Efficient Generation of Different Topological Representations of Graphs Beyond-Planarity

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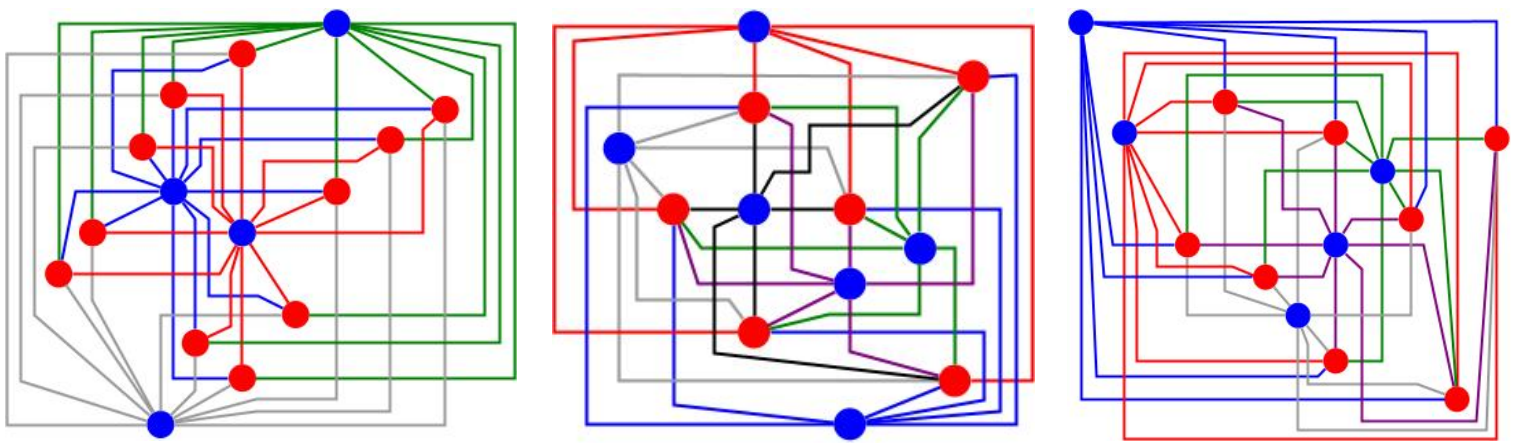
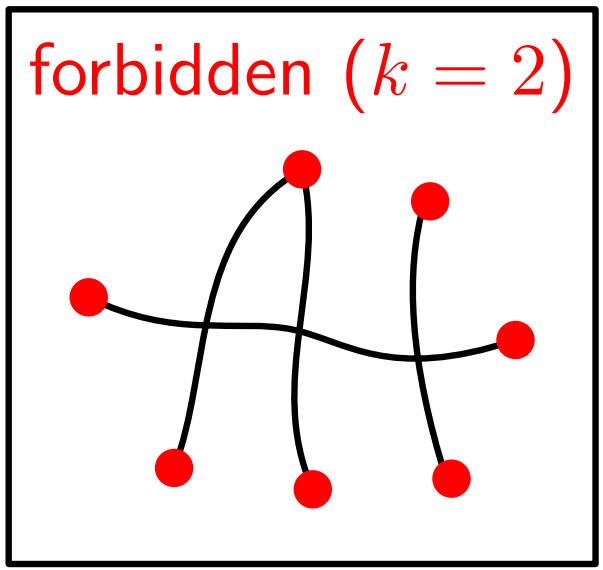


Graph Classes Beyond Planarity

k-planar graphs:
Each edge is crossed at most k times.



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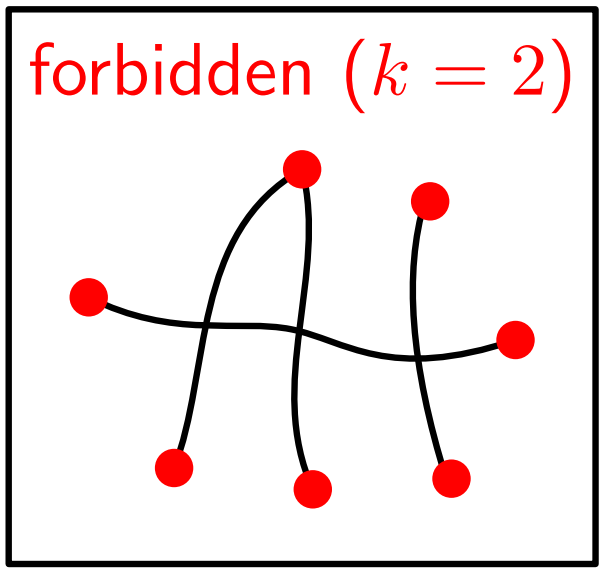

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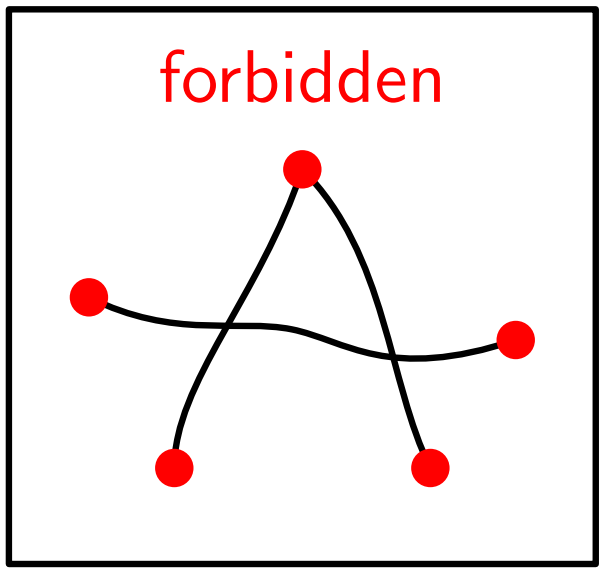

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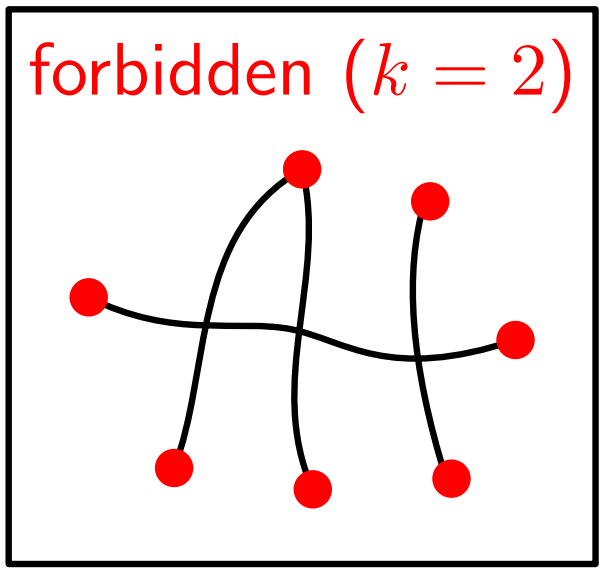

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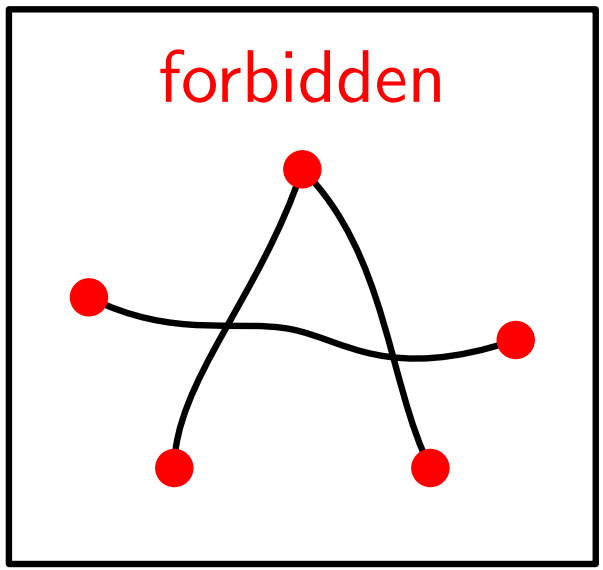

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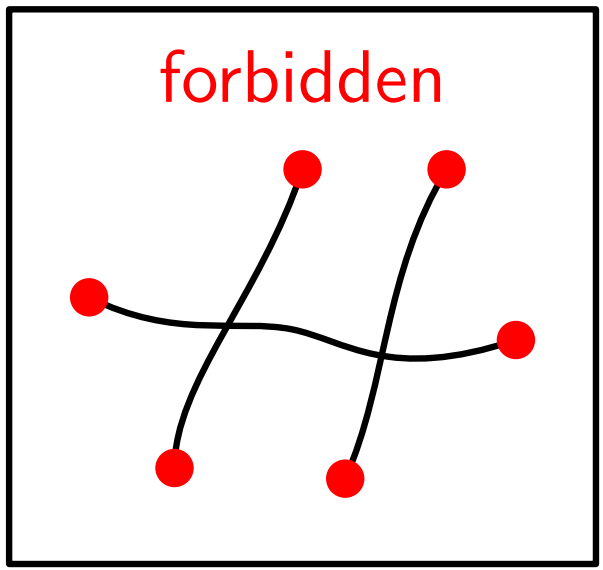

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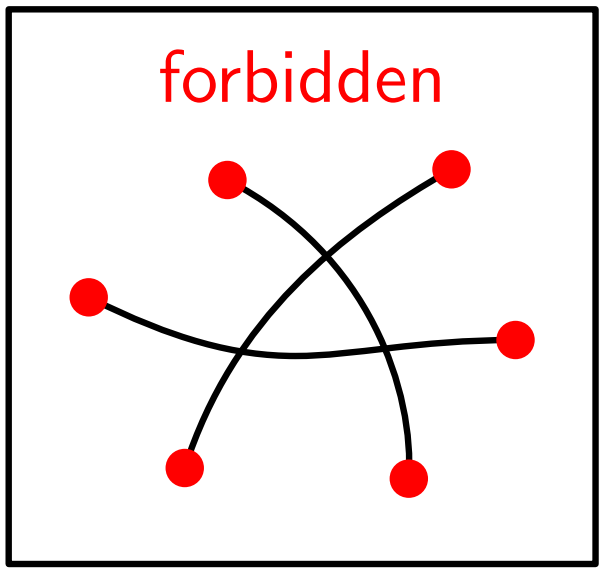

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Graph Classes Beyond Planarity

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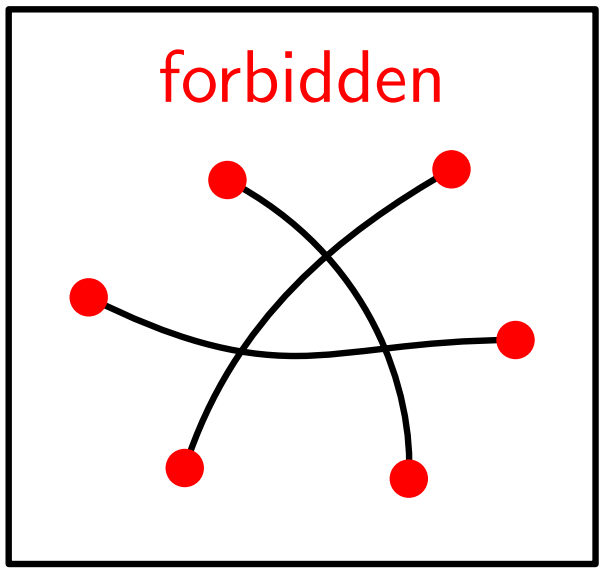
No three edges cross pairwise.

gap-planar graphs: Each crossing must be assigned to one of the two crossing edges, and each edge can have at most one crossing assigned to it.



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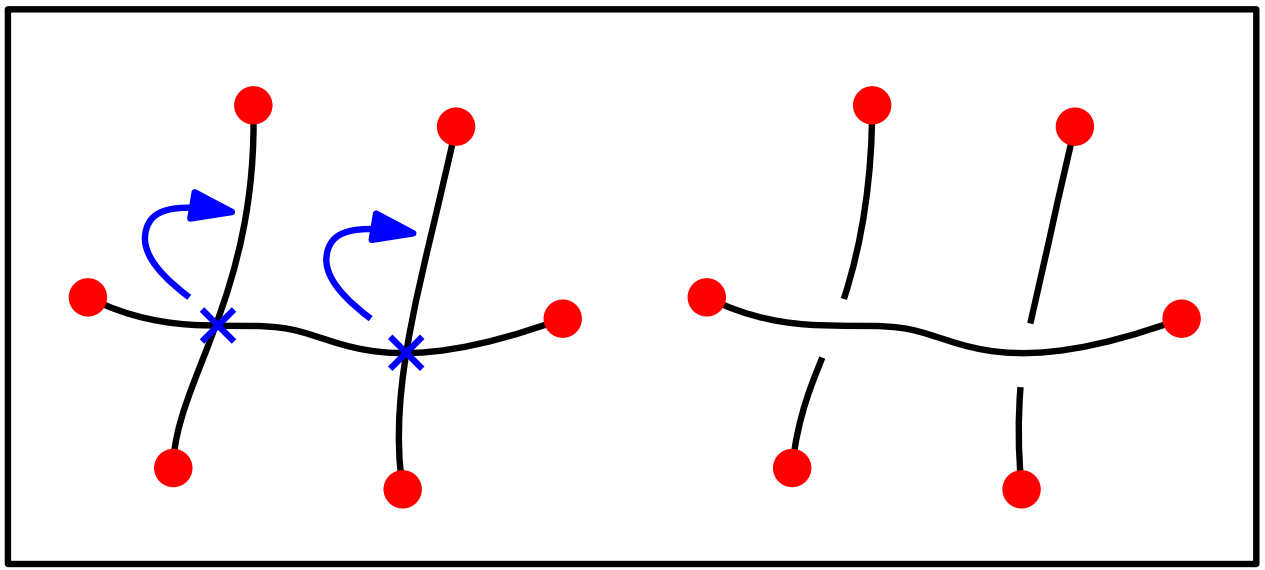

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- $K_{a,a} \notin \mathcal{C} \Rightarrow$ not all graphs with max-degree a in $\mathcal{C}$

Methods

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  - Computer program can do it.

