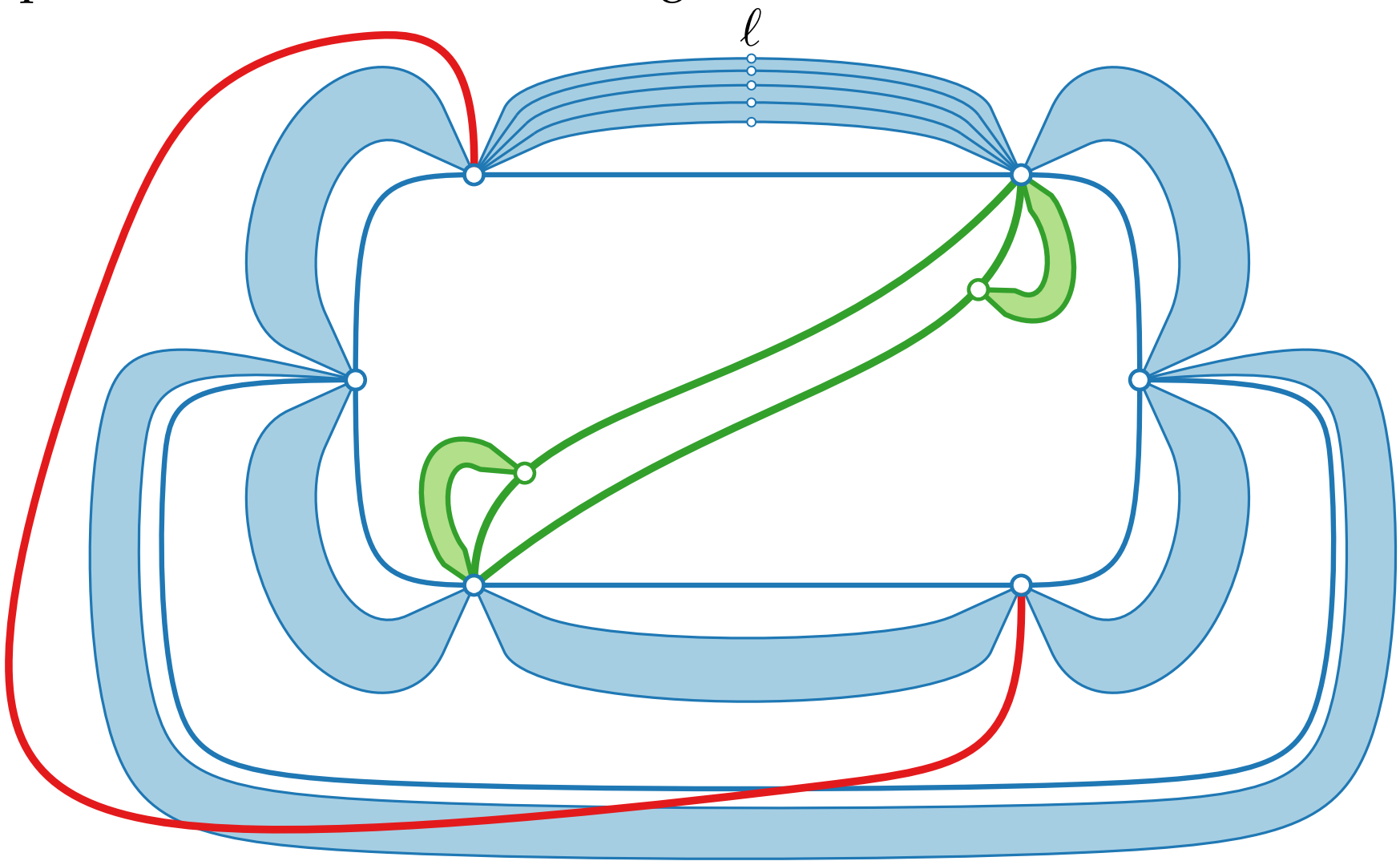

Fan-Planar Graphs

at most $5n - 10$ edges $\Rightarrow$ $O(n^2)$ crossings $\Rightarrow$ $\varrho_{\text{fan}} \in O(n^2)$

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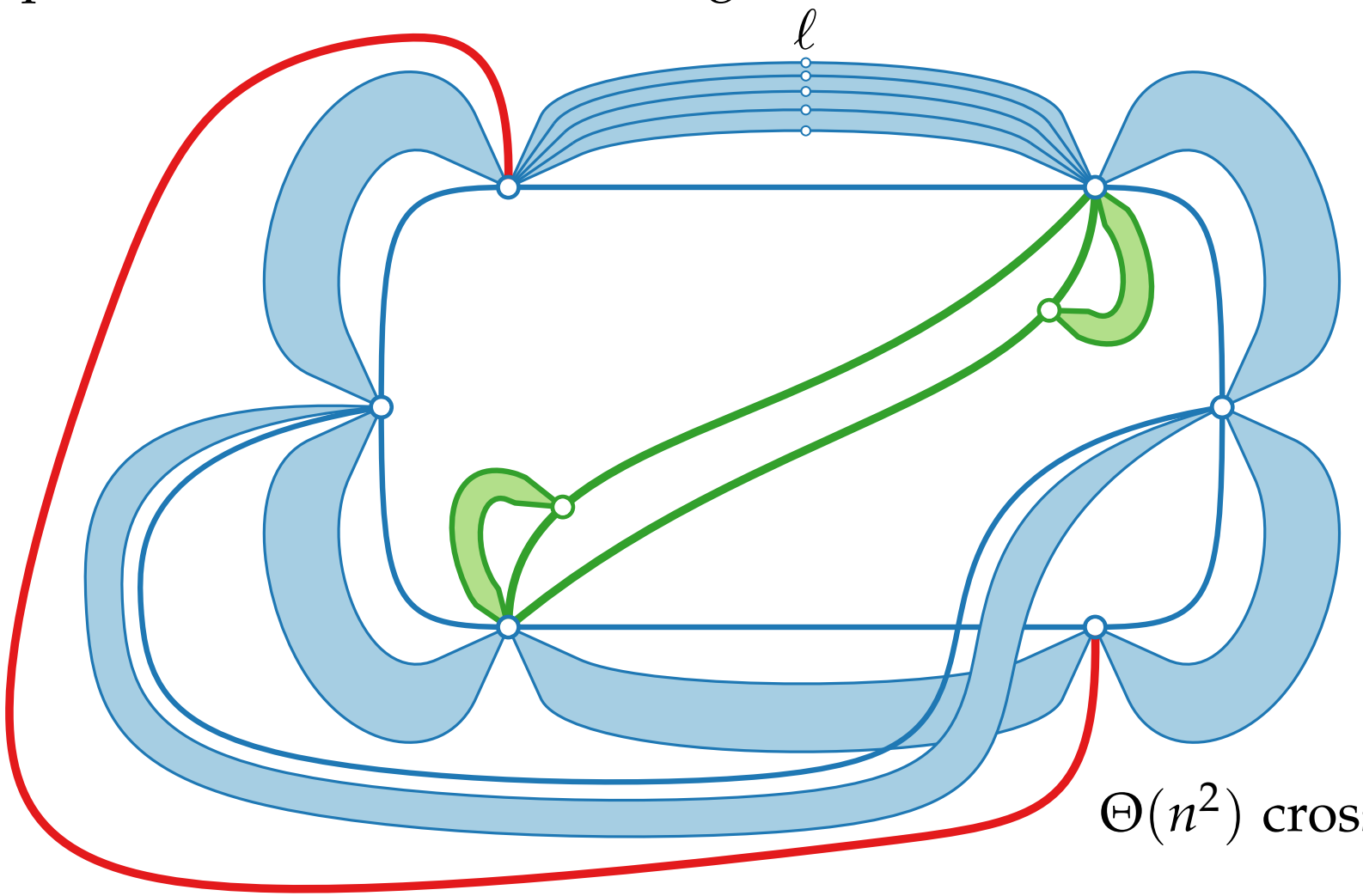

