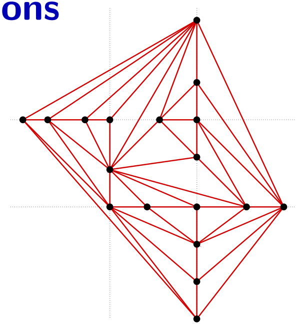
4-Connected Triangulations on Few Lines



GD 2019

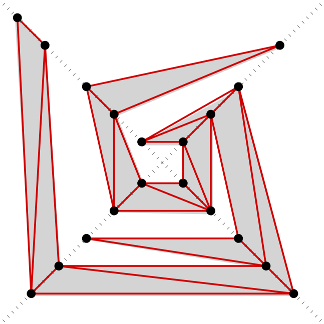
September 20., 2019

Průhonice/Prague

Stefan Felsner (TUB, Berlin)

Line Cover Number

$$\pi(G) = \min\left(\ell \ : \ \exists\text{plane drawing of } G \text{ with vertices covered by } \ell \text{ lines}\right)$$

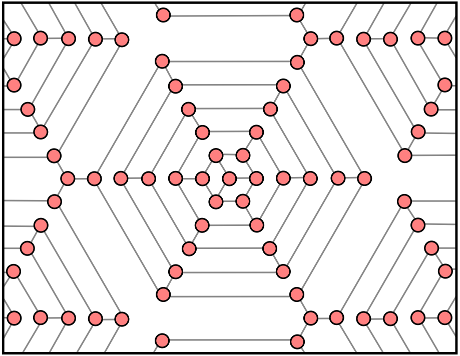


Classes with $\pi(G) = 2$:

- trees
- outerplanar
- grids

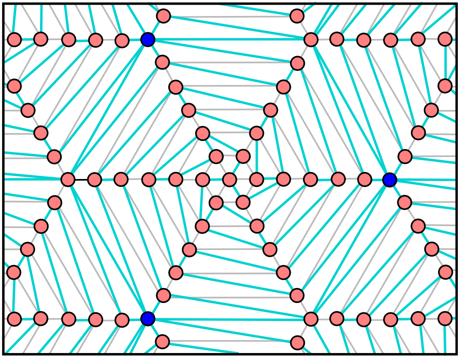
Lower bound

Theorem [Eppstein, SoCG 19 **].** $\exists$ planar, bipartite, cubic, 3-connected graphs G_n with $\pi(G_n) \in \Omega(n^{1/3})$.



Lower bound

Corollary. $\exists$ planar 4-connected graphs G_n with $\pi(G_n) \in \Omega(n^{1/3})$.



Our contribution

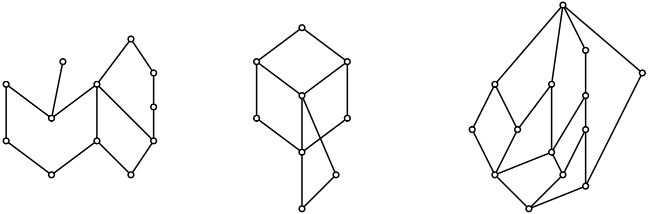
Theorem. For all G planar 4-connected $\pi(G) \leq \sqrt{2n}$.

Tools:

- planar lattices
- orthogonal partitions of posets
- transversal structures

Posets & Lattices

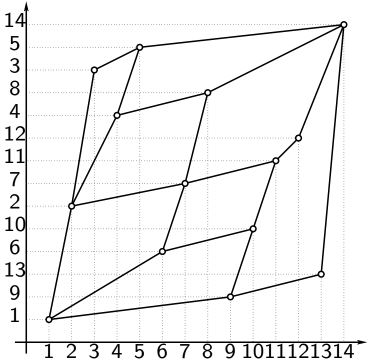
3 diagrams.

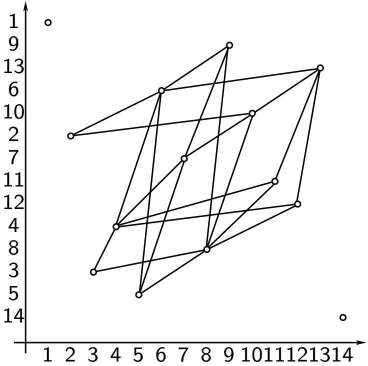


- a planar poset of dimension 3
- a non-planar lattice
- a planar lattice.