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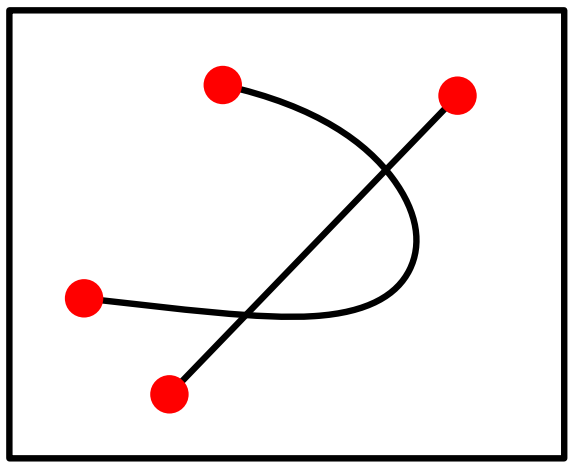

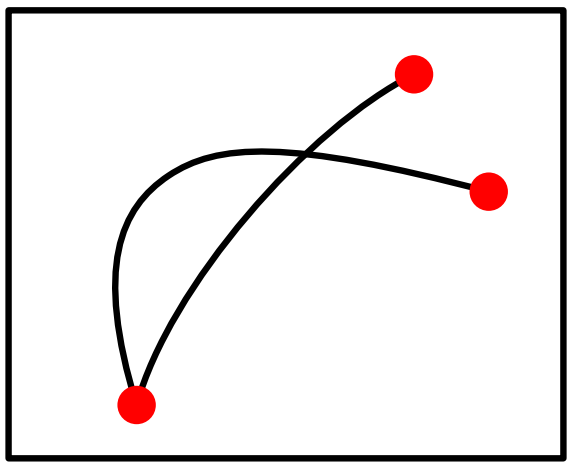

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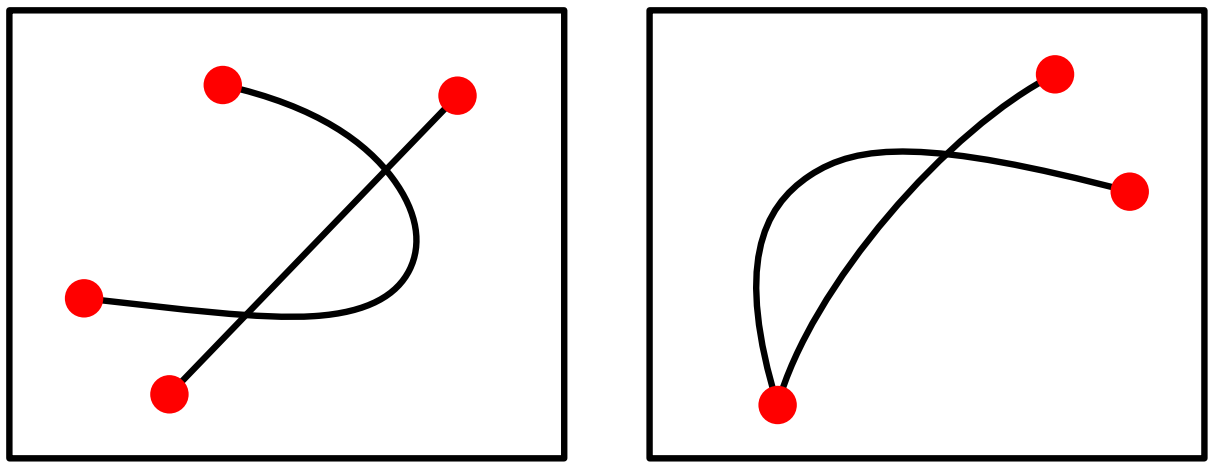

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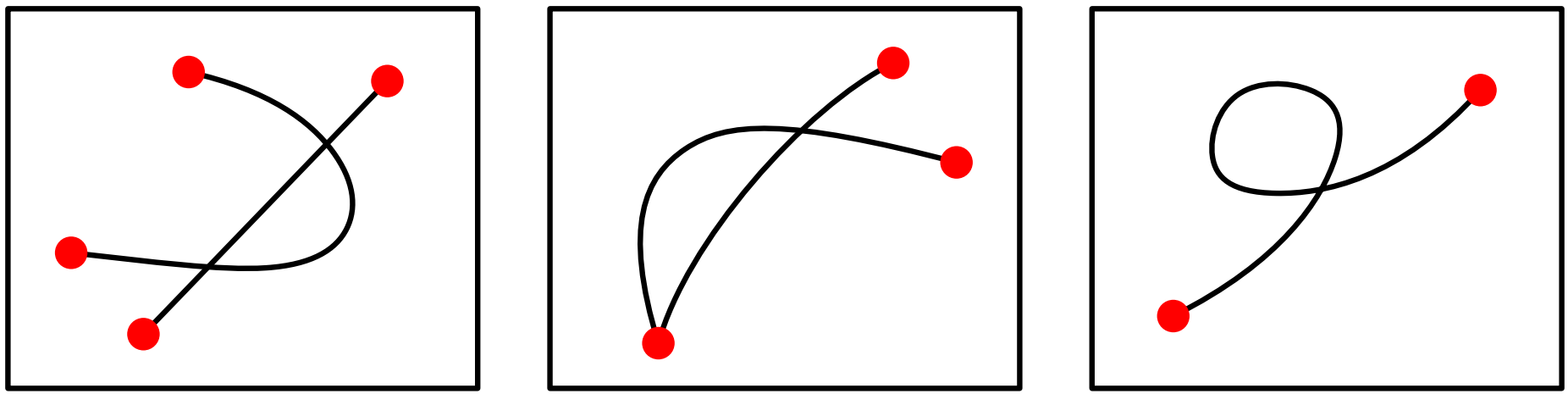

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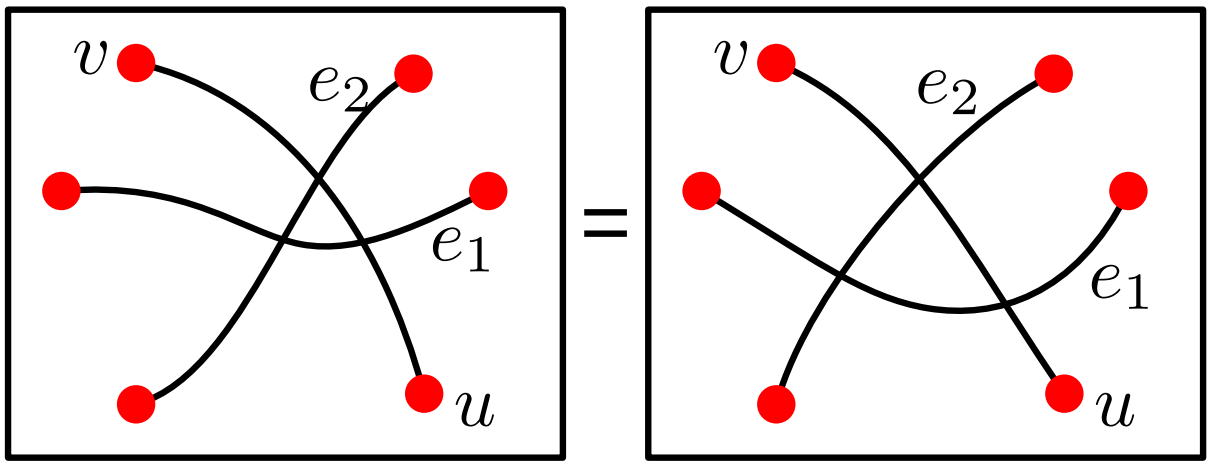

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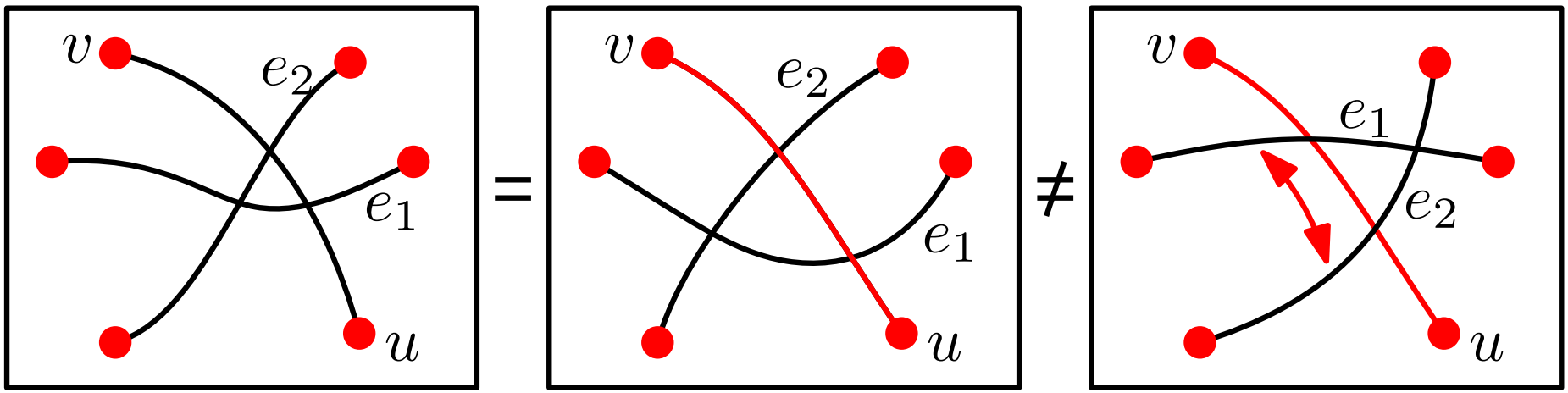

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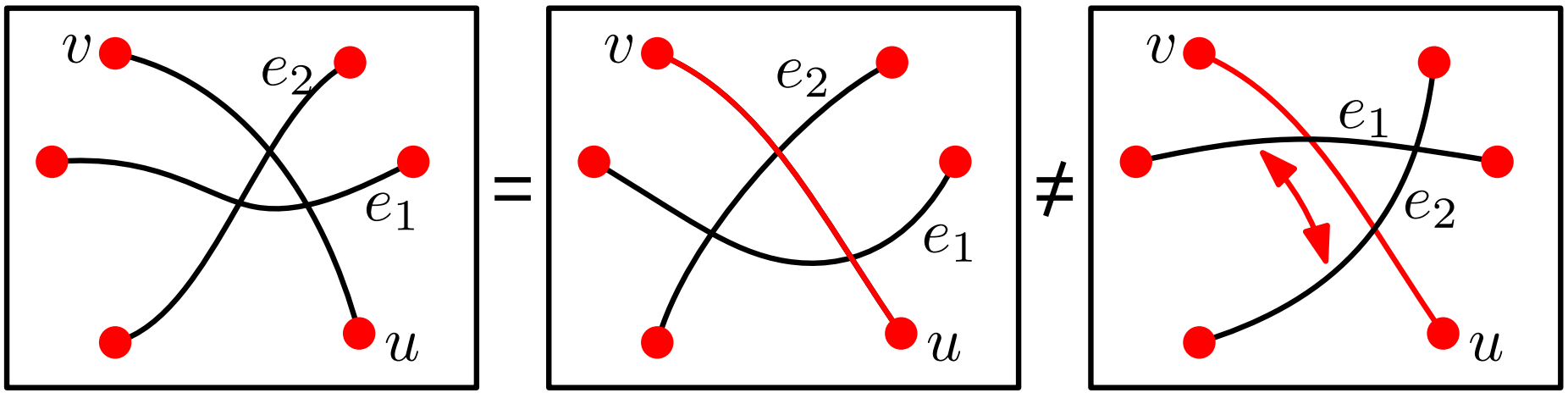

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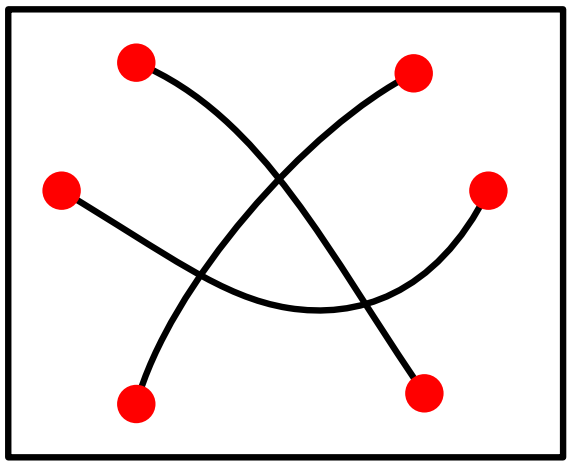

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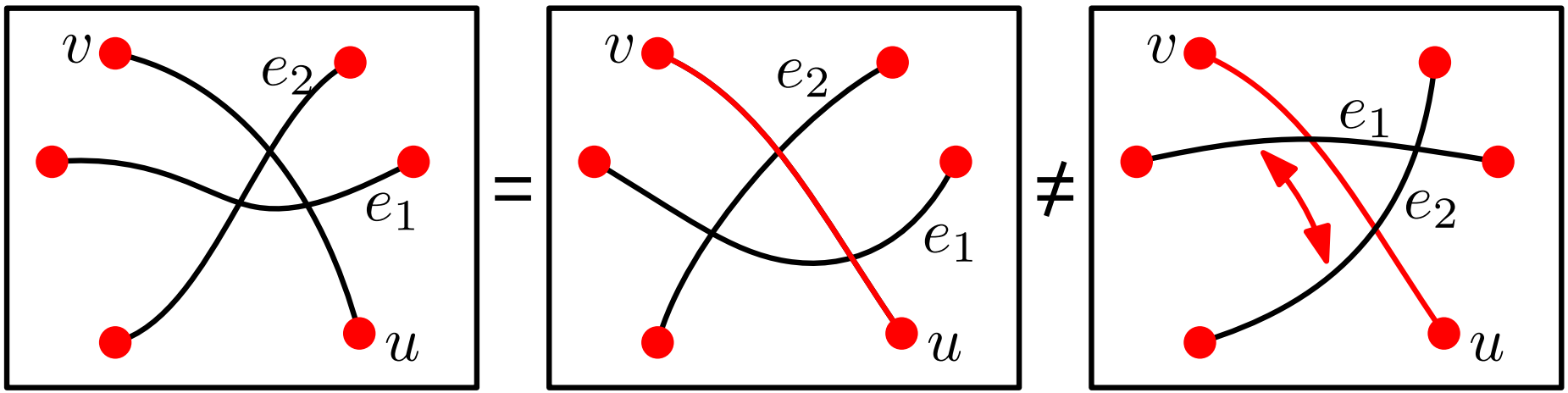

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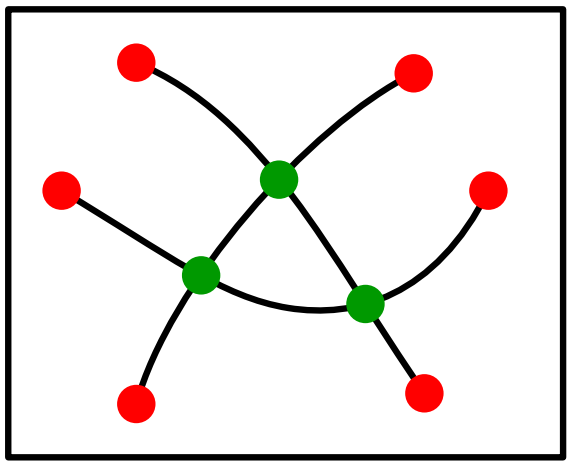

Algorithm

1. Insert all drawings of K_3 / $K_{2,2}$ in a list L
2. **While** $L \neq \emptyset$ **do**
3. $L' \leftarrow \emptyset$
4. **for all** $D \in L$
5. Insert 'next' vertex into D in all possible ways and save new drawings in L'
6. Delete duplicates from L'
7. Save drawings of L' (on hard disk)
8. $L \leftarrow L'$