$\Theta(n^2)$ crossings

Fan-Planar Graphs

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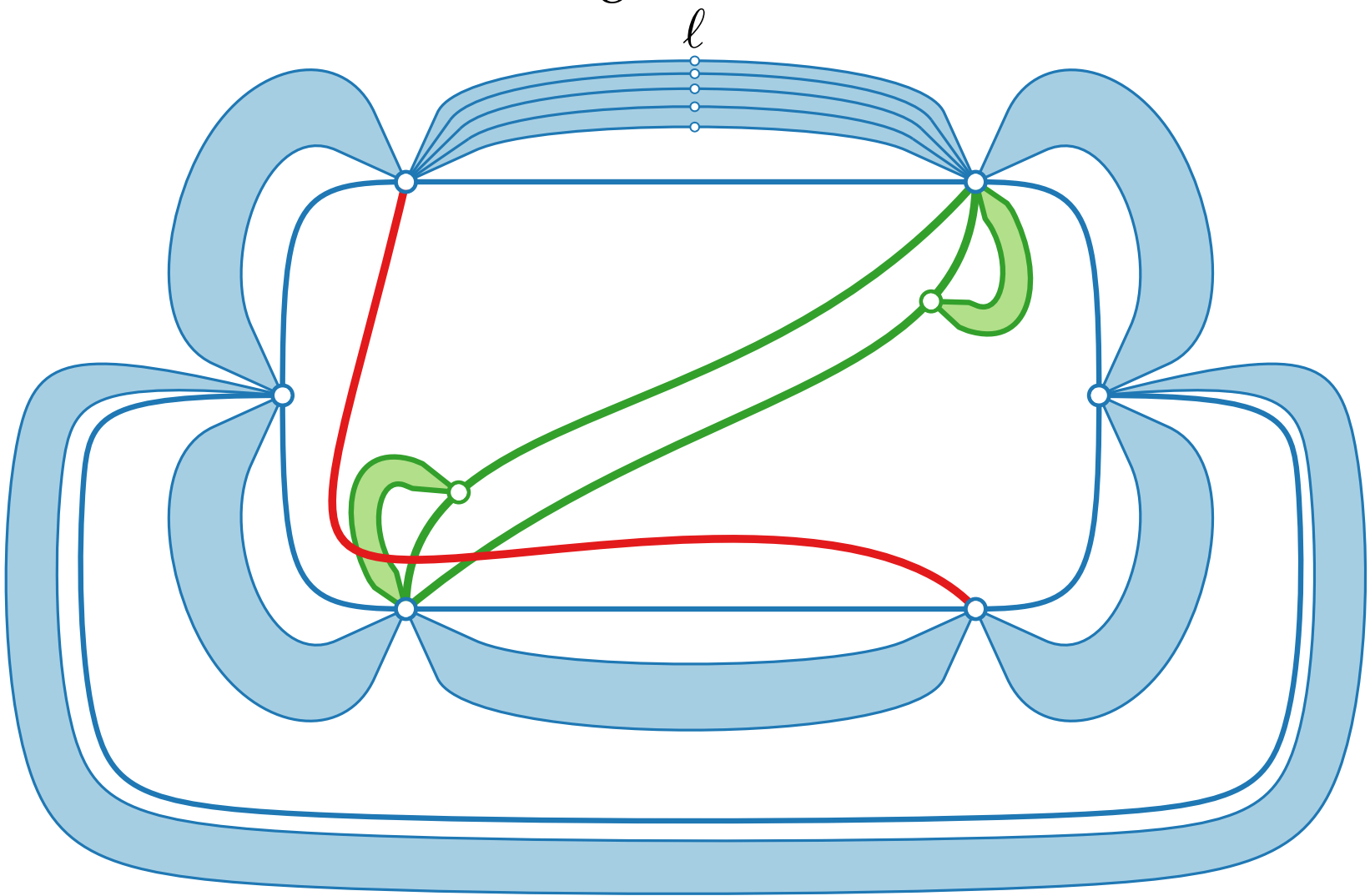
fan-planar, $\ell + 1 \approx n/12$ crossings

$\Rightarrow$ $\varrho_{\text{fan}} \in \Omega(n)$



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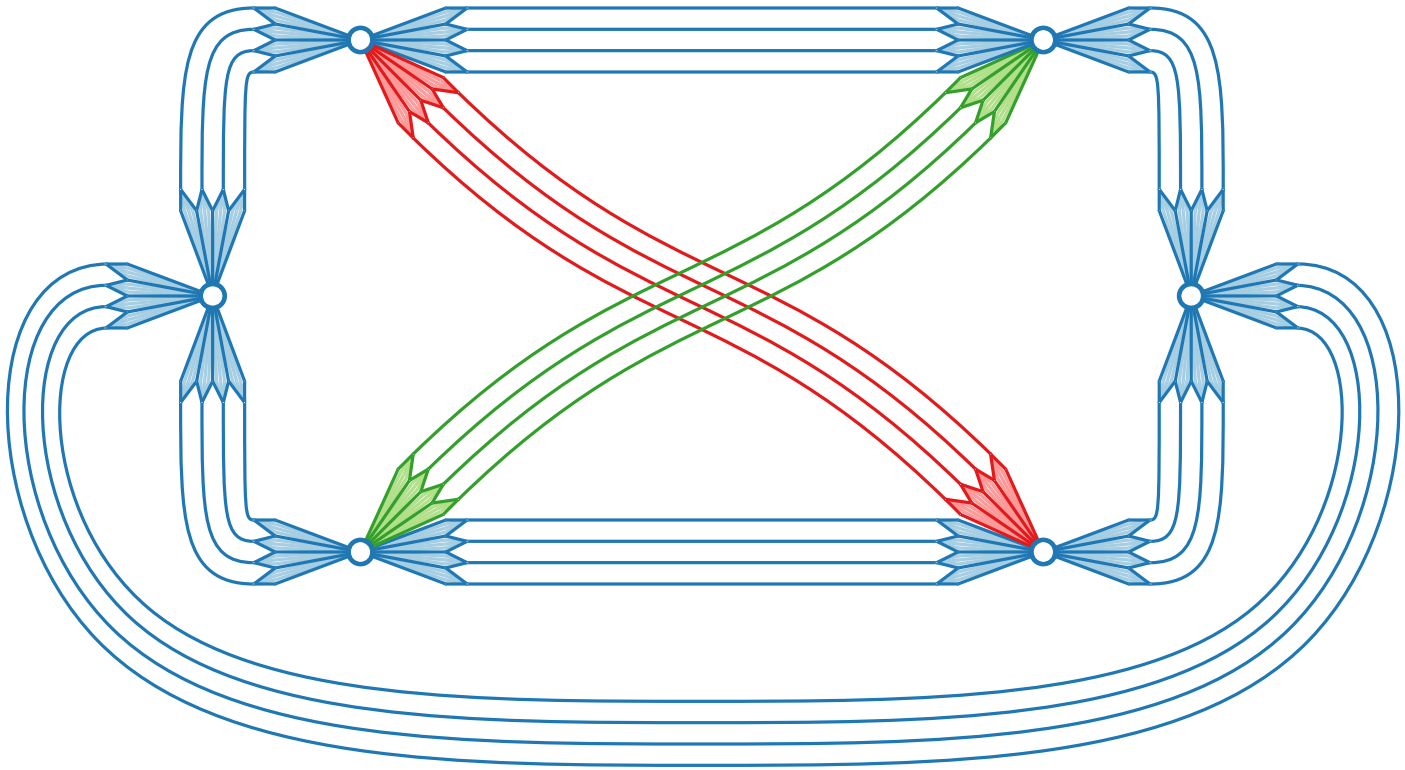