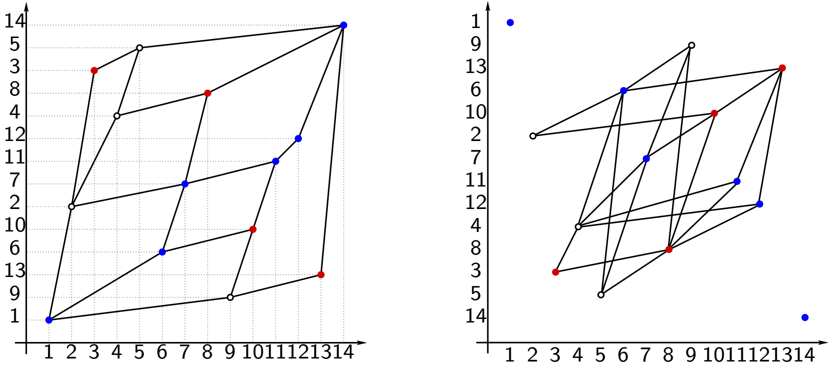
Theorem.
A finite lattice is planar if and only if its dimension is ≤ 2.

A planar lattice and its conjugate

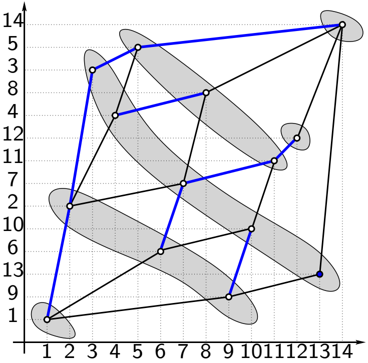


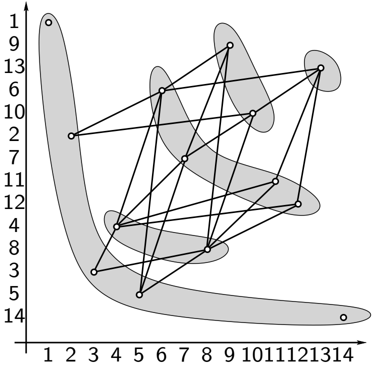


Chains, antichains, width, and height



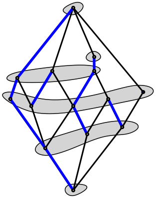
Canonical chain and antichain partitions





Planar lattices on h lines

Proposition. A planar lattice of height h has a diagram with points on h horizontal lines.



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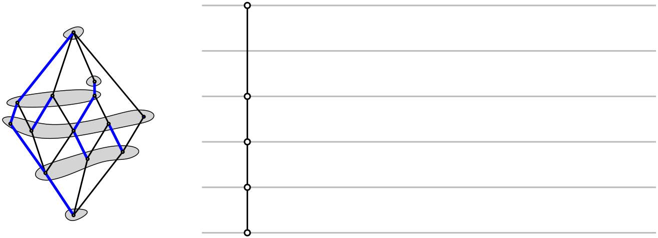

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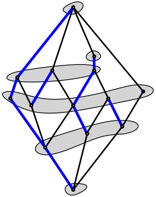

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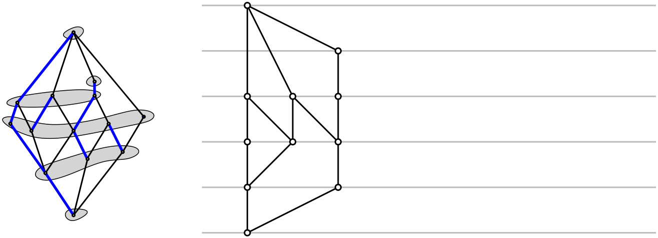

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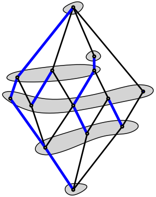
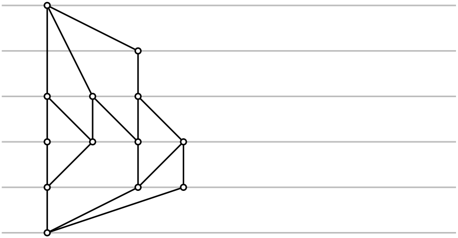