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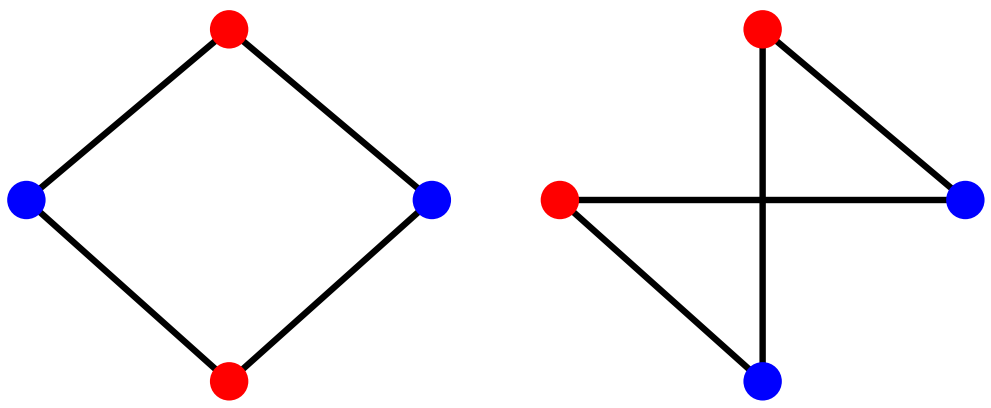

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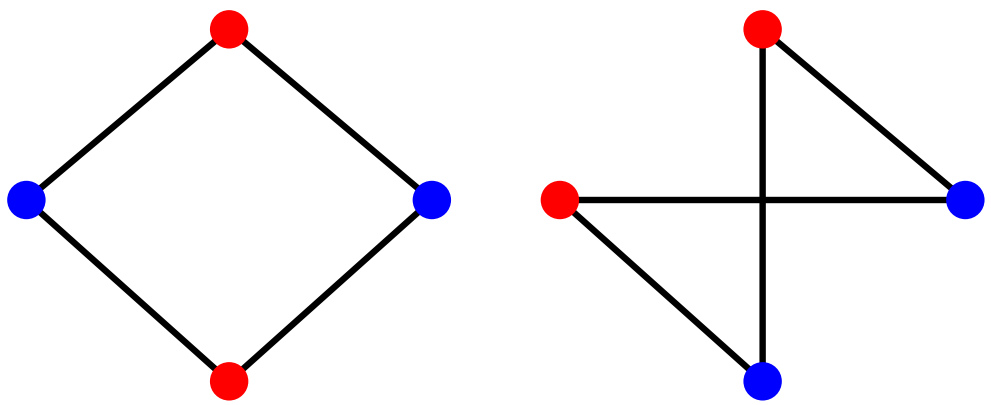

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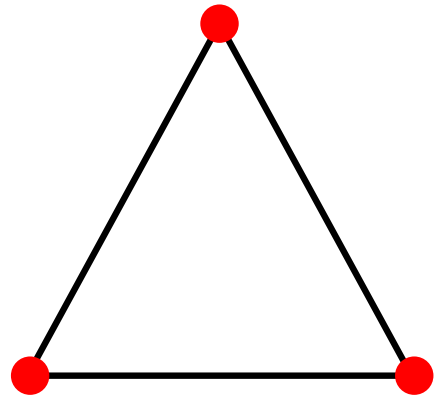

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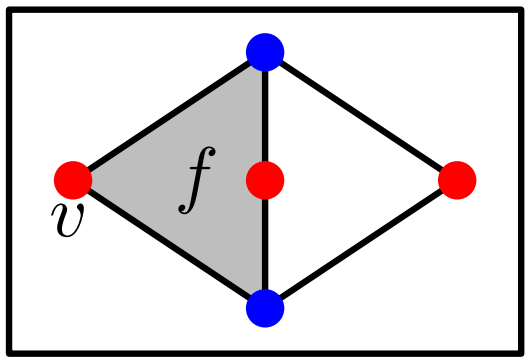
- Choose $v \in V$ (red)



Example for
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Algorithm – Insertion

- Choose $v \in V$ (red)
- **For all** faces f around v:
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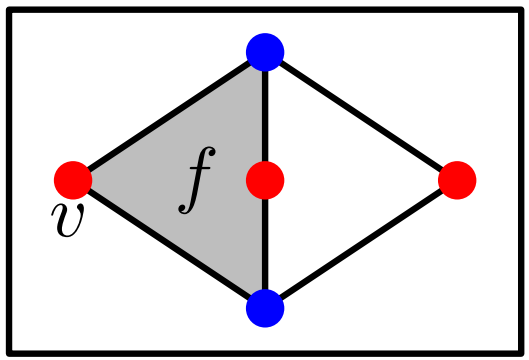

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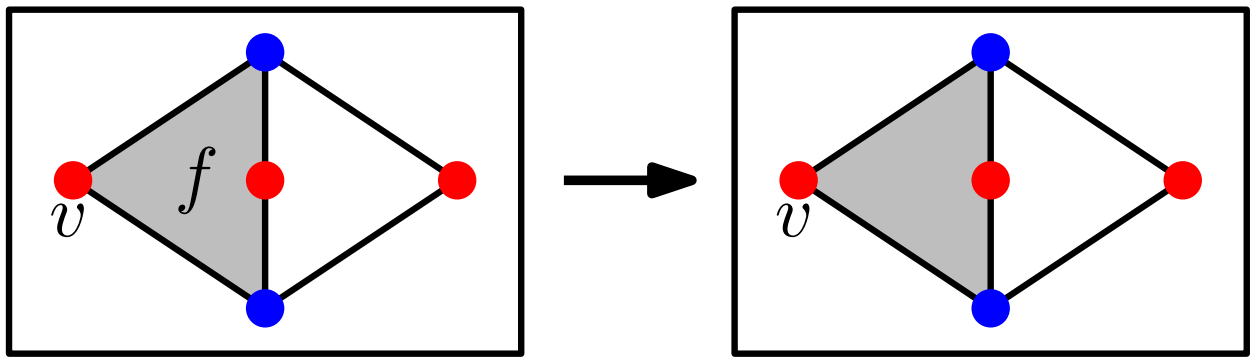

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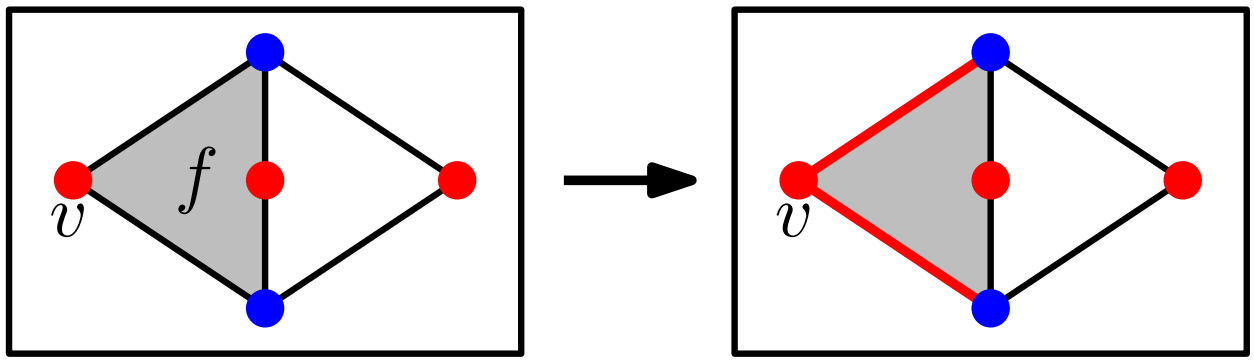

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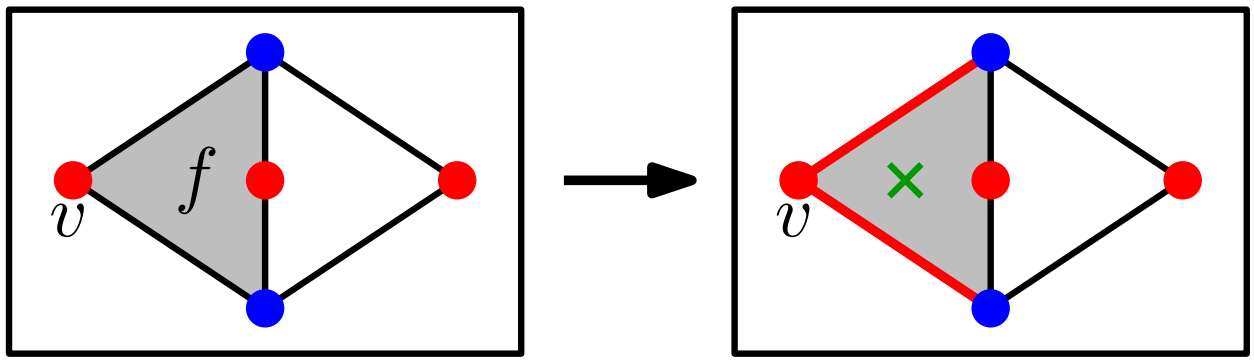

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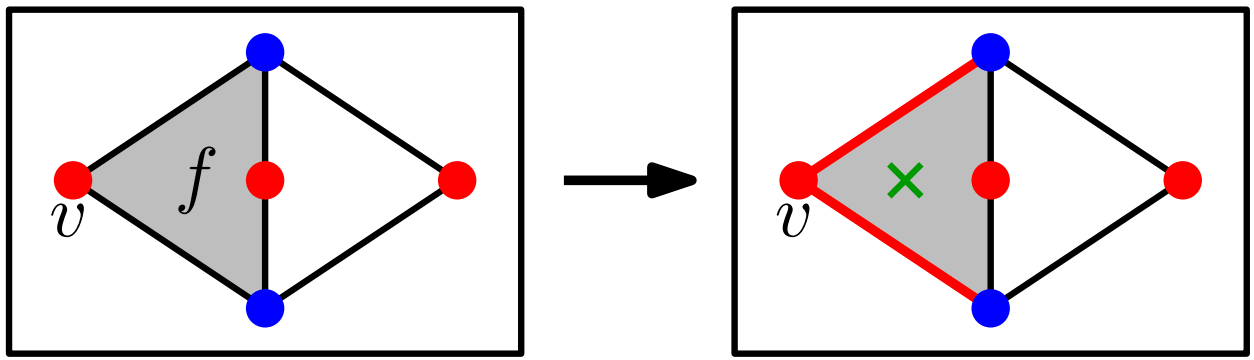

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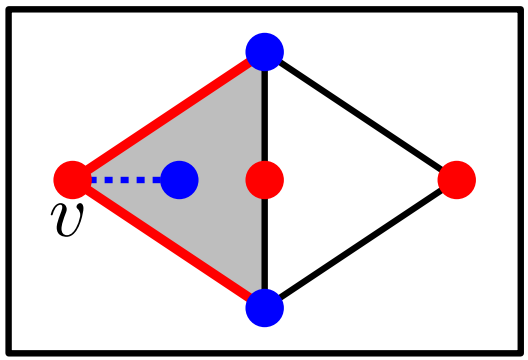

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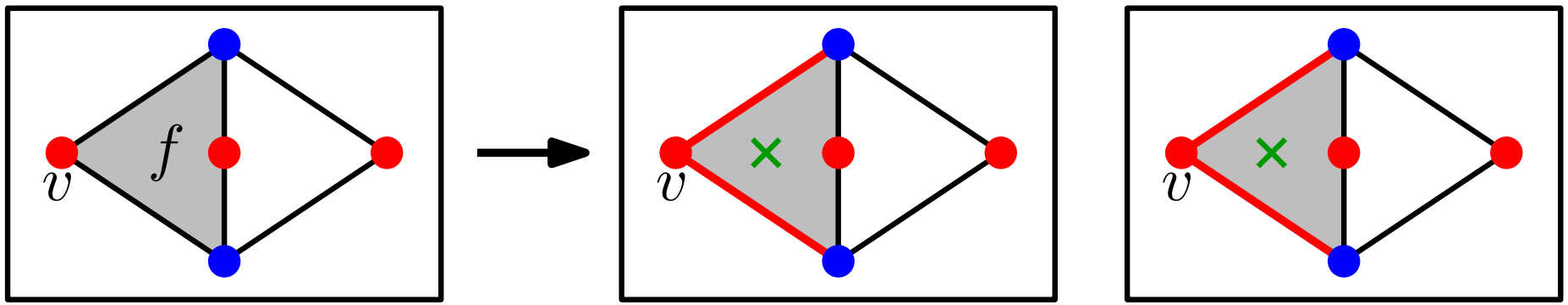

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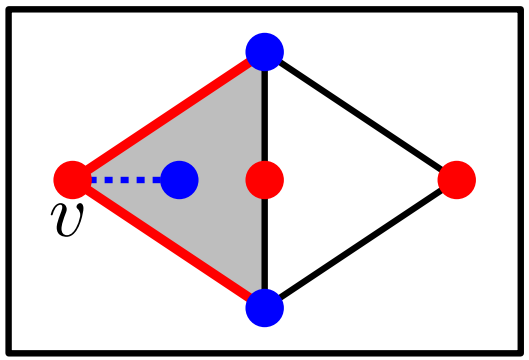

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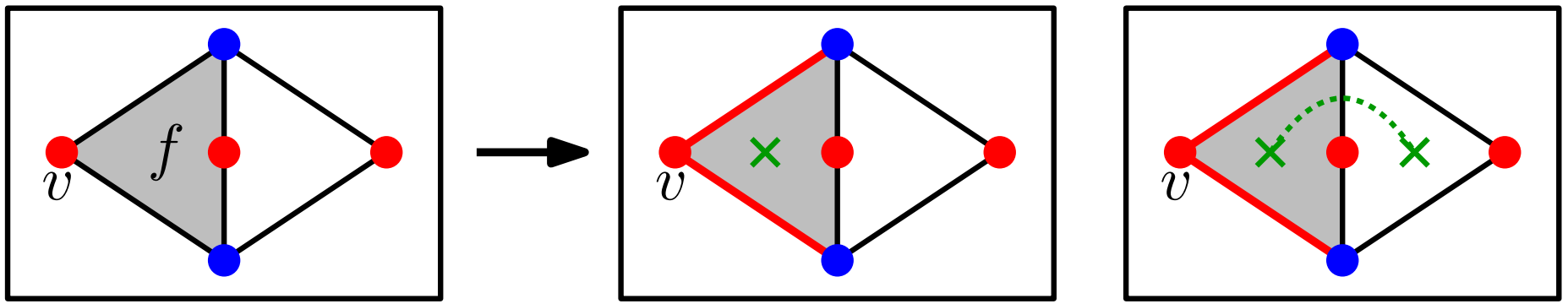
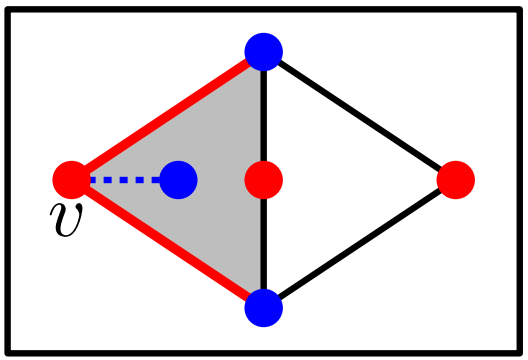