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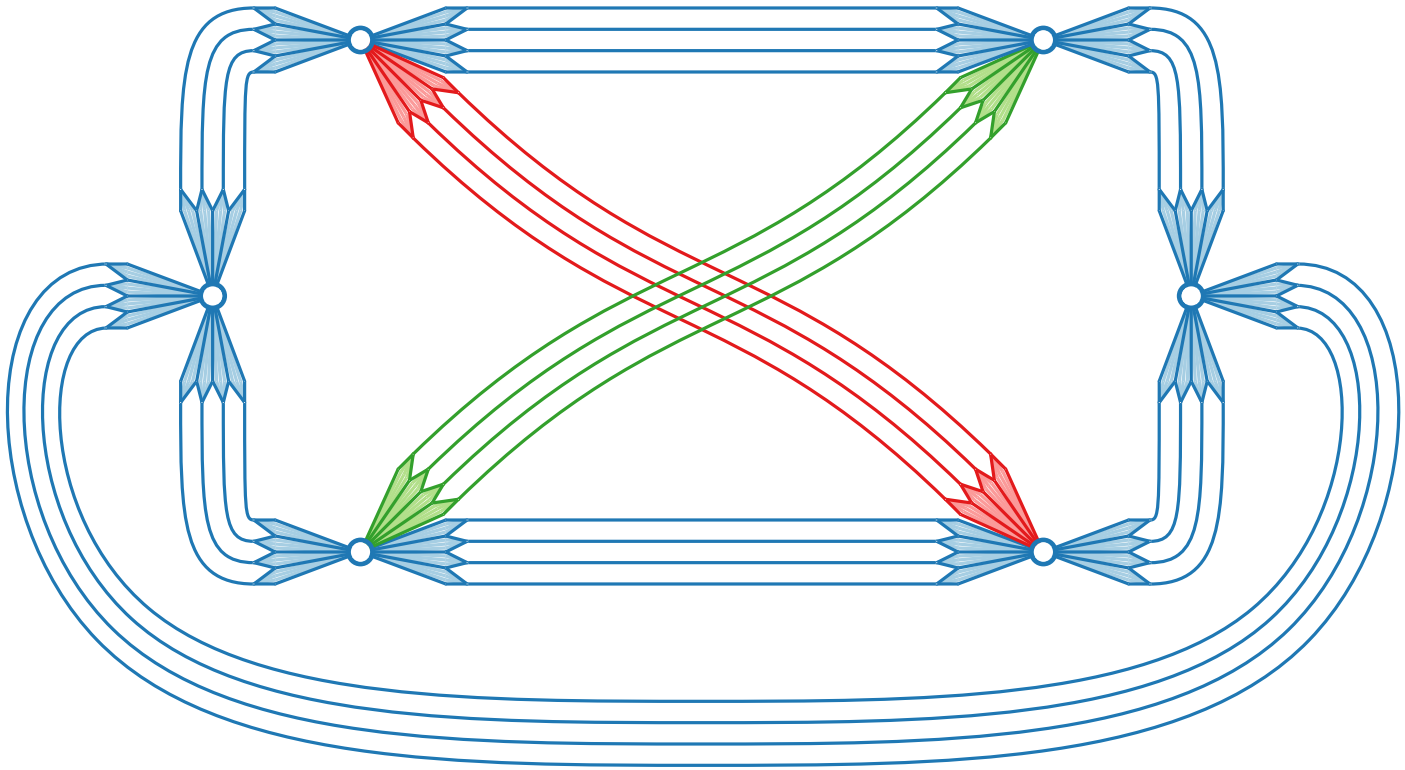

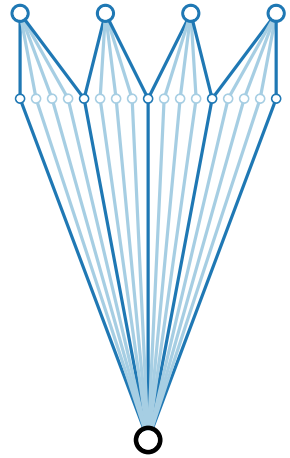

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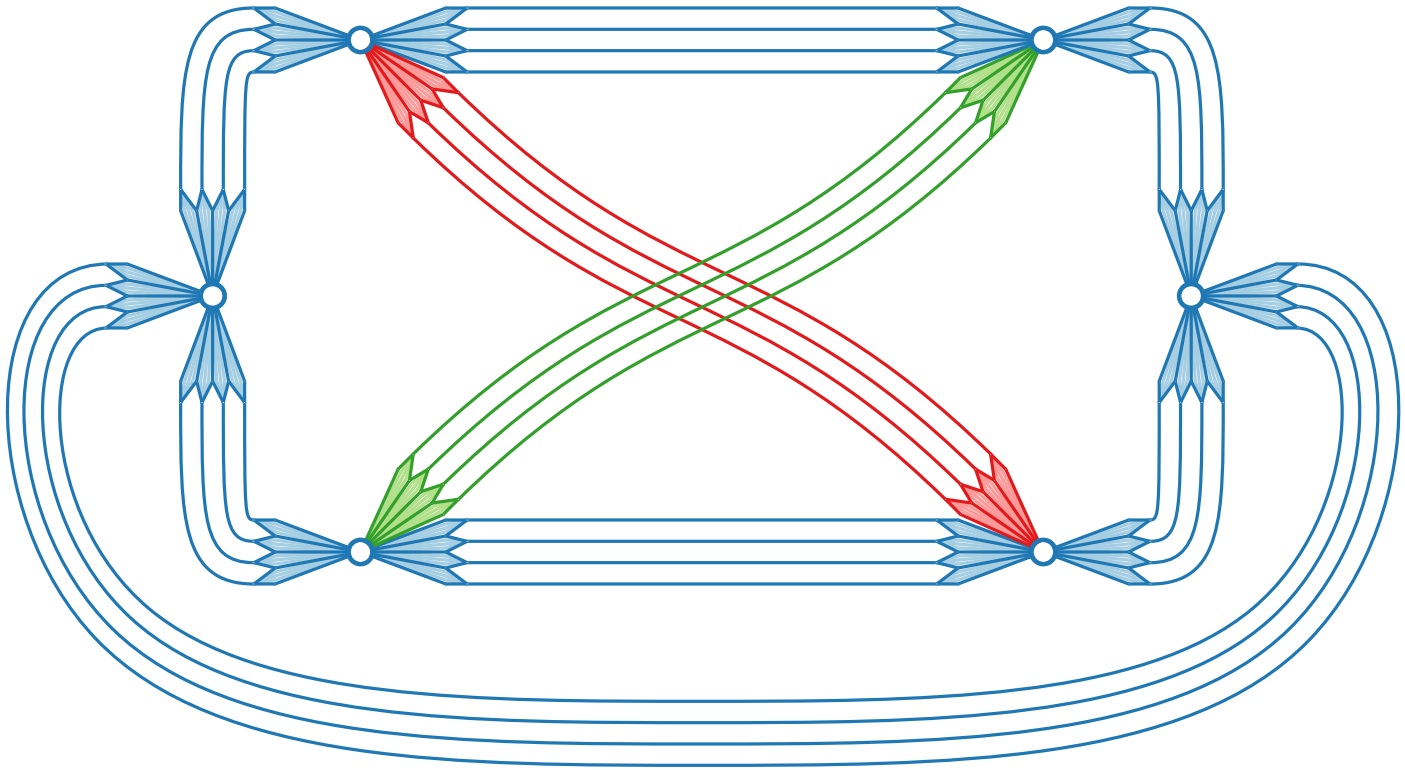