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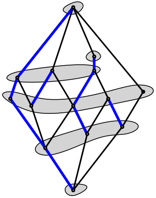
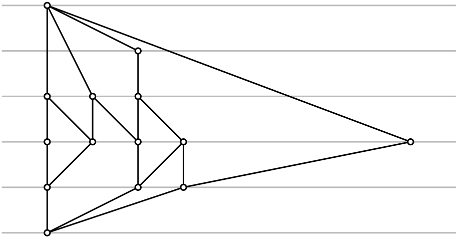
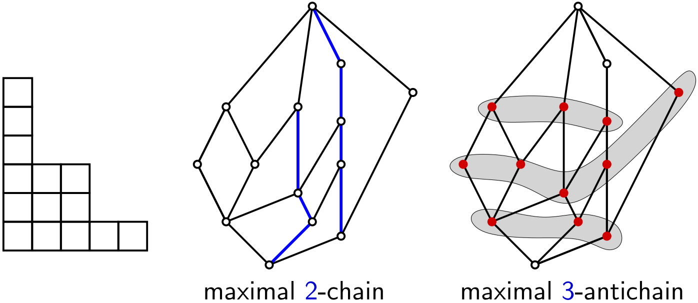




Greene–Kleitman theory

With a poset P with n elements there is a partition λ of n, such that for the Ferrer's diagram $G(P)$ of λ we have:

- The number of squares in the ℓ longest columns of $G(P)$ equals the maximal number of elements covered by an ℓ-chain.
- The number of squares in the k longest rows of $G(P)$ equals the maximal number of elements covered by a k-antichain.

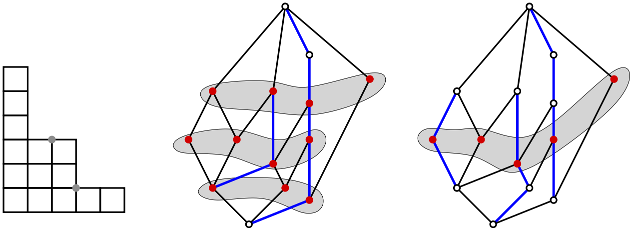


Orthogonal pairs

A chain family $\mathcal{C}$ and an antichain family $\mathcal{A}$ are orthogonal iff

1. $$P = \Big(\bigcup_{A\in\mathcal{A}} A\Big) \cup \Big(\bigcup_{C\in\mathcal{C}} C\Big), \text{ and}$$
2. $|A \cap C| = 1$ for all $A \in \mathcal{A},\ C \in \mathcal{C}$.

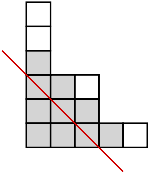
Theorem [Frank '80]. If (ℓ, k) is a corner of $G(P)$, then there is an orthogonal pair consisting of a ℓ-chain $\mathcal{C}$ and a k-antichain $\mathcal{A}$.



Orthogonal pairs

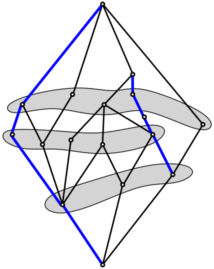
Theorem [Frank '80]. If (ℓ, k) is on the boundary of $G(P)$, then there is an orthogonal pair consisting of a ℓ-chain $\mathcal{C}$ and a k-antichain $\mathcal{A}$.

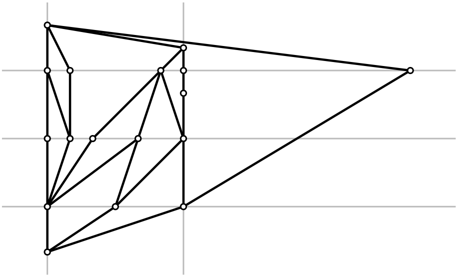
Corollary. A poset with n elements has (ℓ, k) an orthogonal pair consisting of a ℓ-chain $\mathcal{C}$ and a k-antichain with $k + \ell \leq \sqrt{2n} - 1$.



Planar lattices on $\leq \sqrt{2n} - 1$ lines

Proposition. A planar lattice with n elements has a diagram with points on k horizontal lines and ℓ vertical lines where $k + \ell \leq \sqrt{2n} - 1$.



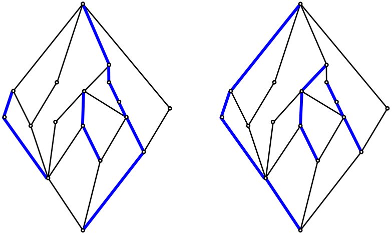


Adjusting chains and antichains

Lemma. $\mathcal{C}, \mathcal{A}$ an orthogonal pair of P

- $\mathcal{C}'$ the canonical chain partition of $P_{\mathcal{C}}$
- $\mathcal{A}'$ the canonical antichain partition of $P_{\mathcal{A}}$

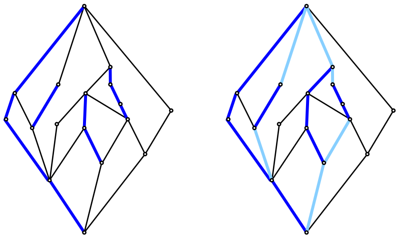
$\implies \mathcal{C}', \mathcal{A}'$ is an orthogonal pair of P.