Example for
1-planar drawings



Algorithm – Insertion

- Choose $v \in V$ (red)
- **For all** faces f around v:
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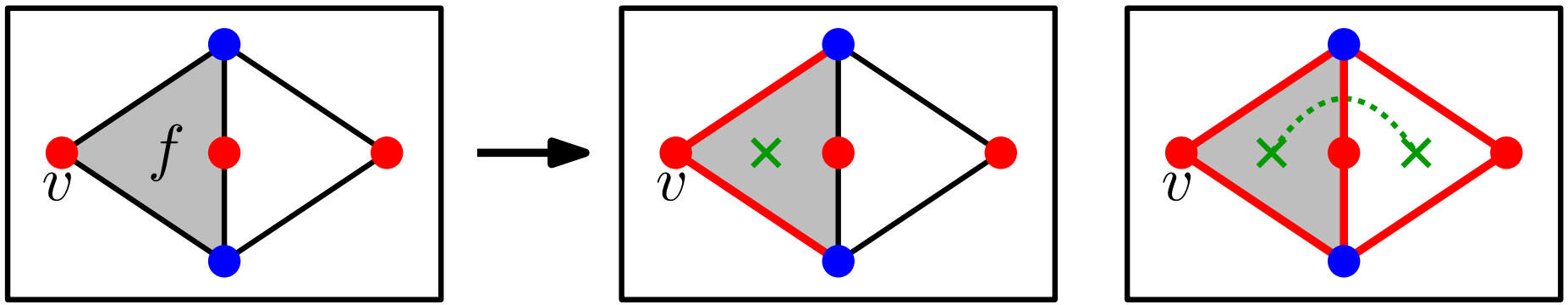

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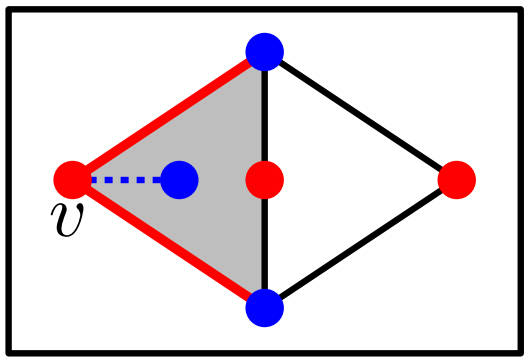

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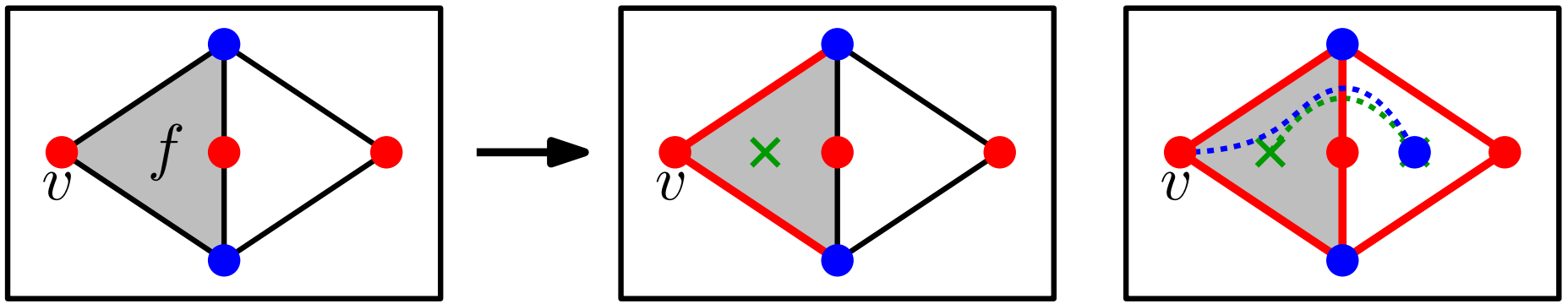

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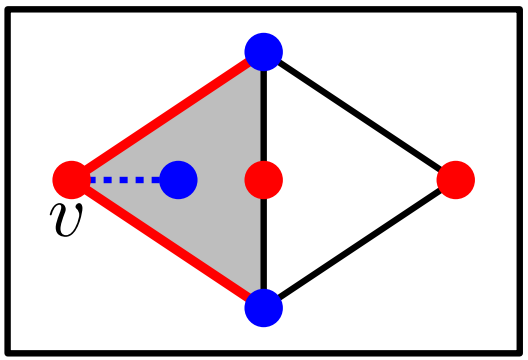

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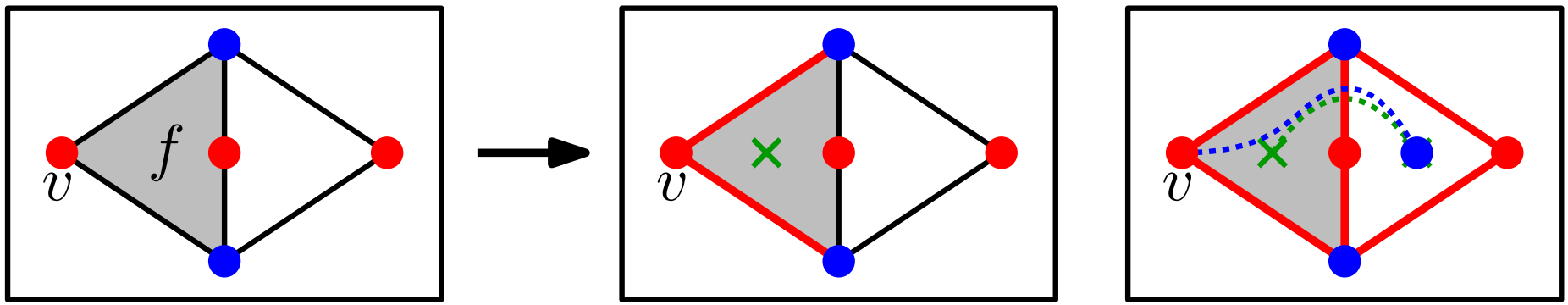

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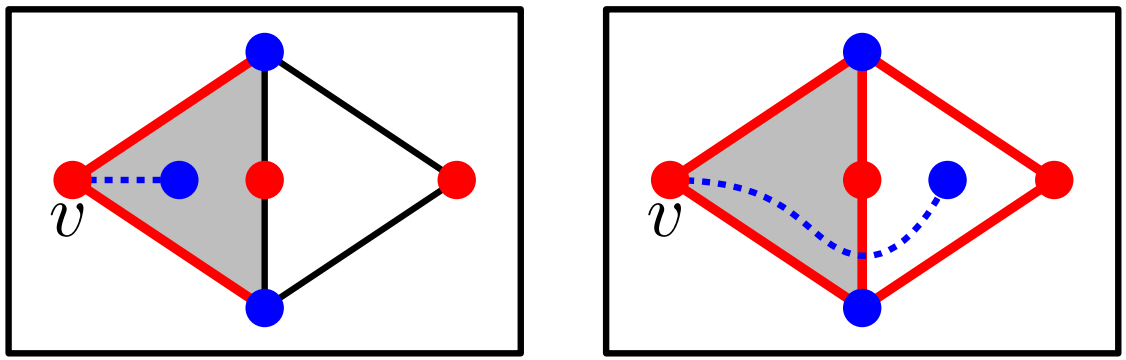

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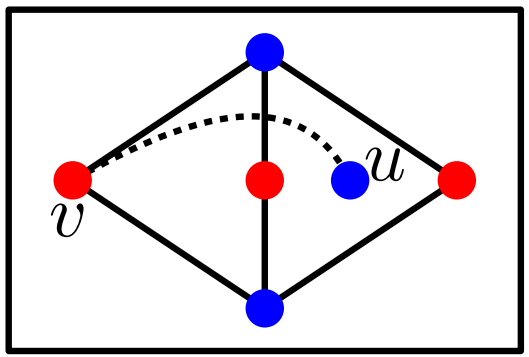

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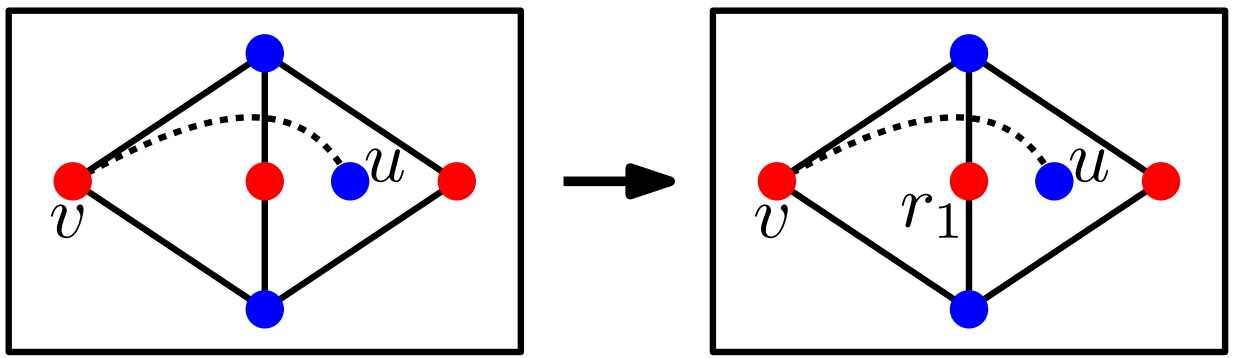

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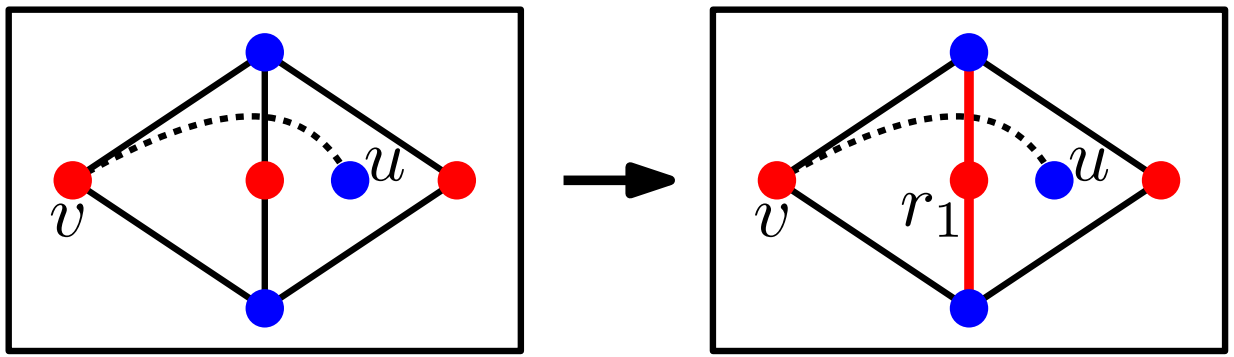

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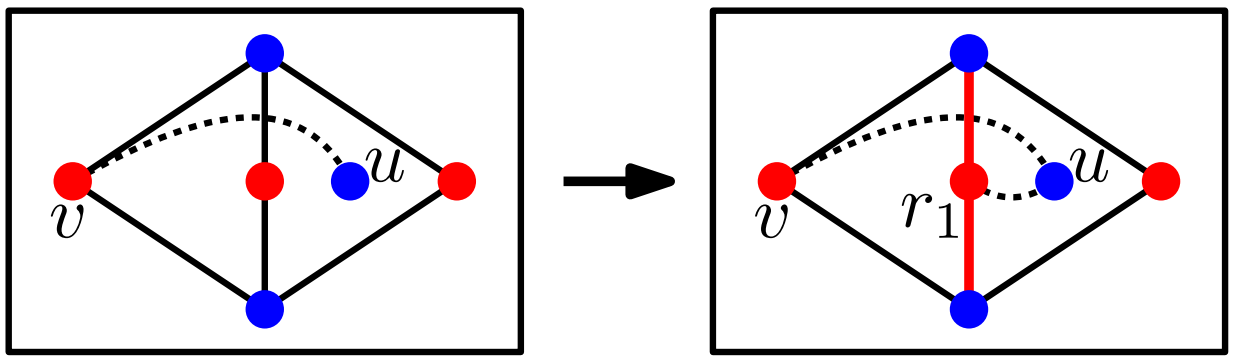

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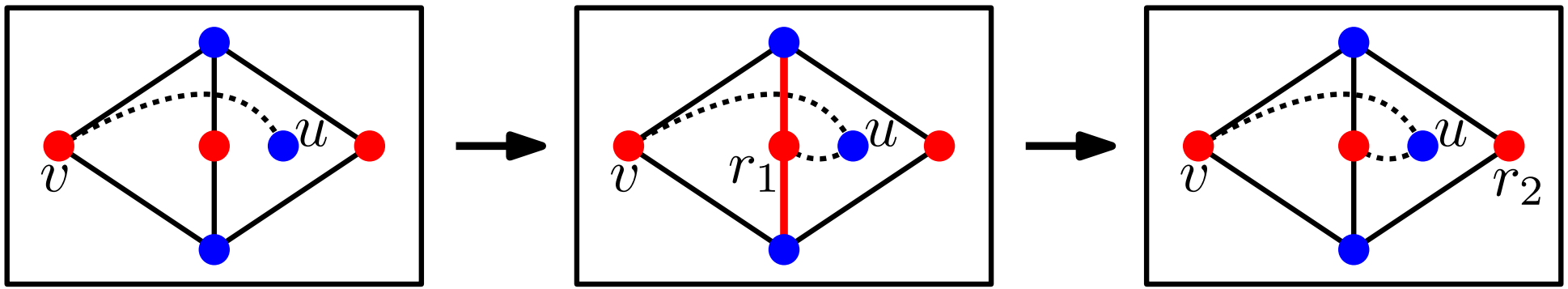

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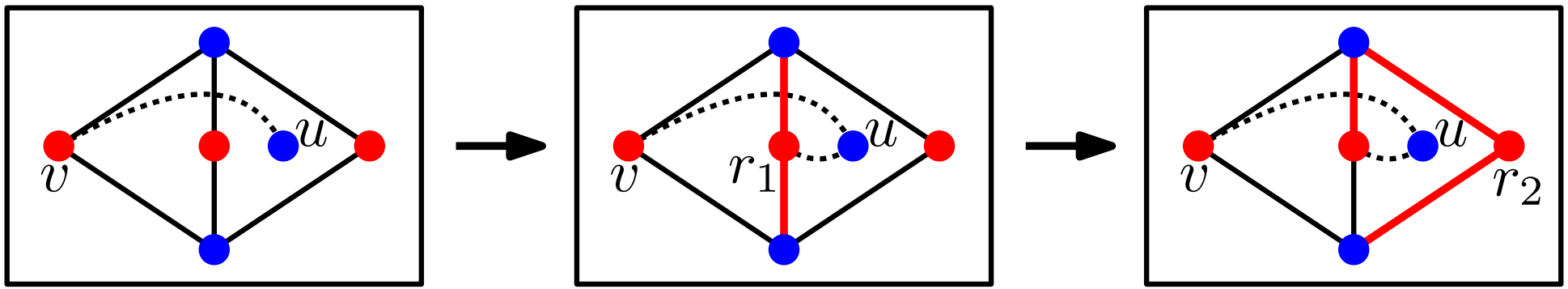