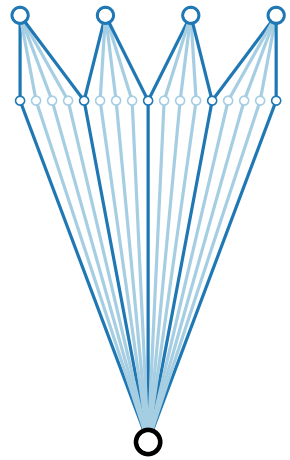

not fan-planar, 16 crossings

Fan-Planar Graphs

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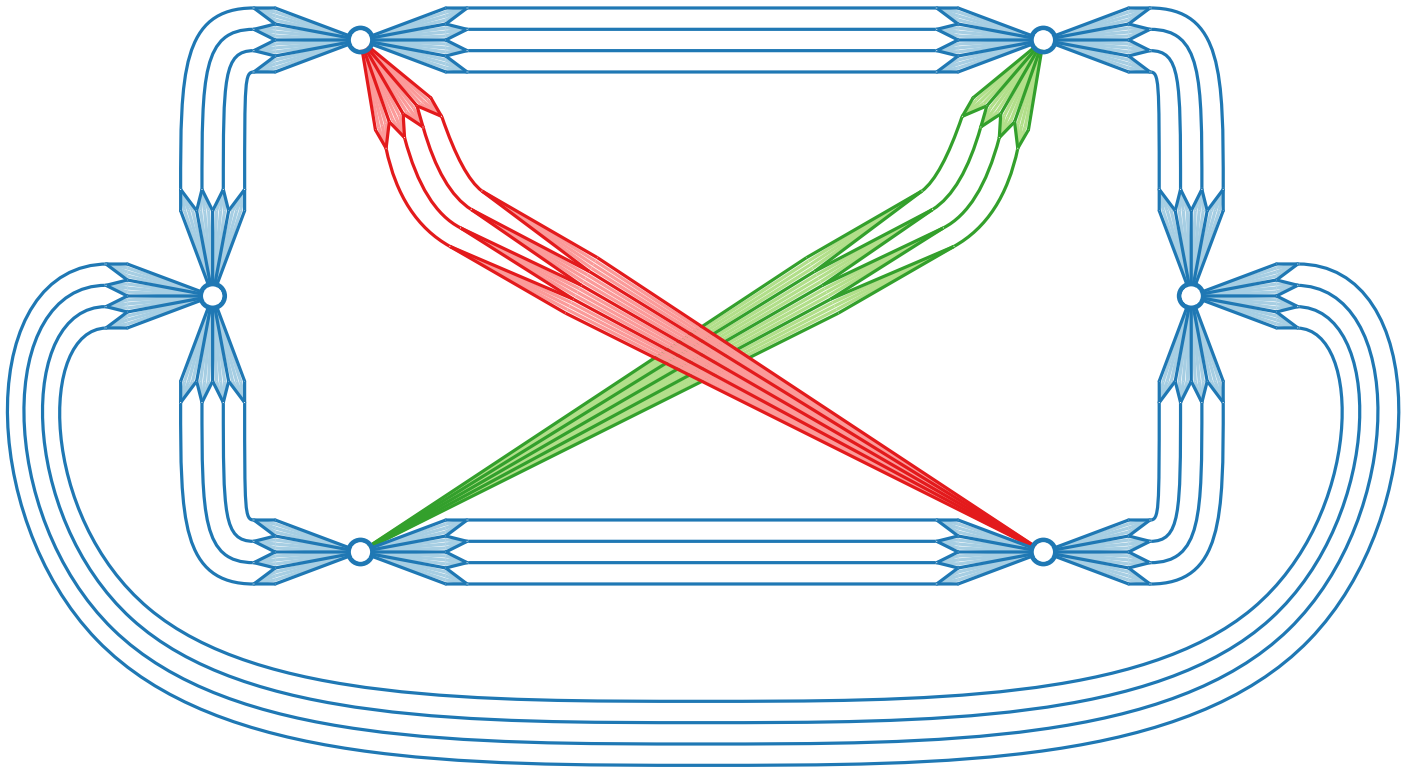

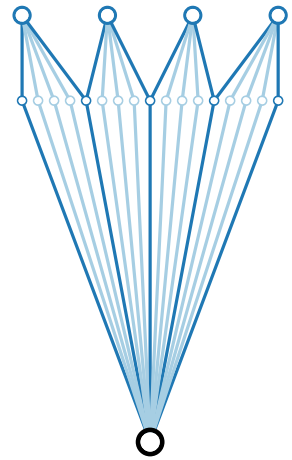

not fan-planar, 16 crossings

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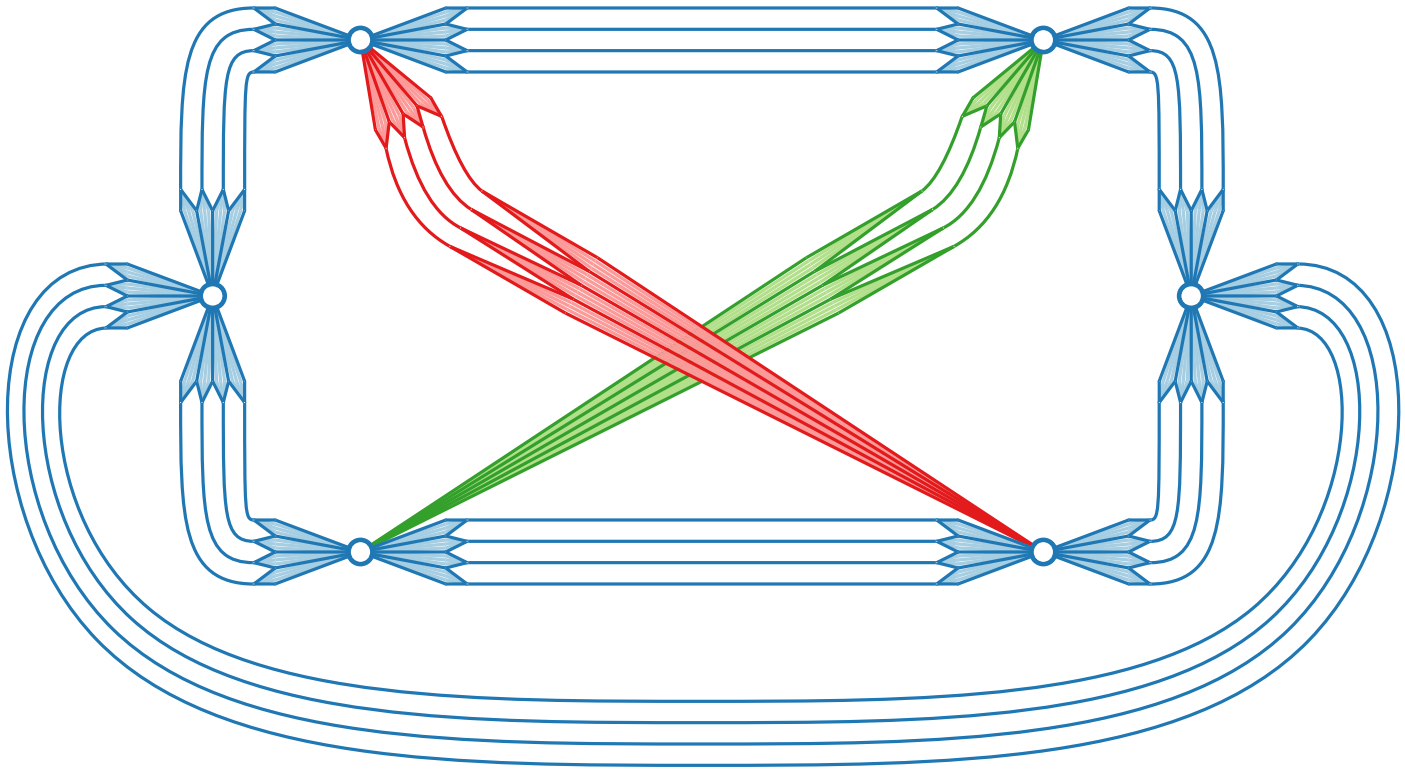
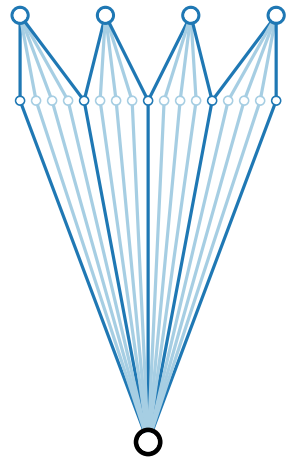
fan-planar, $\ell + 1 \approx n/12$ crossings