Canonical chain cover

Lemma. $(C_1, \ldots, C_\ell)$ the canonical chain partition of $P_{\mathcal{C}}$ $\implies$ there are extensions C_i^+ of C_i such that

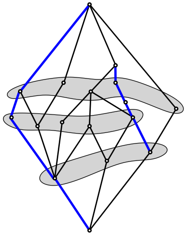
- C_i^+ is a maximal chain of $P_{\mathcal{C}}$
- $C_i^+ \subseteq \bigcup_{j \leq i} C_j$

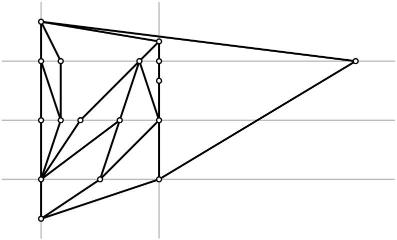


Phase i

In phase i we add all elements between C_{i-1}^+ and C_i^+ including C_i^+

- add right ears to C_{i-1}^+
- draw C_i and add left ears to C_i
- add the connecting edges, chains, components

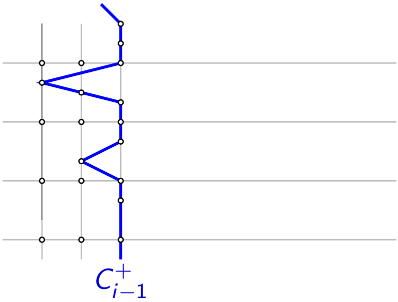




Phase i

In phase i we add all elements between C_{i-1}^{+} and C_{i}^{+} including C_{i}^{+}

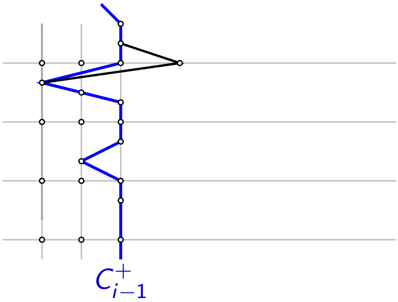
- add right ears to C_{i-1}^{+}



Phase i

In phase i we add all elements between C^+_{i-1} and C^+_i including C^+_i

- add right ears to C^+_{i-1}



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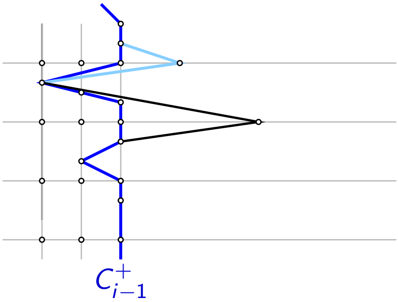
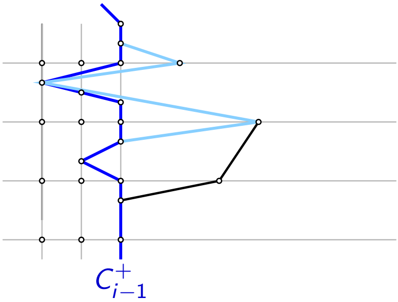


Phase i

In phase i we add all elements between C_{i-1}^{+} and C_i^{+} including C_i^{+}

- add right ears to C_{i-1}^{+}



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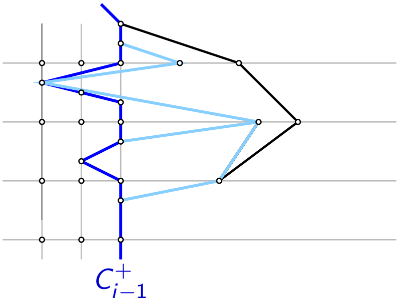

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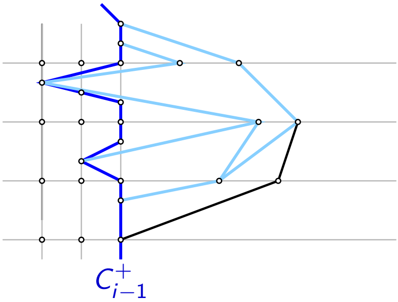