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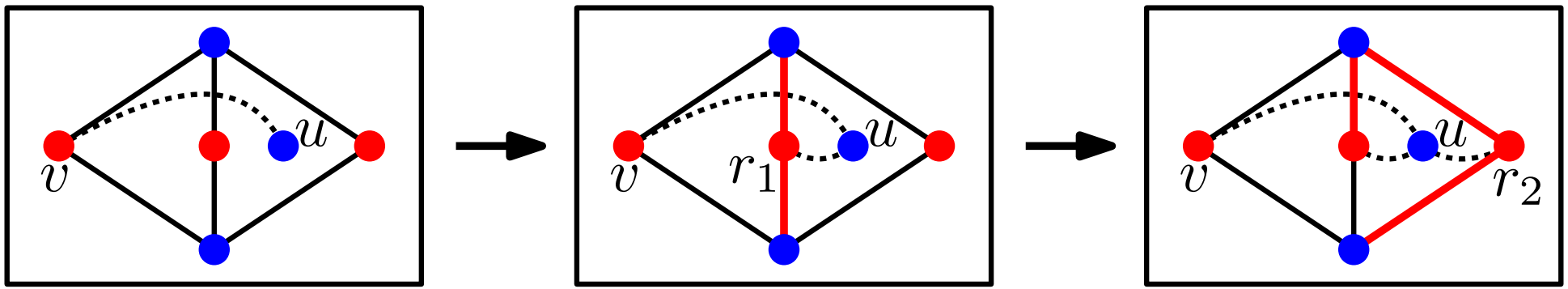

Example for
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Algorithm – Iso. Test

6. Delete duplicates from L'

Definition: D_1 and D_2 *isomorphic* $\Leftrightarrow$ D_1 and D_2 are equivalent after renaming vertices

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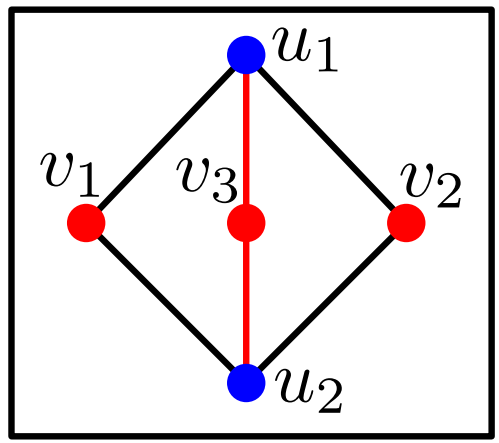

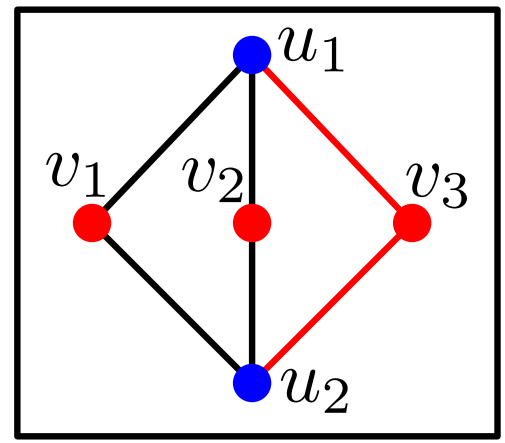

Algorithm – Iso. Test

Valid bijective mapping between planarizations Γ_1 of D_1 and Γ_2 of D_2 has following properties:

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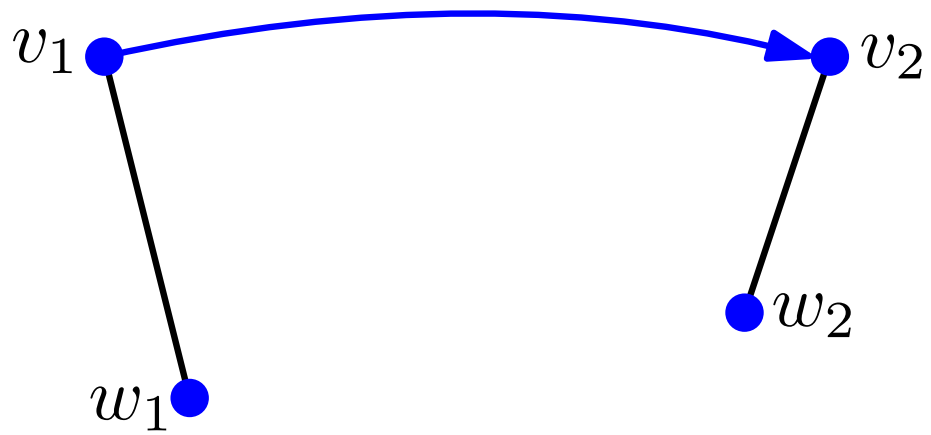

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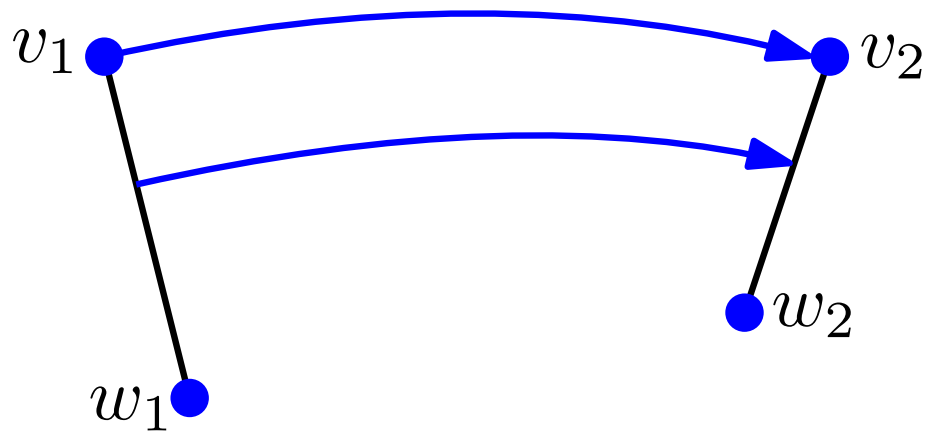

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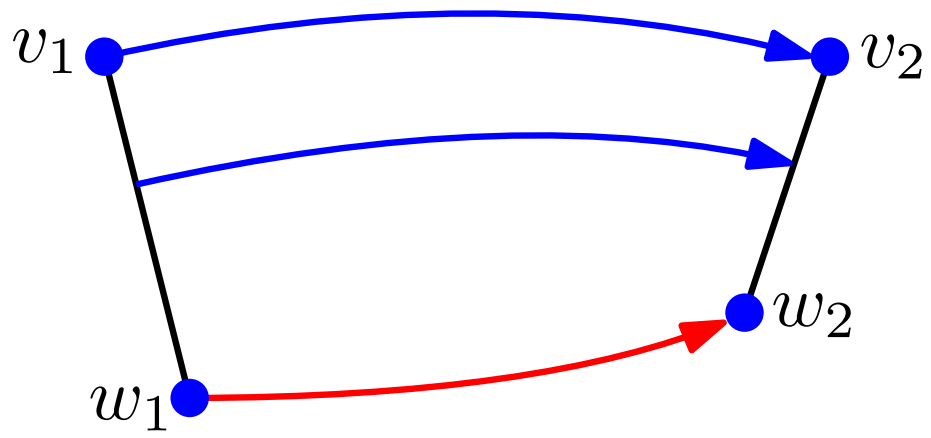

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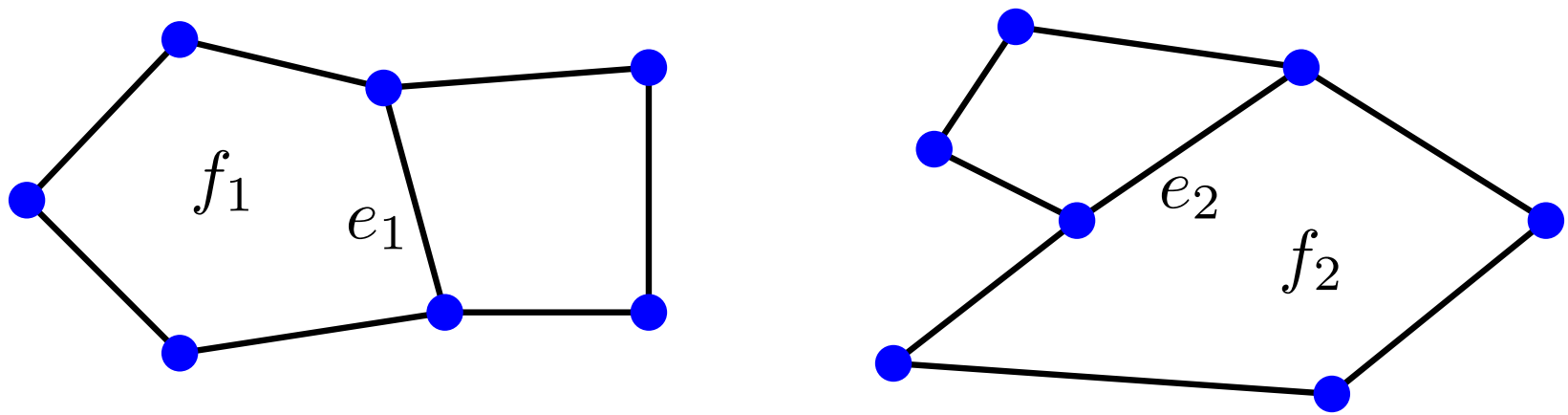

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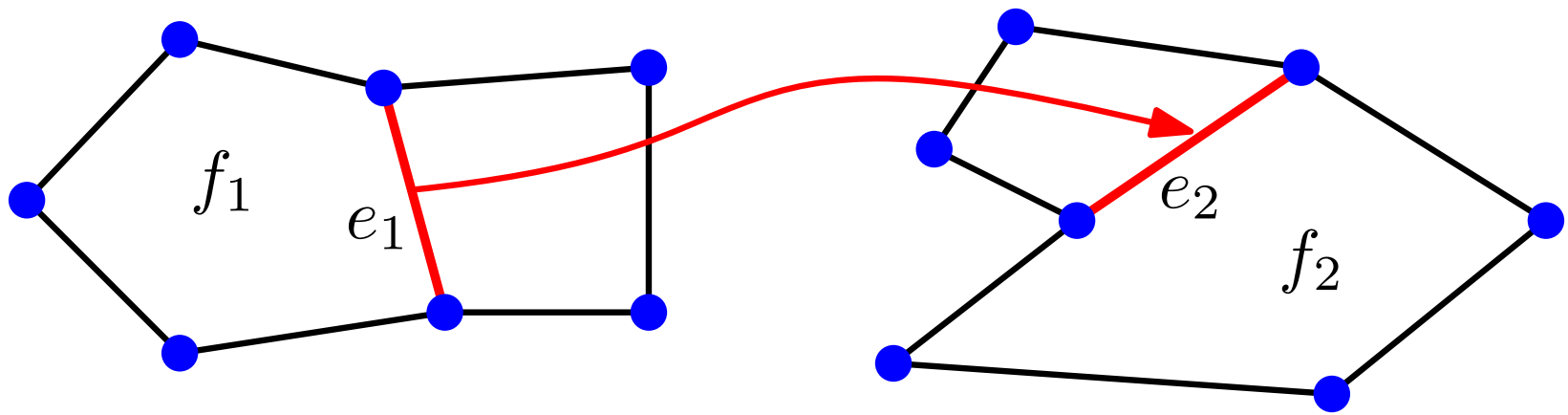

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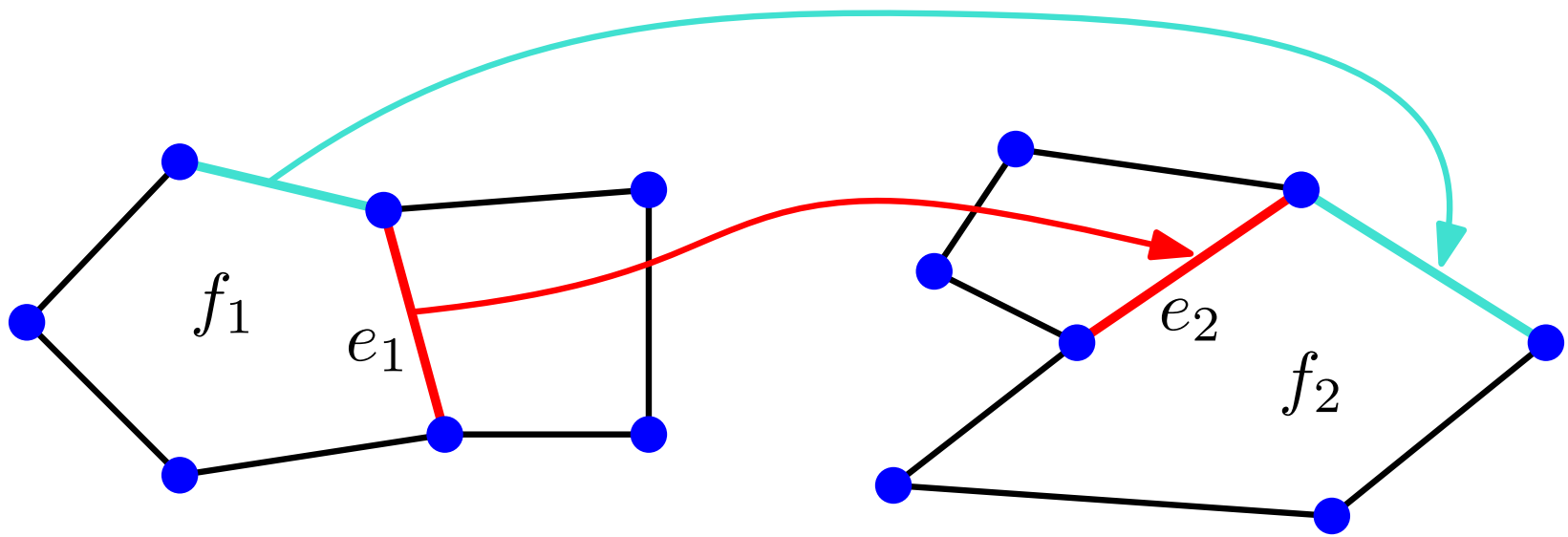