$\Rightarrow \varrho_{\text{fan}} \in \Omega(n)$





not fan-planar, 16 crossings
fan-planar, $\Omega(n^2)$ crossings

Fan-Planar Graphs

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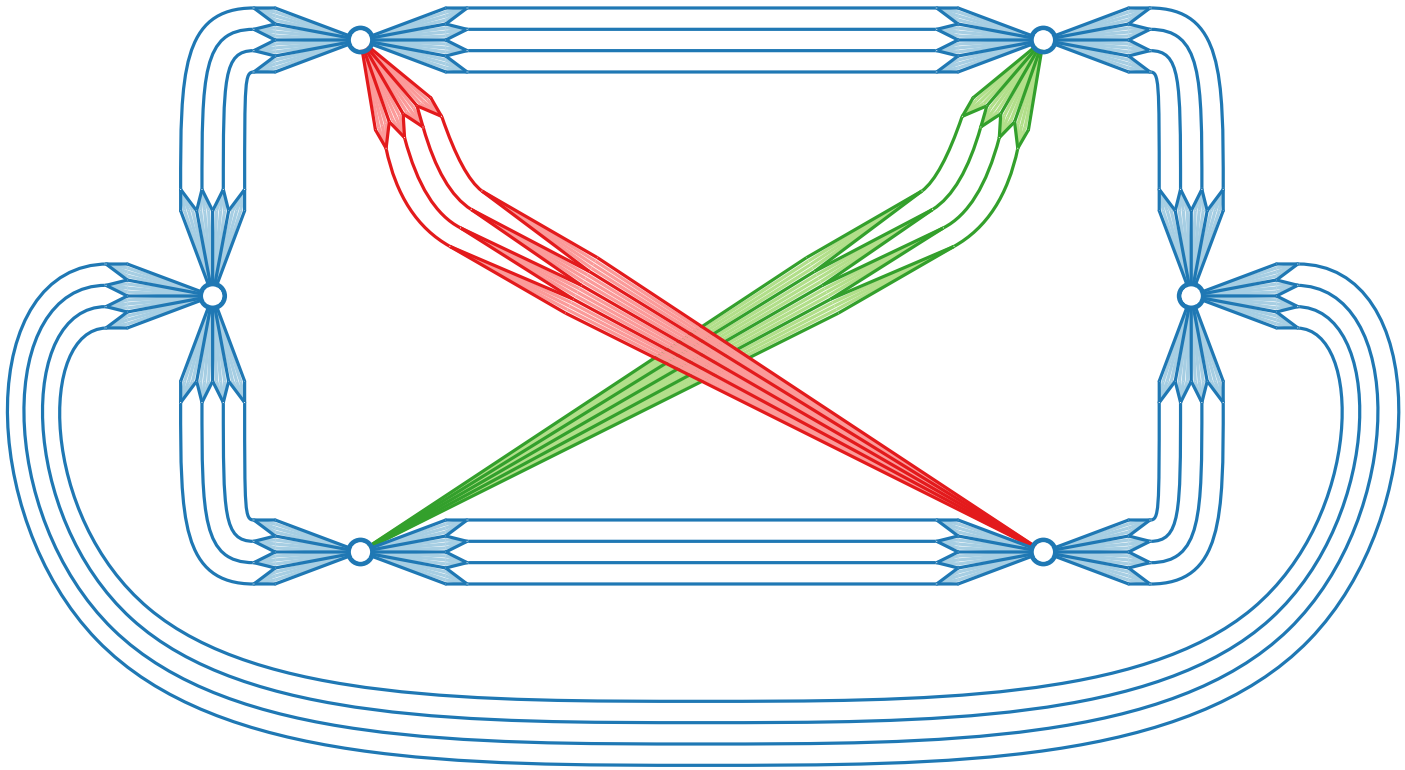

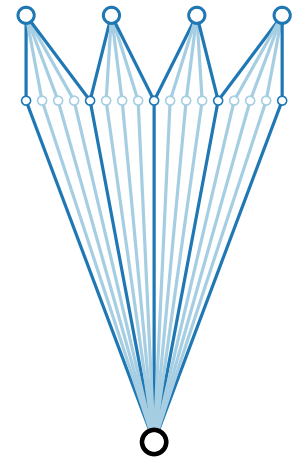

not fan-planar, 16 crossings
fan-planar, $\Omega(n^2)$ crossings

$\Rightarrow$ $\varrho_{\text{fan}} \in \Omega(n^2)$?

k-Quasi-Planar Graphs

at most $f(k)n \log n$ edges

k-Quasi-Planar Graphs

at most $f(k)n \log n$ edges $\Rightarrow O(f'(k)n^2 \log^2 n)$ crossings

k-Quasi-Planar Graphs

at most $f(k)n\log n$ edges $\Rightarrow O(f'(k)n^2\log^2 n)$ crossings

$\Rightarrow \varrho_{k\text{-pl}} \in O(f''(k)n^2\log^2 n)$

k-Quasi-Planar Graphs

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$2k$-wheel

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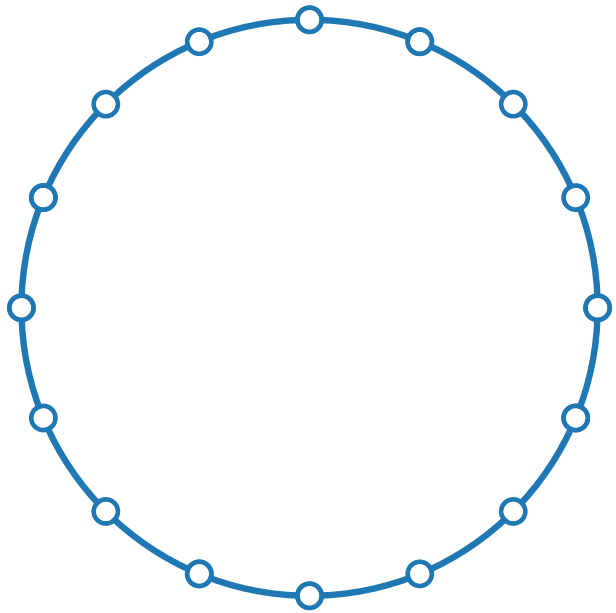

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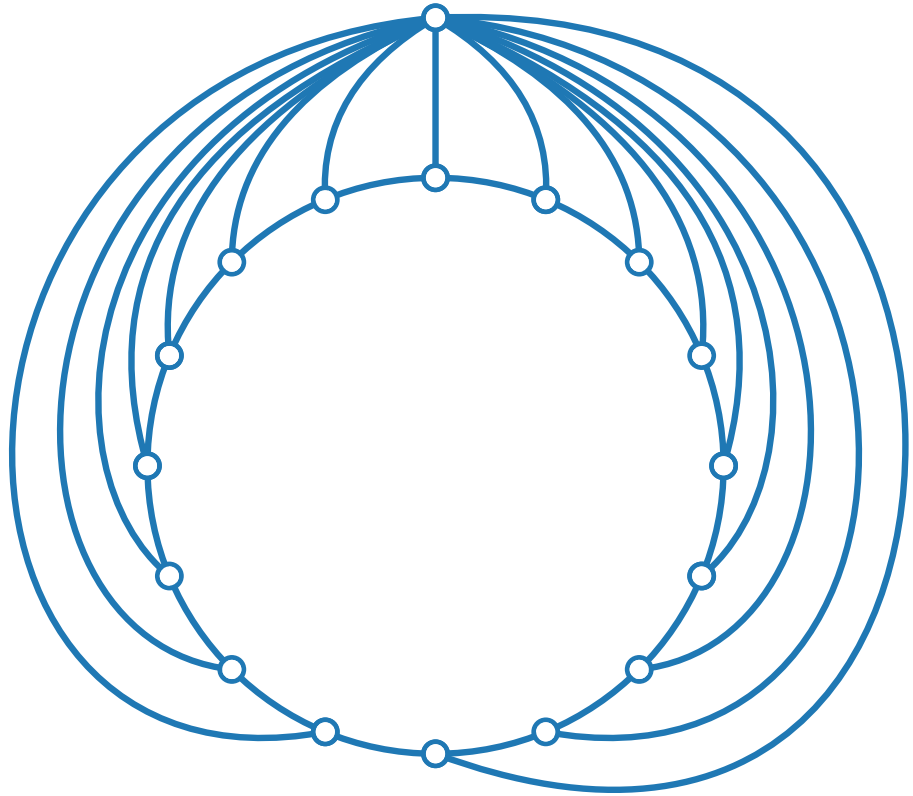