Phase i

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- add right ears to C_{i-1}^+
- draw C_i and add left ears to C_i

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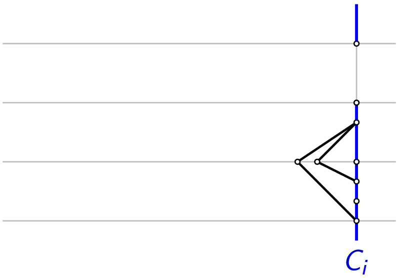
- draw C_i and add left ears to C_i



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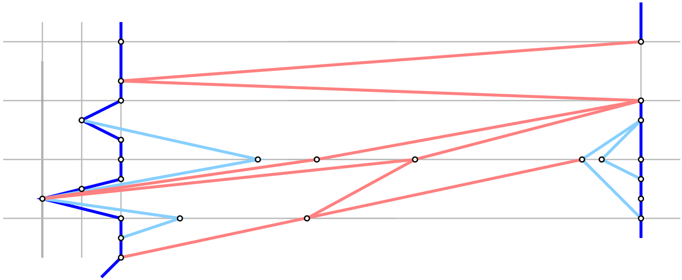
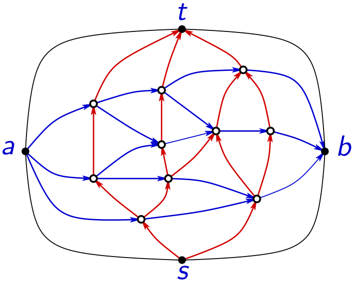
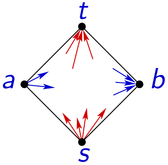
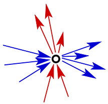


Transversal structures

Proposition. 4-connected inner triangulations of a quadrangle admit transversal structures (a.k.a. regular edge labeling).

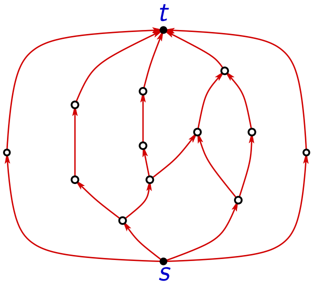






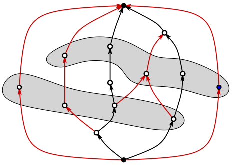
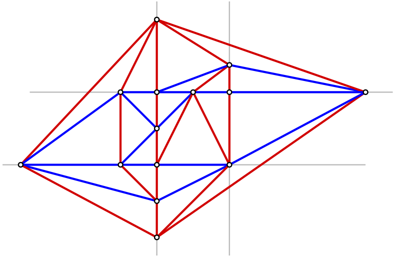
Transversal structures and planar lattices

Proposition. The red graph of a transversal structure is the diagram of a planar lattice.



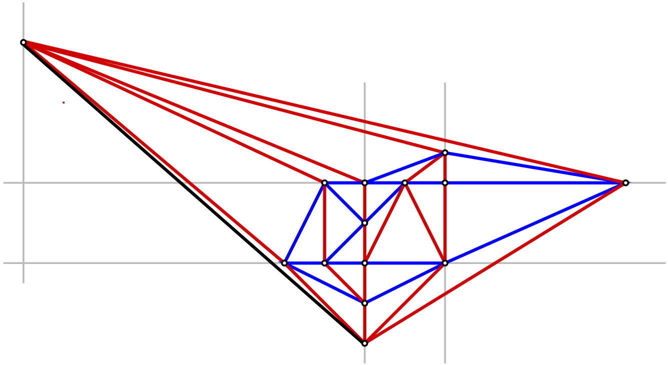
4-connected planar

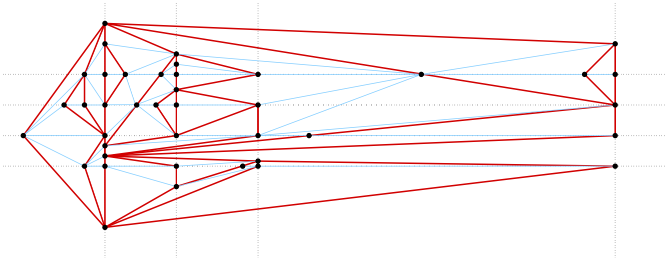
Proposition. The blue edges can be included while drawing the red lattice.

4-connected planar

1 extra line for the missing edge.





Thank you