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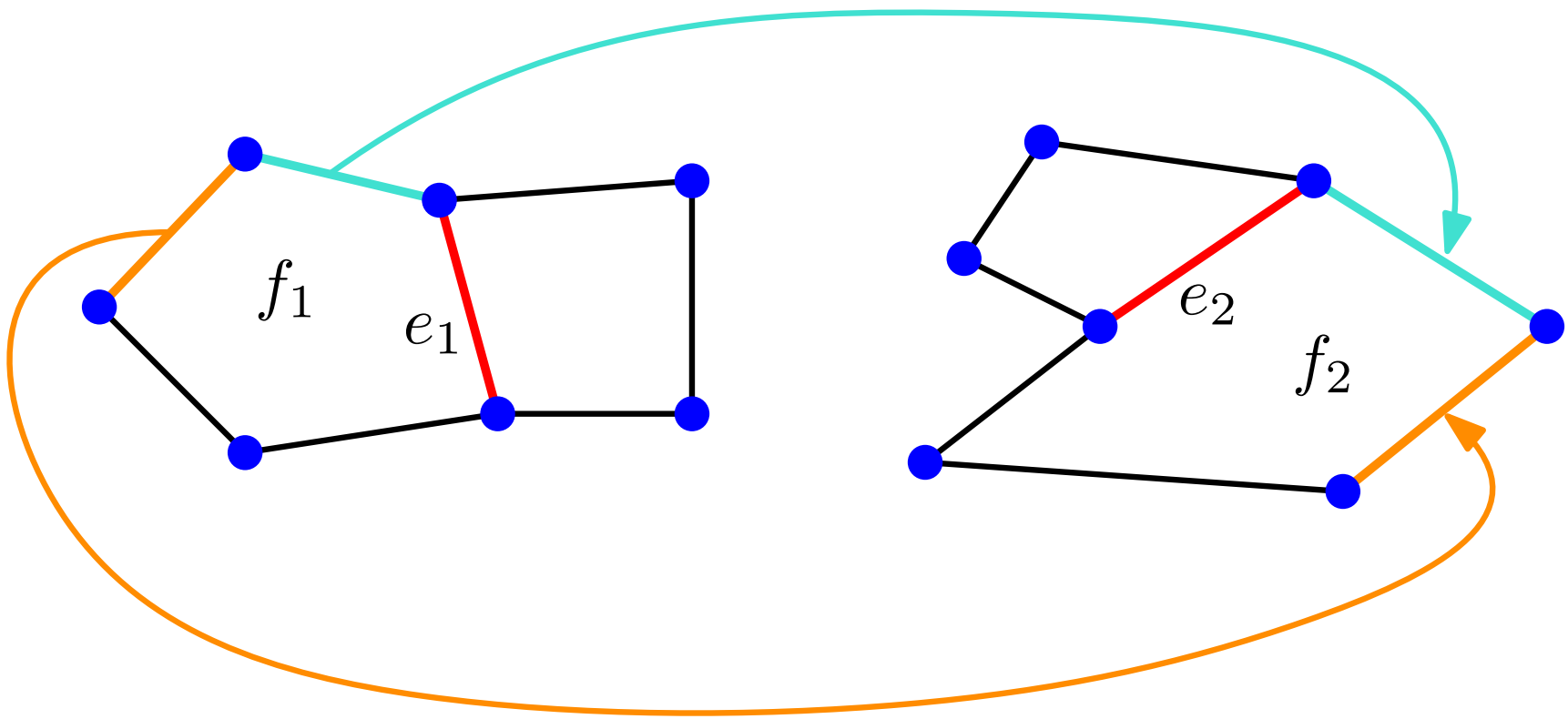

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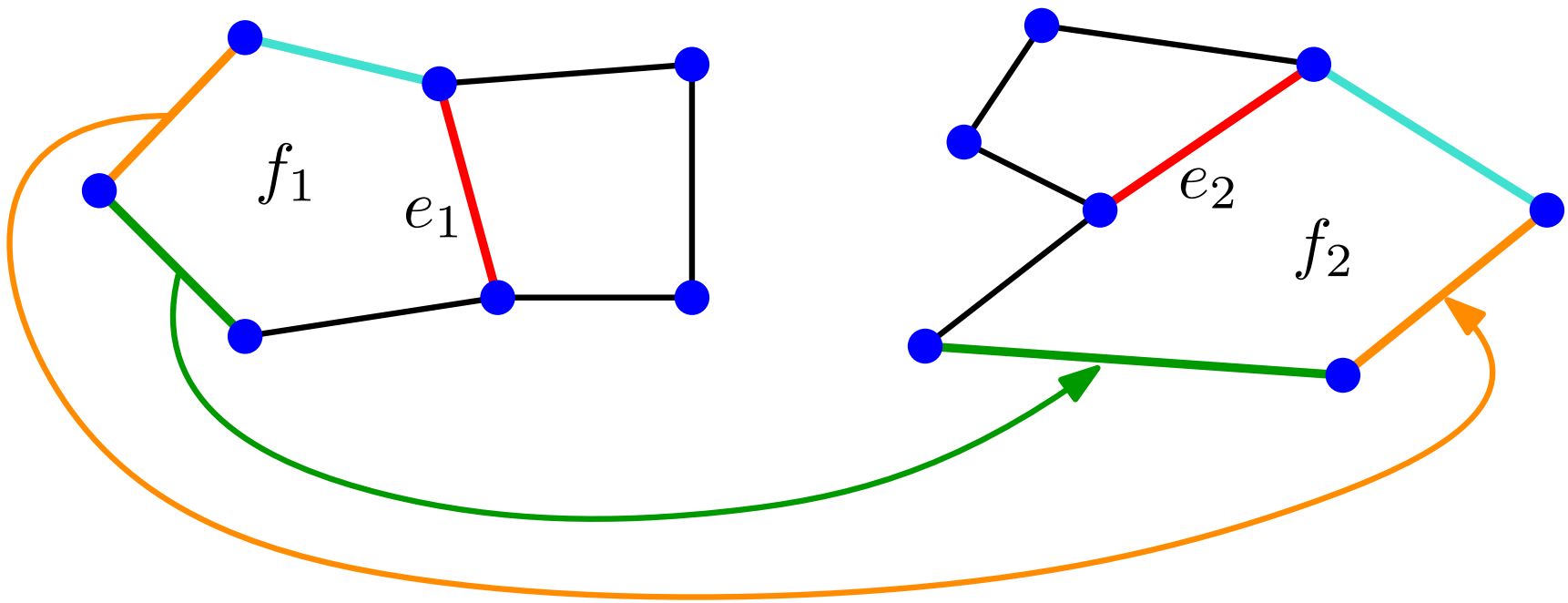

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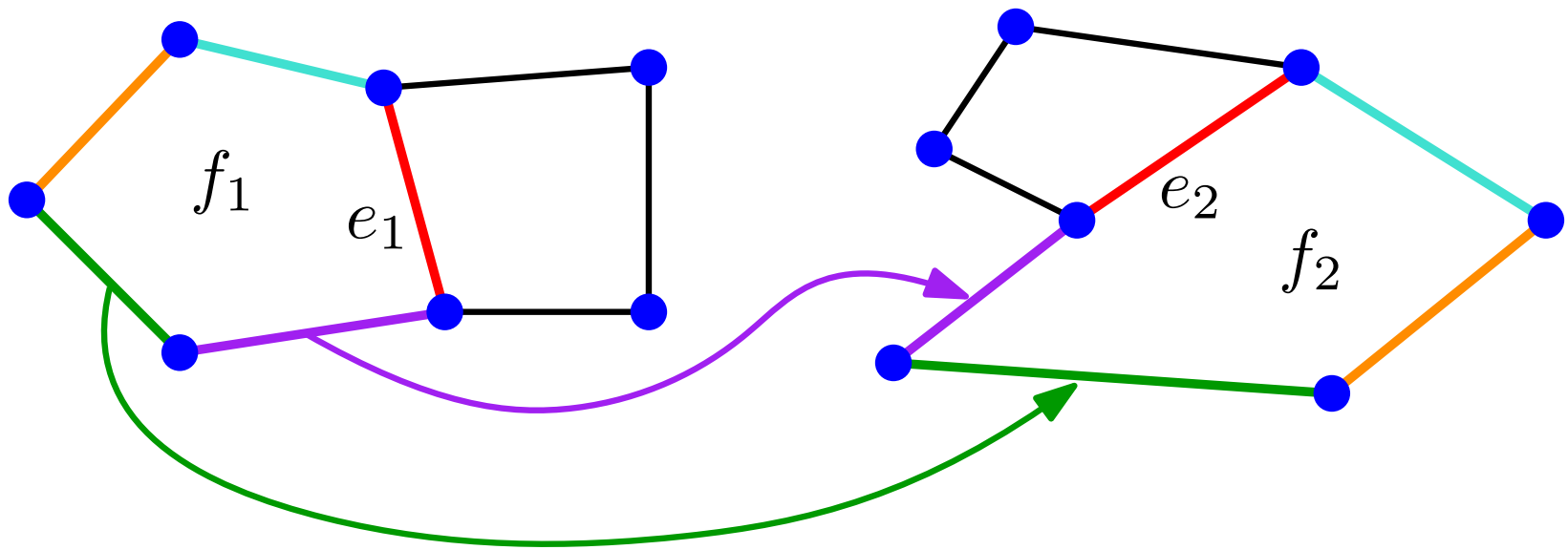

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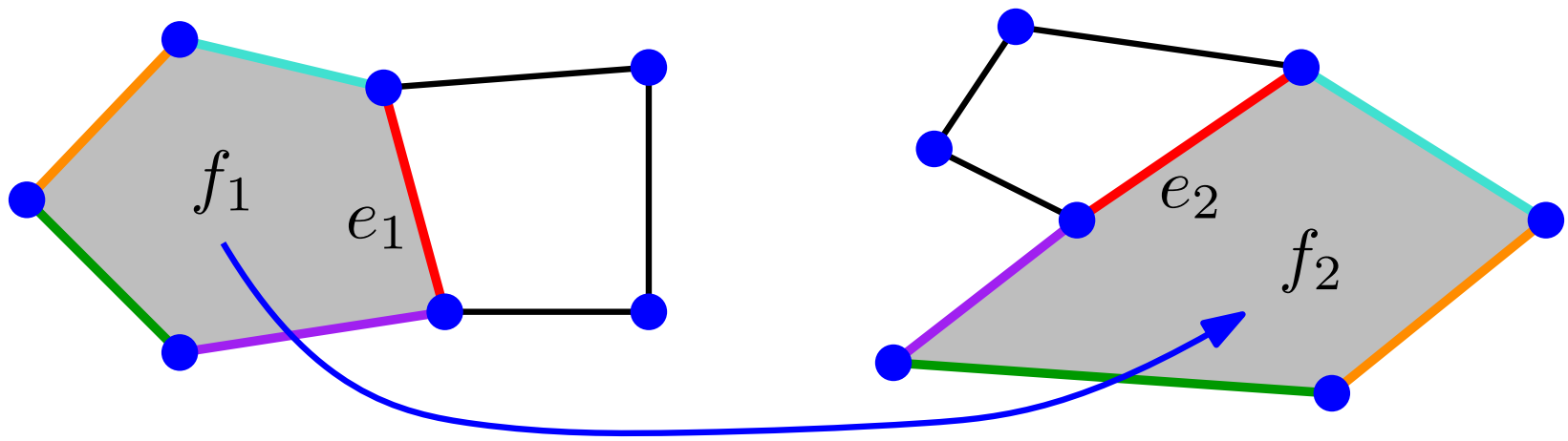

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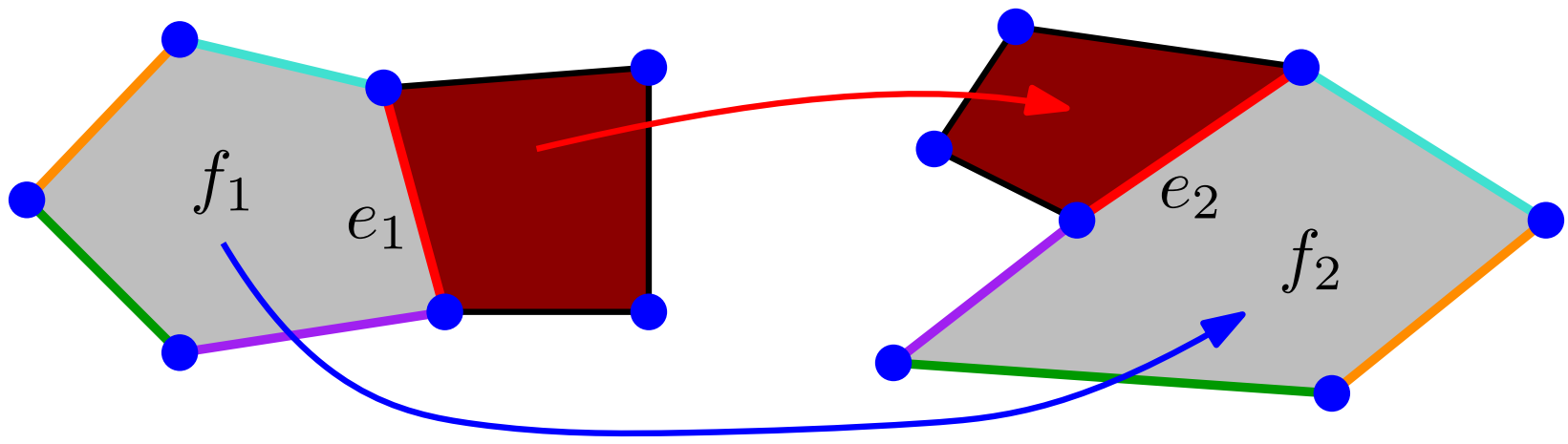

Results – Complete Graphs

class $\mathcal{C}$	$\in \mathcal{C}$	$\notin \mathcal{C}$
1-planar	K_6	K_7
2-planar	K_7	K_9
3-planar	K_8	K_{10}
4-planar	K_9	K_{11}
5-planar	K_9	K_{19}
fan-planar	K_7	K_9
fan-crossing free	K_6	K_7
gap-planar	K_8	K_9
quasi-planar	K_{10}	K_{11}

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$\rightarrow$ Answers a question by Bae et al (2018):
gap-planar $\not\subset$ fan-planar

Results – Complete Bipartite Graphs

class $\mathcal{C}$	$\in \mathcal{C}$	$\notin \mathcal{C}$
1-planar	$K_{3,6}$	$K_{3,7}$
	$K_{4,4}$	$K_{4,5}$
2-planar	$K_{3,10}$	$K_{3,11}$
	$K_{4,6}$	$K_{4,7}$
	$K_{4,5}$	$K_{5,5}$
3-planar	$K_{3,14}$	$K_{3,15}$
	$K_{4,9}$	$K_{4,10}$
	$K_{5,6}$	$K_{5,7}$
	$K_{5,6}$	$K_{6,6}$

Results – Complete Bipartite Graphs

class $\mathcal{C}$	$\in \mathcal{C}$	$\notin \mathcal{C}$
fan-planar	$K_{4,n}$	$K_{5,5}$
fan-crossing free	$K_{3,6}$	$K_{3,7}$
	$K_{4,6}$	$K_{4,7}$
	$K_{4,5}$	$K_{5,5}$
gap-planar	$K_{3,12}$	$K_{3,14}$
	$K_{4,8}$	$K_{4,10}$
	$K_{5,6}$	$K_{5,7}$
	$K_{5,6}$	$K_{6,6}$

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gap-planar	$K_{3,12}$	$K_{3,14}$
	$K_{4,8}$	~~$K_{4,10}$~~ $K_{4,9}$
	$K_{5,6}$	$K_{5,7}$
	$K_{5,6}$	$K_{6,6}$

Still open: $K_{3,13}$

Effect of Isomorphic Test

test	with iso. test		without iso. test	
	drawings	time	drawings	time
1-planar K_7	25	$\approx 0.1s$	158	$\approx 1.4s$
1-planar $K_{4,5}$	69	$\approx 0.2s$	2662	$\approx 24.3s$
2-planar K_8	191	$\approx 0.5s$	11366	$\approx 113s$
2-planar $K_{5,5}$	1473	$\approx 5s$	1423684	$\approx 6h$

Effect of Isomorphic Test

test	with iso. test		without iso. test	
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1-planar K_7	25	$\approx 0.1s$	158	$\approx 1.4s$
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2-planar $K_{5,5}$	1473	$\approx 5s$	1423684	$\approx 6h$

Statistics

2-planar	non-iso	time	3-planar	non-iso	time
$K_{2,3}$	6	90	$K_{2,3}$	6	234
$K_{3,3}$	19	254	$K_{3,3}$	69	1802
$K_{3,4}$	71	1458	$K_{3,4}$	1188	16969
$K_{4,4}$	38	1152	$K_{4,4}$	2704	97801
$K_{4,5}$	37	1826	$K_{4,5}$	7653	310194
$K_{5,5}$	0	357	$K_{5,5}$	1899	199908
			$K_{5,6}$	438	47396
			$K_{6,6}$	0	4822
Sum	171	5137	Sum	13957	679126

Time in milliseconds

Statistics

2-planar	non-iso	time	3-planar	non-iso	time
$K_{2,3}$	6	90	$K_{2,3}$	6	234
$K_{3,3}$	19	254	$K_{3,3}$	69	1802
$K_{3,4}$	71	1458	$K_{3,4}$	1188	16969
$K_{4,4}$	38	1152	$K_{4,4}$	2704	97801
$K_{4,5}$	37	1826	$K_{4,5}$	7653	310194
$K_{5,5}$	0	357	$K_{5,5}$	1899	199908
			$K_{5,6}$	438	47396
			$K_{6,6}$	0	4822
Sum	171	5137	Sum	13957	679126
		$\approx$ 5 s			$\approx$ 11 min

Time in milliseconds

Statistics

2-planar	non-iso	time	3-planar	non-iso	time
$K_{2,3}$	6	90	$K_{2,3}$	6	234
$K_{3,3}$	19	254	$K_{3,3}$	69	1802
$K_{3,4}$	71	1458	$K_{3,4}$	1188	16969
$K_{4,4}$	38	1152	$K_{4,4}$	2704	97801
$K_{4,5}$	37	1826	$K_{4,5}$	7653	310194
$K_{5,5}$	0	357	$K_{5,5}$	1899	199908
			$K_{5,6}$	438	47396
			$K_{6,6}$	0	4822
Sum	171	5137	Sum	13957	679126
		$\approx$ 5 s			$\approx$ 11 min

Time in milliseconds

Limits

4-planar	non-iso	time
$K_{2,3}$	6	108
$K_{3,3}$	102	2146
$K_{3,4}$	6194	163000
$K_{4,4}$	81817	34096183
$K_{4,5}$	? ? ?	