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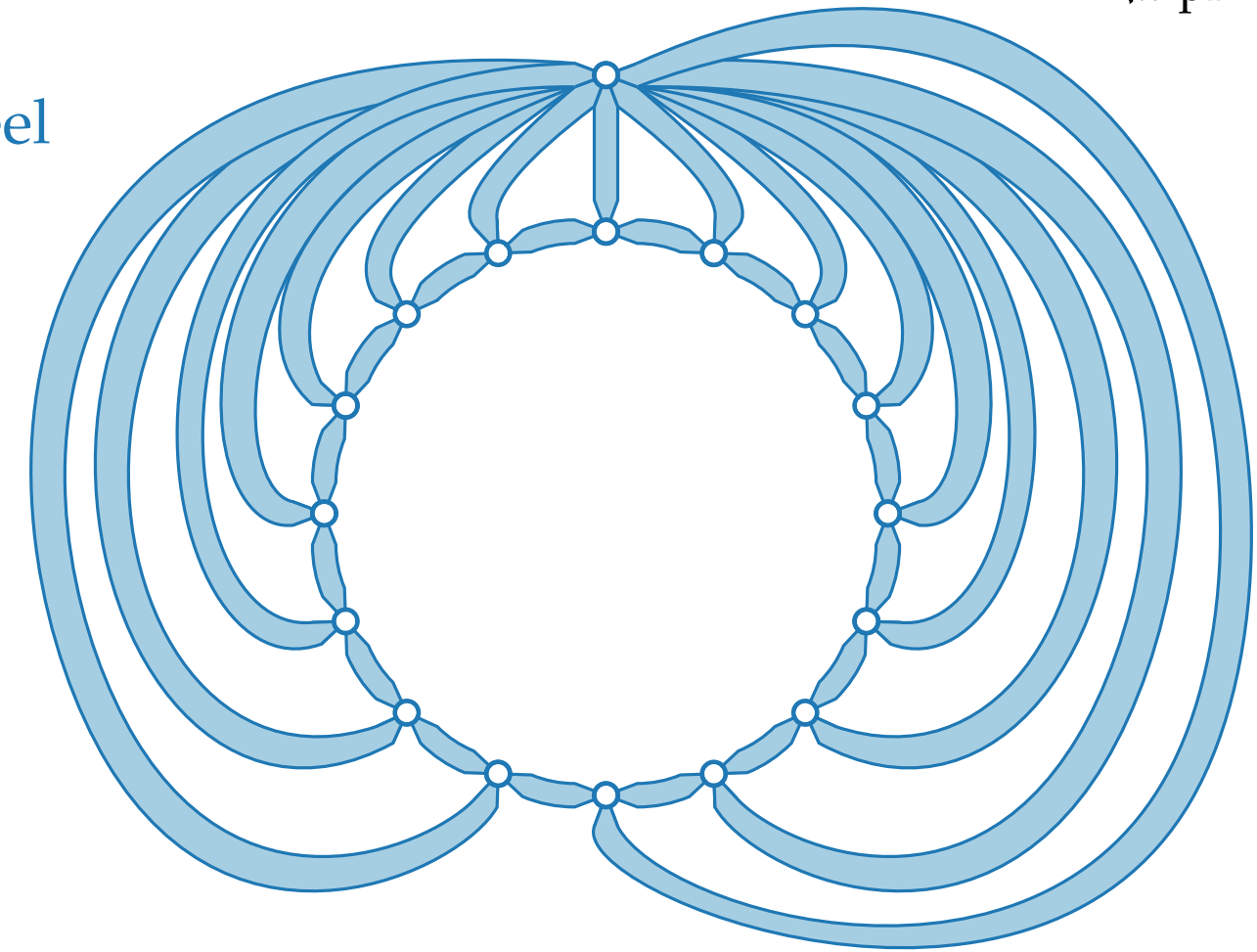

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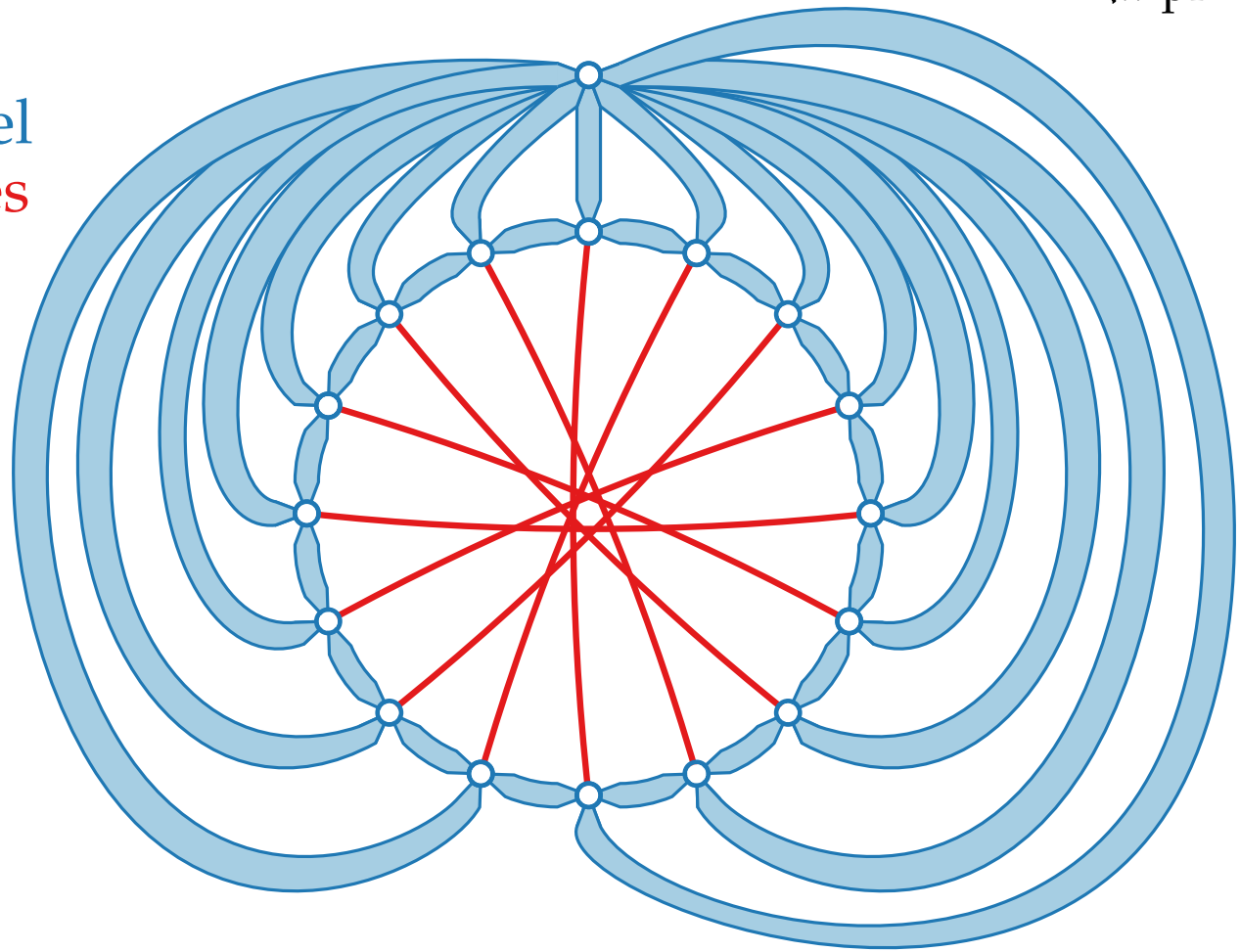
$2k$-wheel
$+k$ edges