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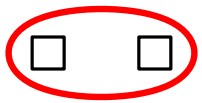
- long time before you have a 'good' (many vertices) drawing
- **New idea:** search only for drawings
- change original algorithm slightly
  $\Rightarrow$ DFS variant

DFS Variant of the Algorithm

Current Variant (BFS)



$K_{2,2}$

$K_{2,3}$

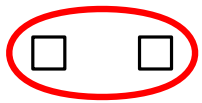
$K_{3,3}$

$K_{3,4}$

$K_{4,4}$

$K_{4,5}$

New Variant (DFS)



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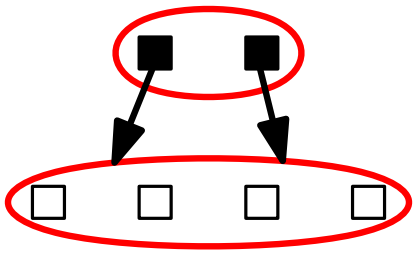
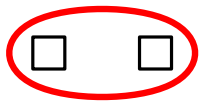


New Variant (DFS)



$K_{2,2}$

$K_{2,3}$

$K_{3,3}$

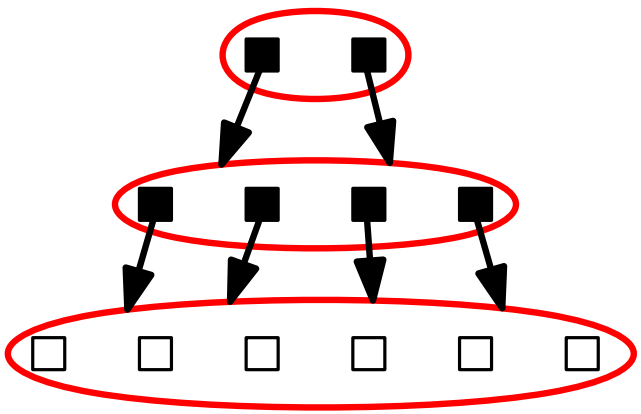
$K_{3,4}$

$K_{4,4}$

$K_{4,5}$

DFS Variant of the Algorithm

Current Variant (BFS)



$K_{2,2}$

$K_{2,3}$

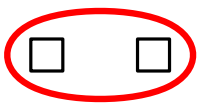
$K_{3,3}$

$K_{3,4}$

$K_{4,4}$

$K_{4,5}$

New Variant (DFS)



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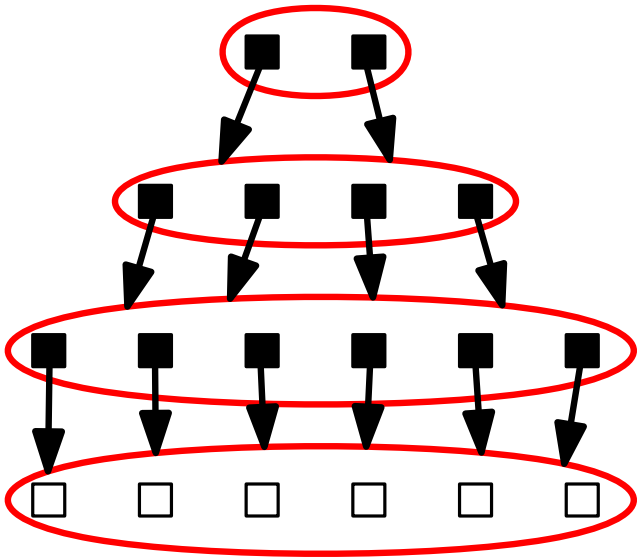


$K_{2,2}$

$K_{2,3}$

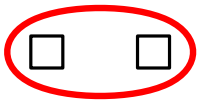
$K_{3,3}$

$K_{3,4}$

$K_{4,4}$

$K_{4,5}$

New Variant (DFS)



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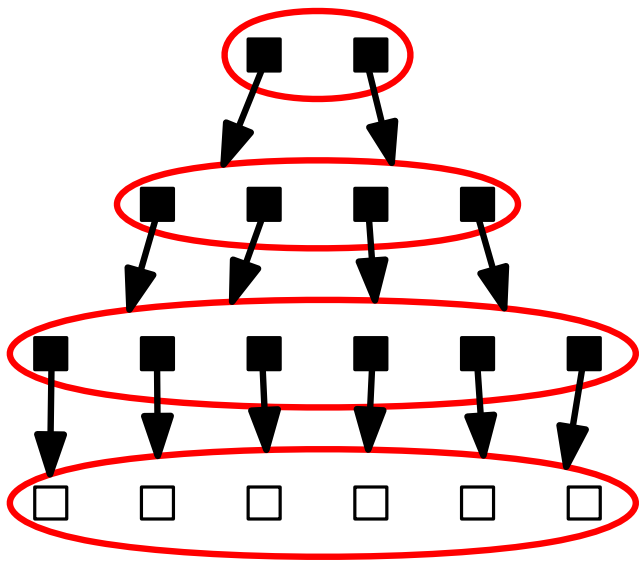


$K_{2,2}$

$K_{2,3}$

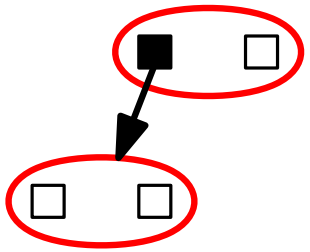
$K_{3,3}$

$K_{3,4}$

$K_{4,4}$

$K_{4,5}$

New Variant (DFS)



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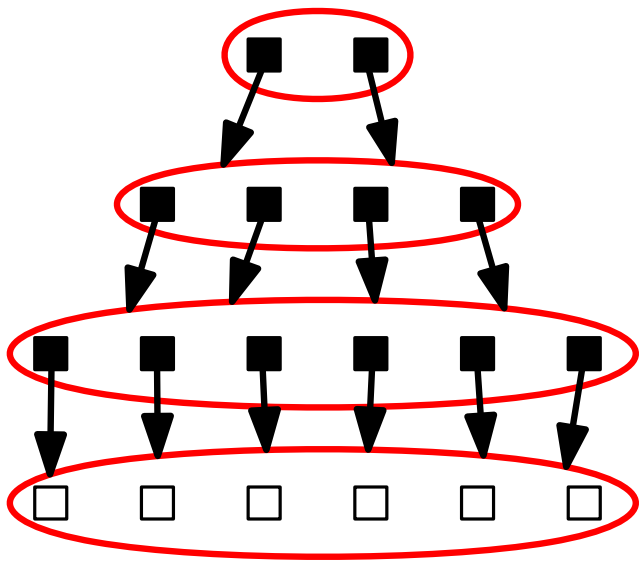

$K_{2,2}$

$K_{2,3}$

$K_{3,3}$

$K_{3,4}$

$K_{4,4}$

$K_{4,5}$

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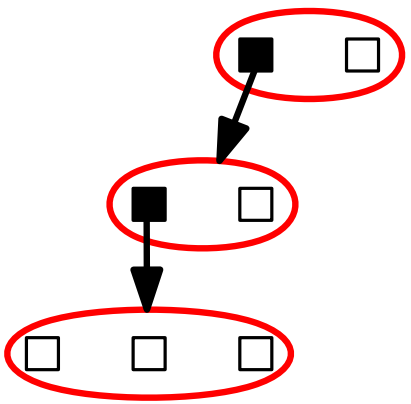

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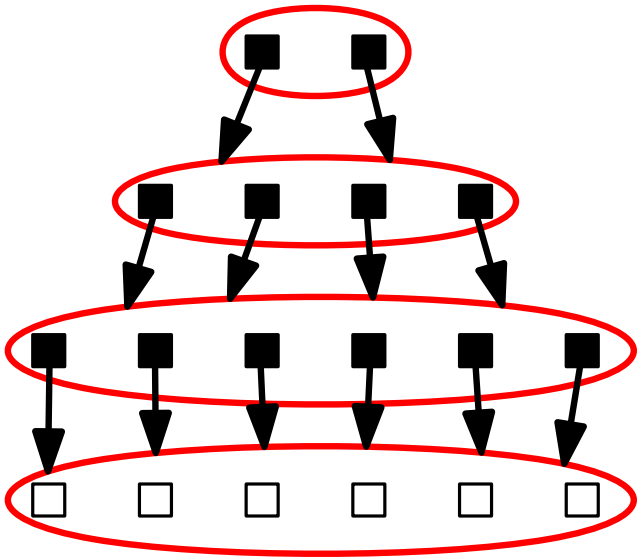

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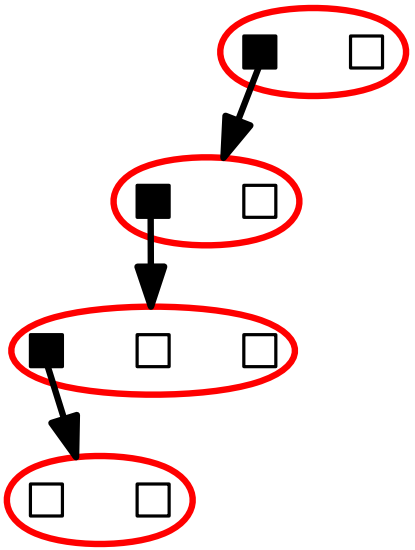

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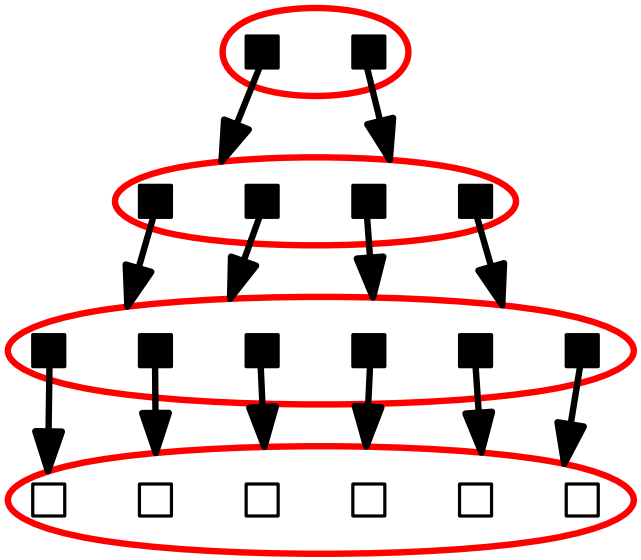

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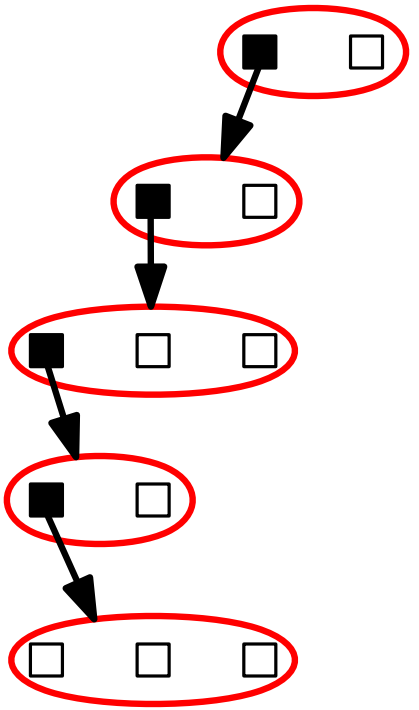

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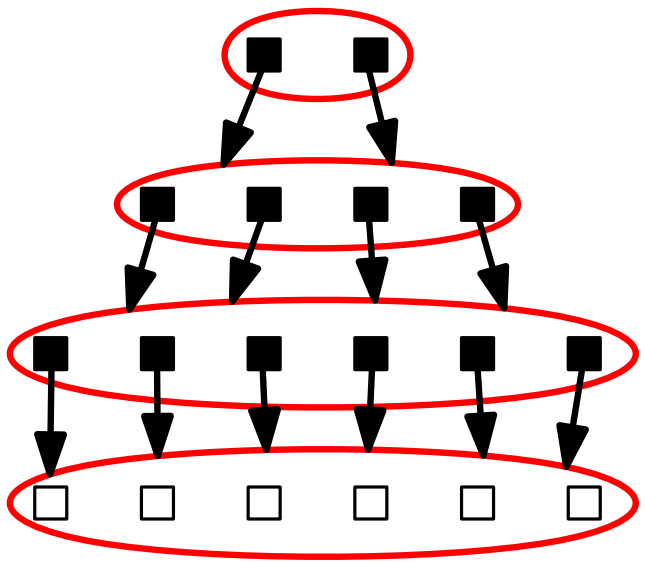

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$K_{3,4}$

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$K_{4,5}$

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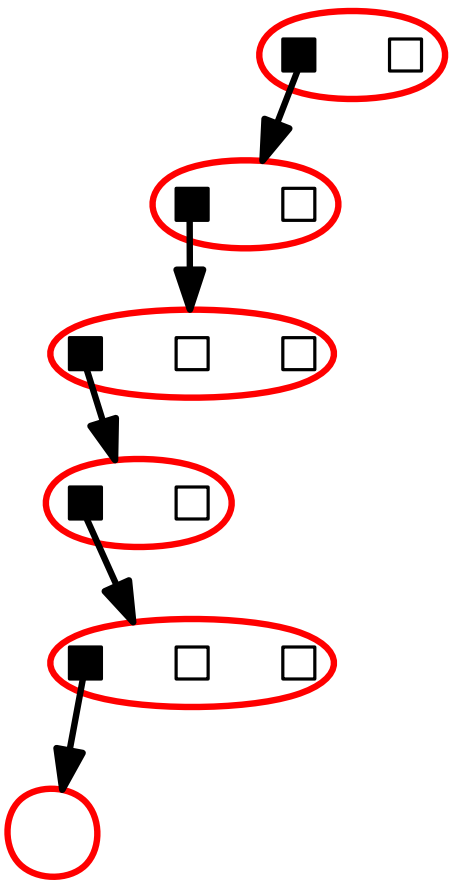

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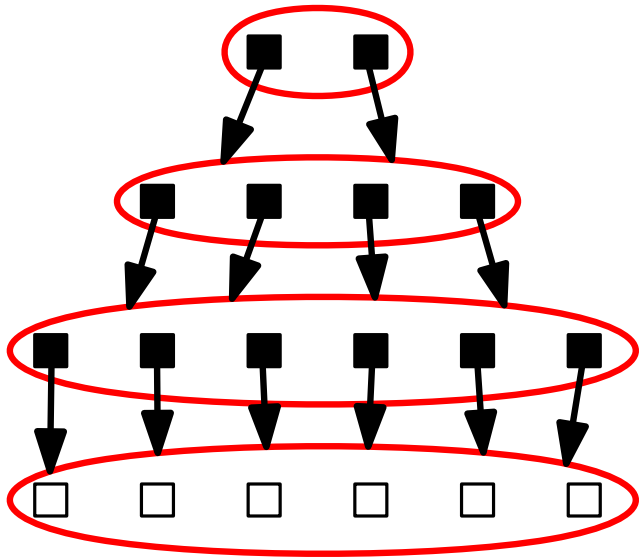

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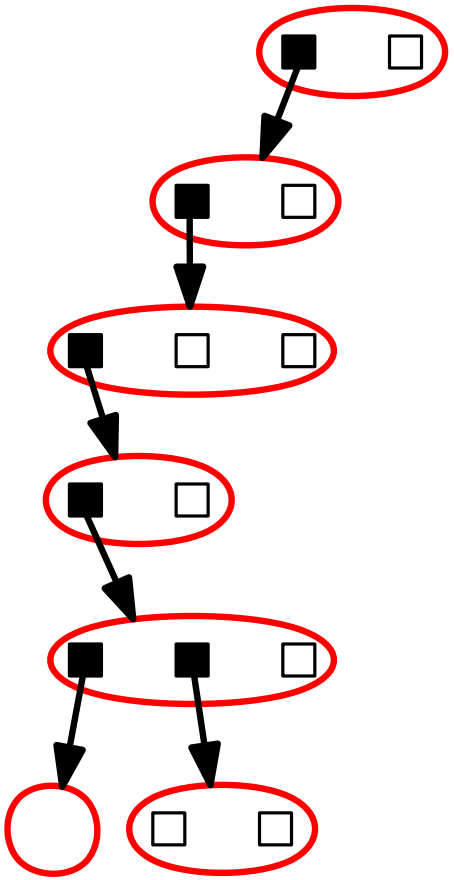

Results

class $\mathcal{C}$	$\in \mathcal{C}$	$\notin \mathcal{C}$
4-planar	$K_{3,18}$	$K_{3,19}$
	$K_{4,11}$	$K_{4,19}$
	$K_{5,8}$	$K_{5,19}$
	$K_{6,6}$	$K_{6,19}$
quasi-planar	$K_{4,n}$	–
	$K_{5,18}$	?
	$K_{6,10}$	?
	$K_{7,7}$	$K_{7,52}$

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quasi-planar	$K_{4,n}$	–
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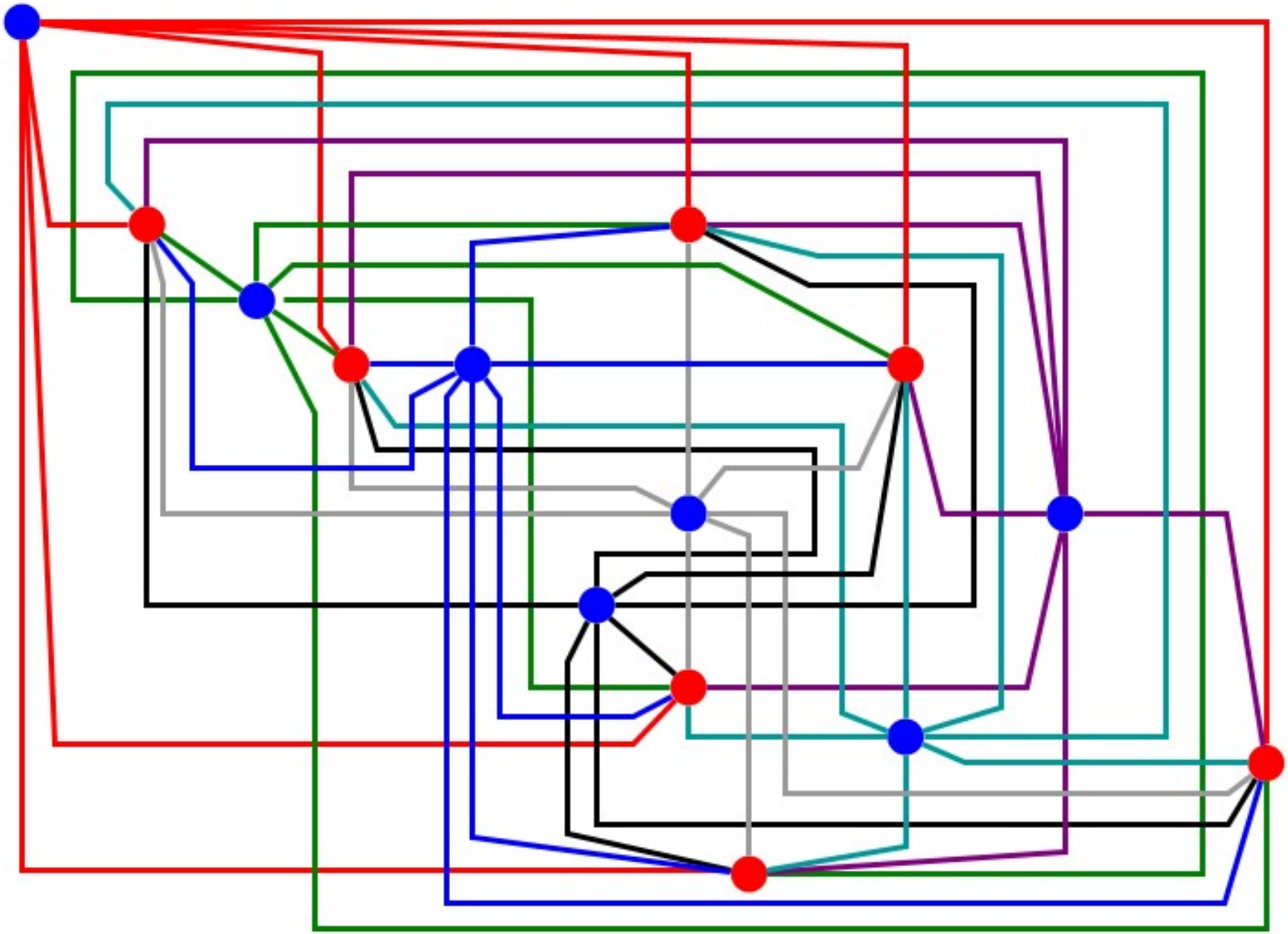
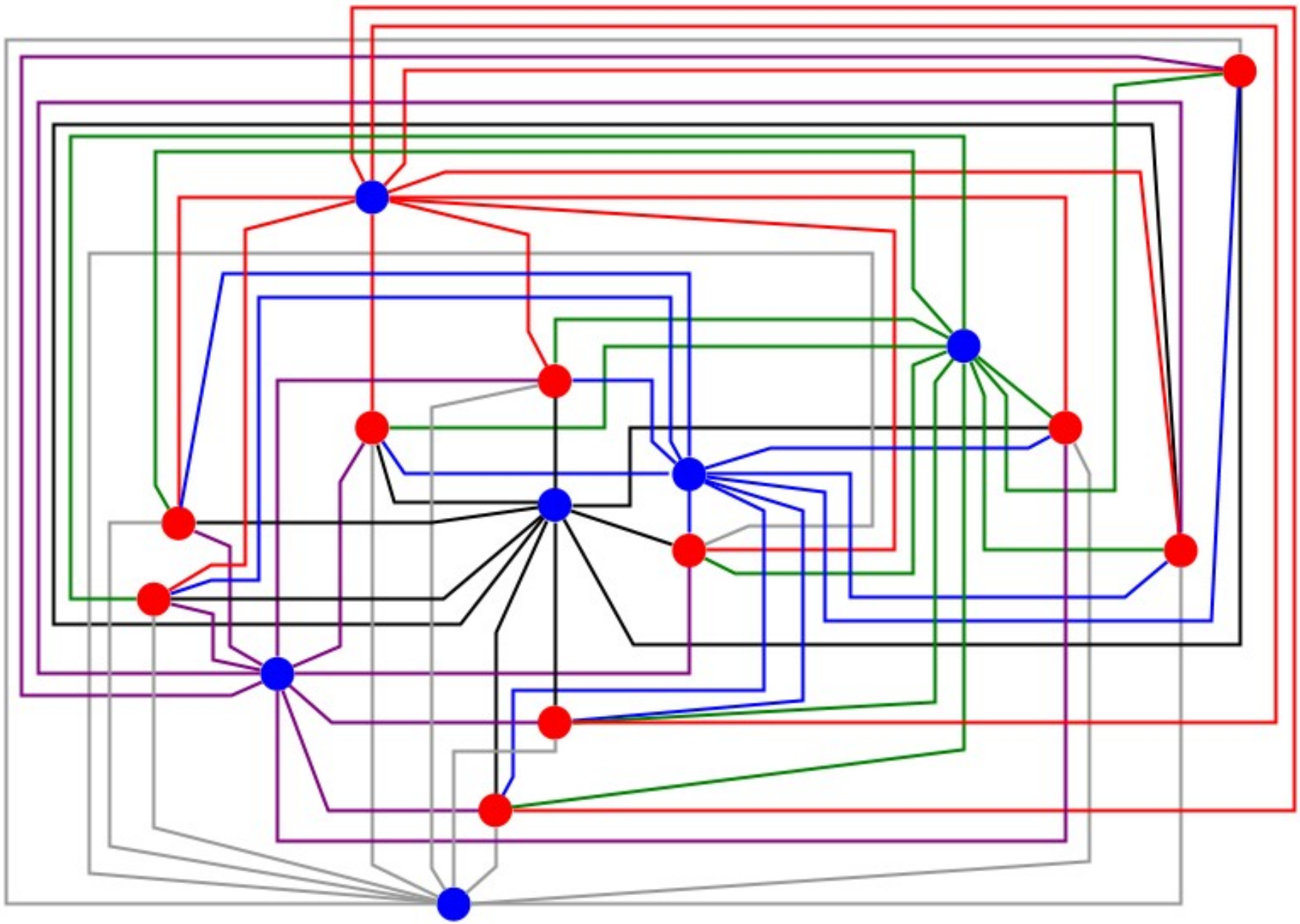
	$K_{6,10}$	?
	$K_{7,7}$	$K_{7,52}$

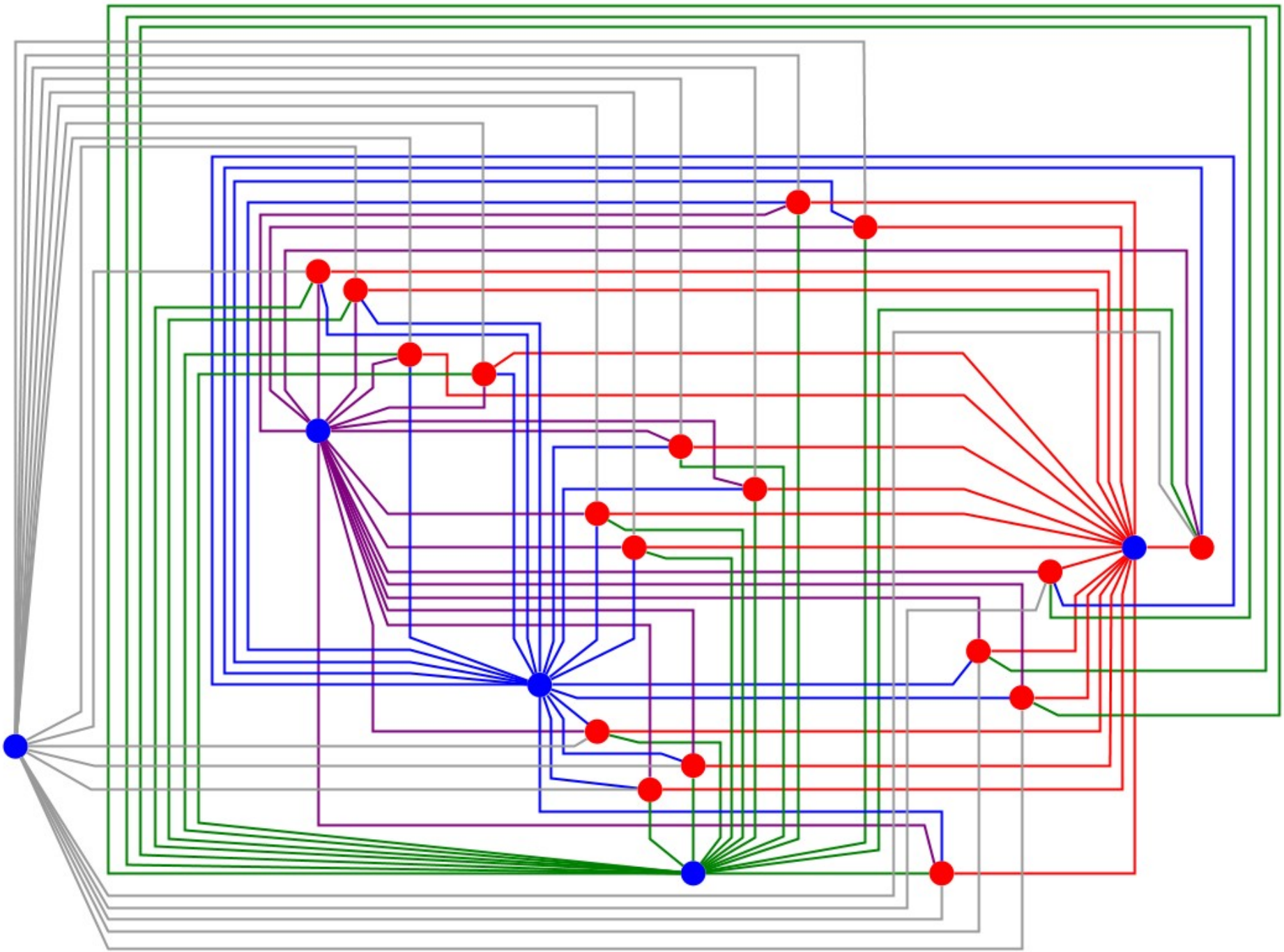
Quasiplanar Drawing of $K_{7,7}$



Quasiplanar Drawing of $K_{6,10}$



Quasiplanar Drawing of $K_{5,18}$



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Thank you for your attention !