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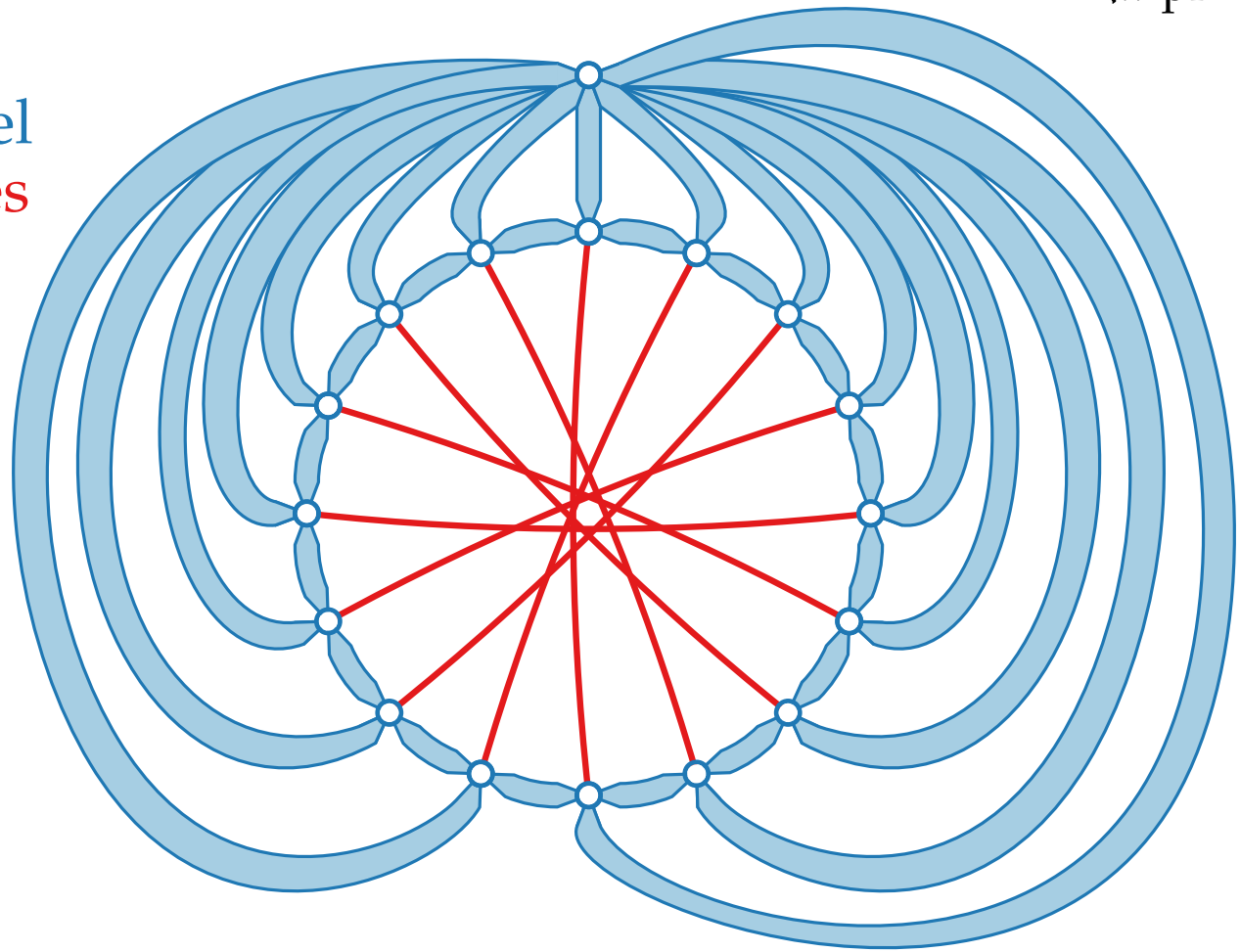
$+k$ edges



not k-quasi-planar, $k(k-1)/2$ crossings

k-Quasi-Planar Graphs

at most $f(k)n\log n$ edges $\Rightarrow O(f'(k)n^2\log^2 n)$ crossings

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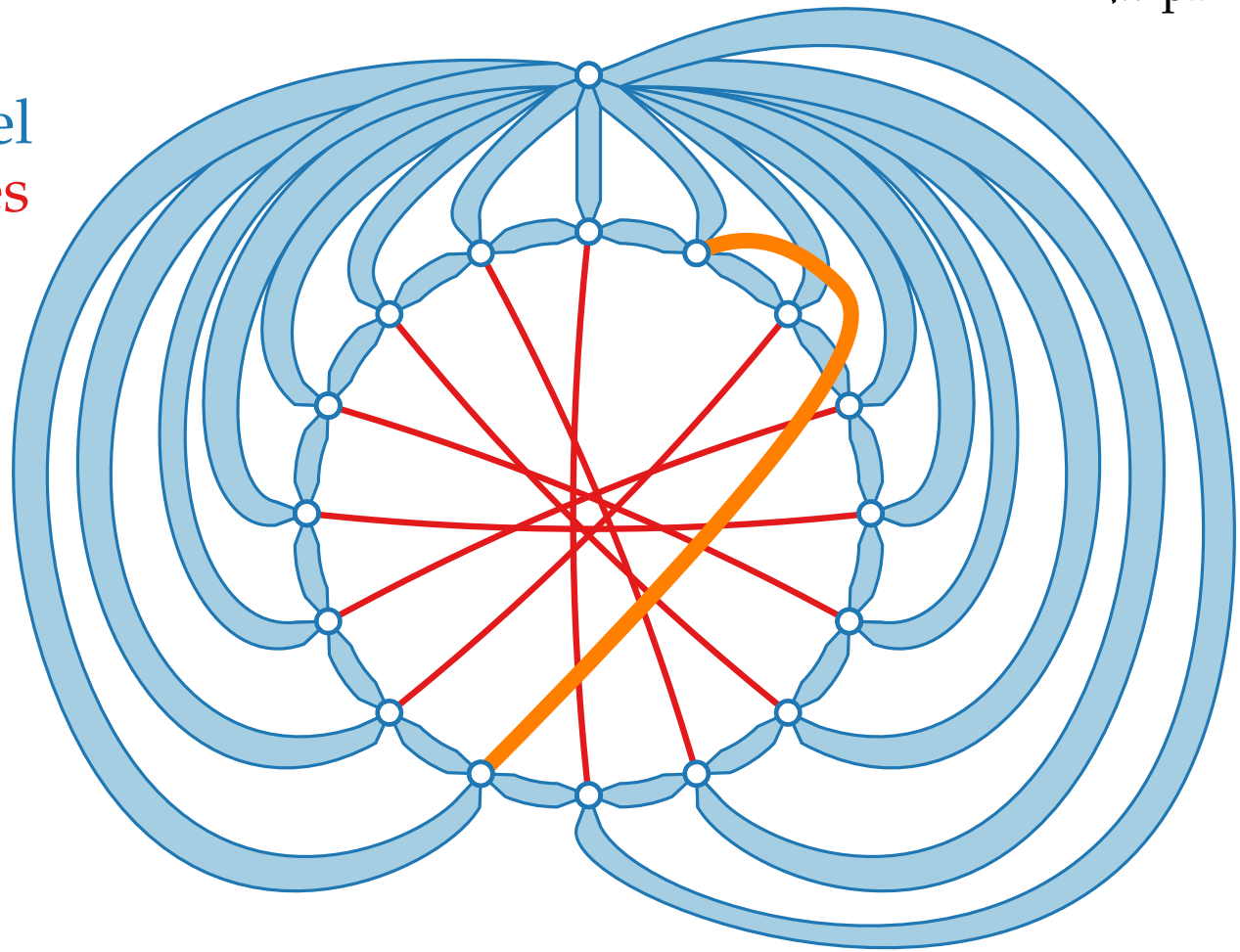
$+k$ edges



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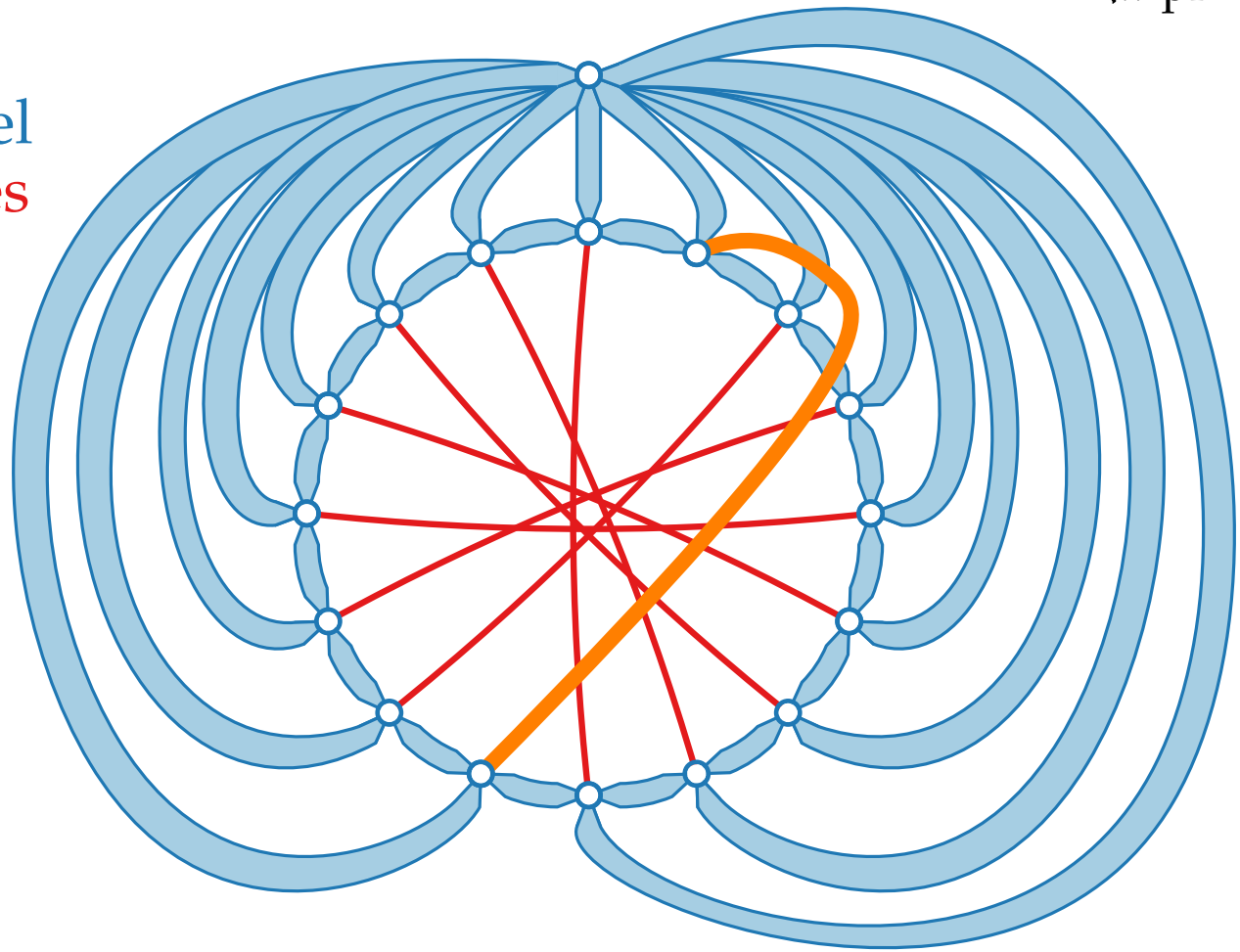
$+k$ edges



not k-quasi-planar, $k(k-1)/2$ crossings

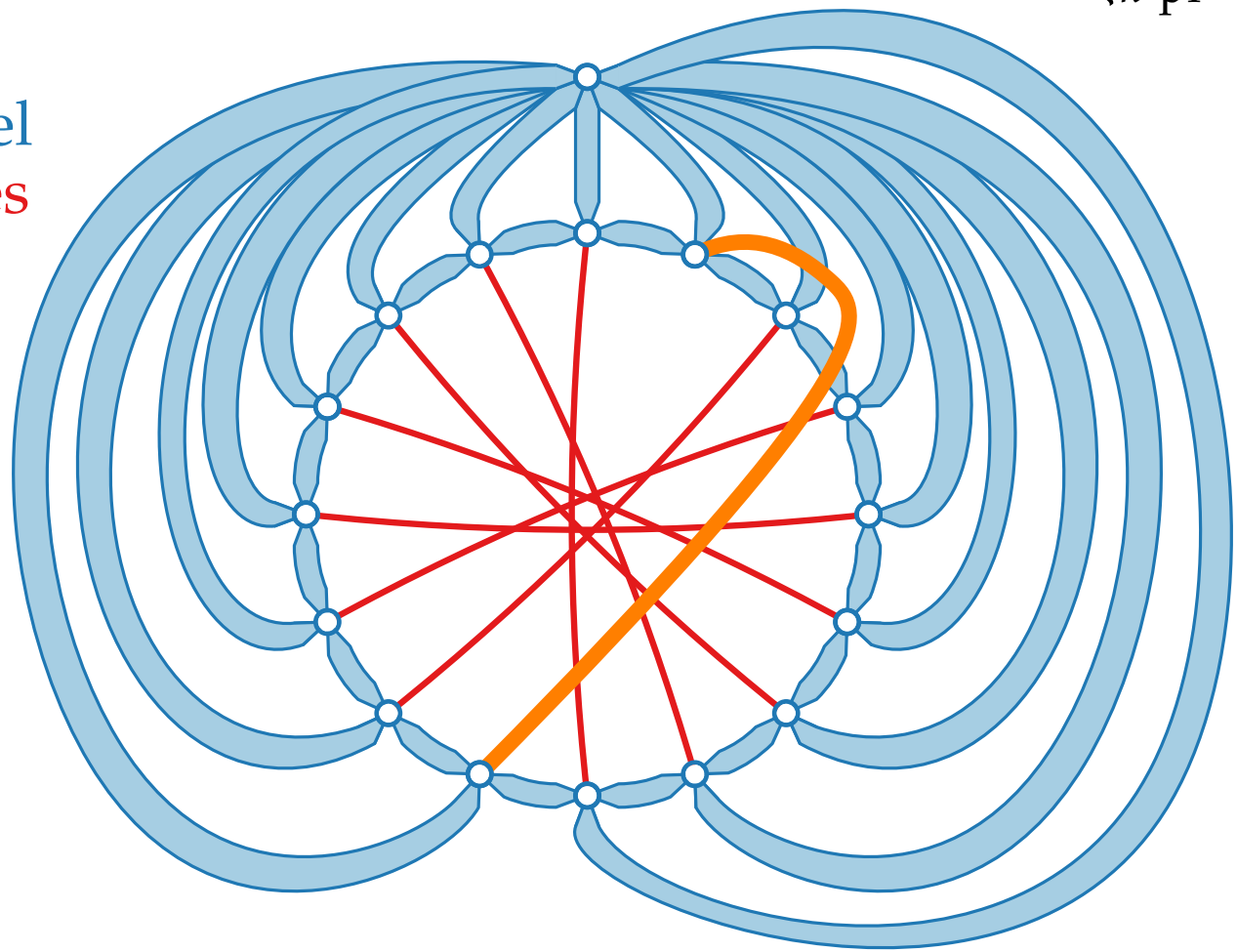
k-quasi-planar, $O(n/k + k^2)$ crossings

k-Quasi-Planar Graphs

at most $f(k)n\log n$ edges $\Rightarrow O(f'(k)n^2\log^2 n)$ crossings

$\Rightarrow \varrho_{k\text{-pl}} \in O(f''(k)n^2\log^2 n)$

$2k$-wheel
$+k$ edges

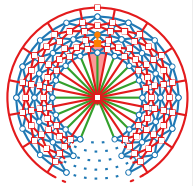


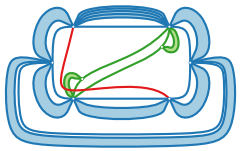
not k-quasi-planar, $k(k-1)/2$ crossings
k-quasi-planar, $O(n/k+k^2)$ crossings

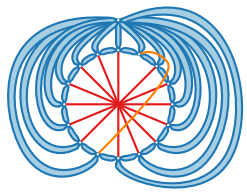
$\Rightarrow$ $\varrho_{k\text{-pl}} \in \Omega(n/k^3)$

Summary

Graph class	lower bound	upper bound
1-planar	$n/2-1$	$n/2-1$
fan-planar	$\Omega(n)$	$O(n^2)$
quasi-planar	$\Omega(n)$	$O(n^2)$
k-quasi-planar	$\Omega(n/k^3)$	$f(k)n^2\log^2 n$







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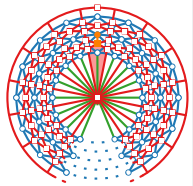

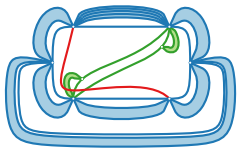

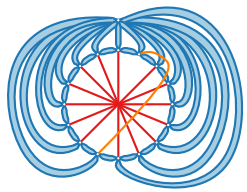

Improve bounds?

Summary

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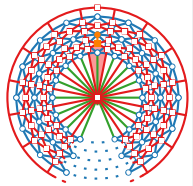

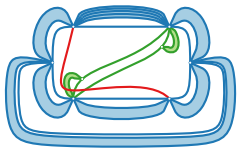

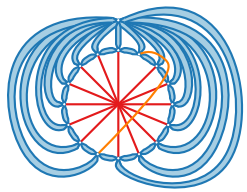

Improve bounds?

Compare between different drawing styles?

Summary

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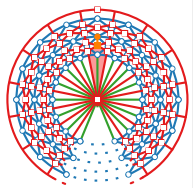
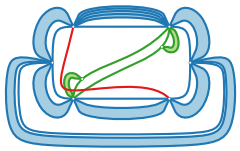
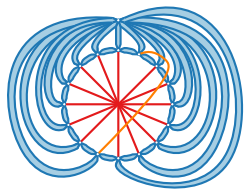
quasi-planar	$\Omega(n)$	$O(n^2)$
k-quasi-planar	$\Omega(n/k^3)$	$f(k)n^2\log^2 n$







Improve bounds?

Compare between different drawing styles?

Restrict to straight-line?