Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Chair for
INFORMATICS I
Efficient Algorithms and
Knowledge-Based Systems

Institute for Informatics

Line & Plane Cover Numbers Revisited

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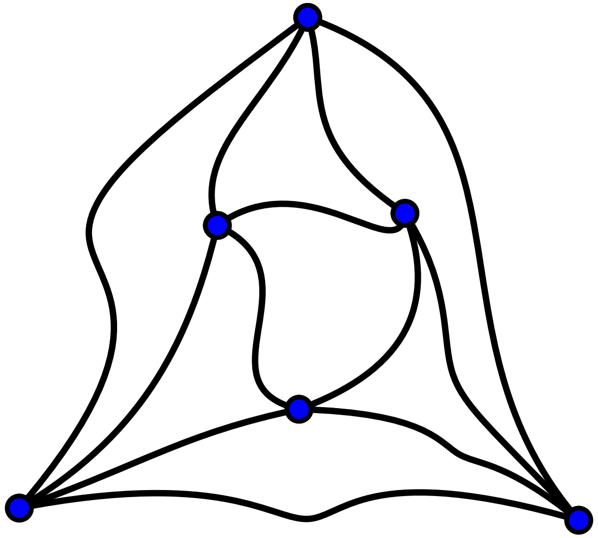
Stefan Felsner — TU Berlin
Henk Meijer — Rooseveldt C
Alexander Wolff — U Würzburg

Weak Line Cover Numbers

[Chaplick, Fleszar, Lipp, Ravsky, Verbitsky, W. GD'16]

Given: graph G



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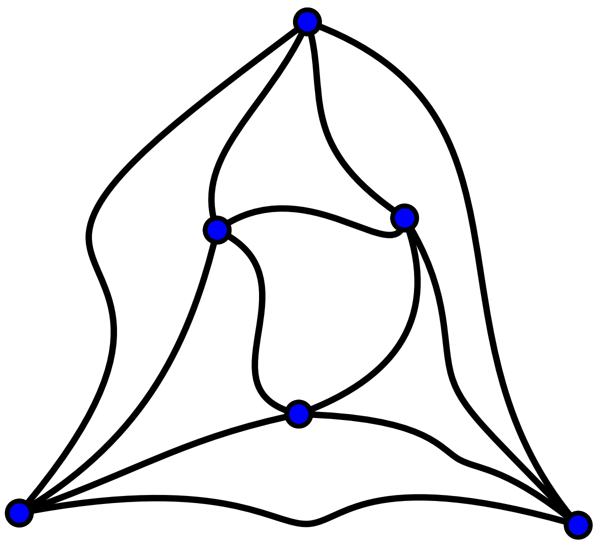

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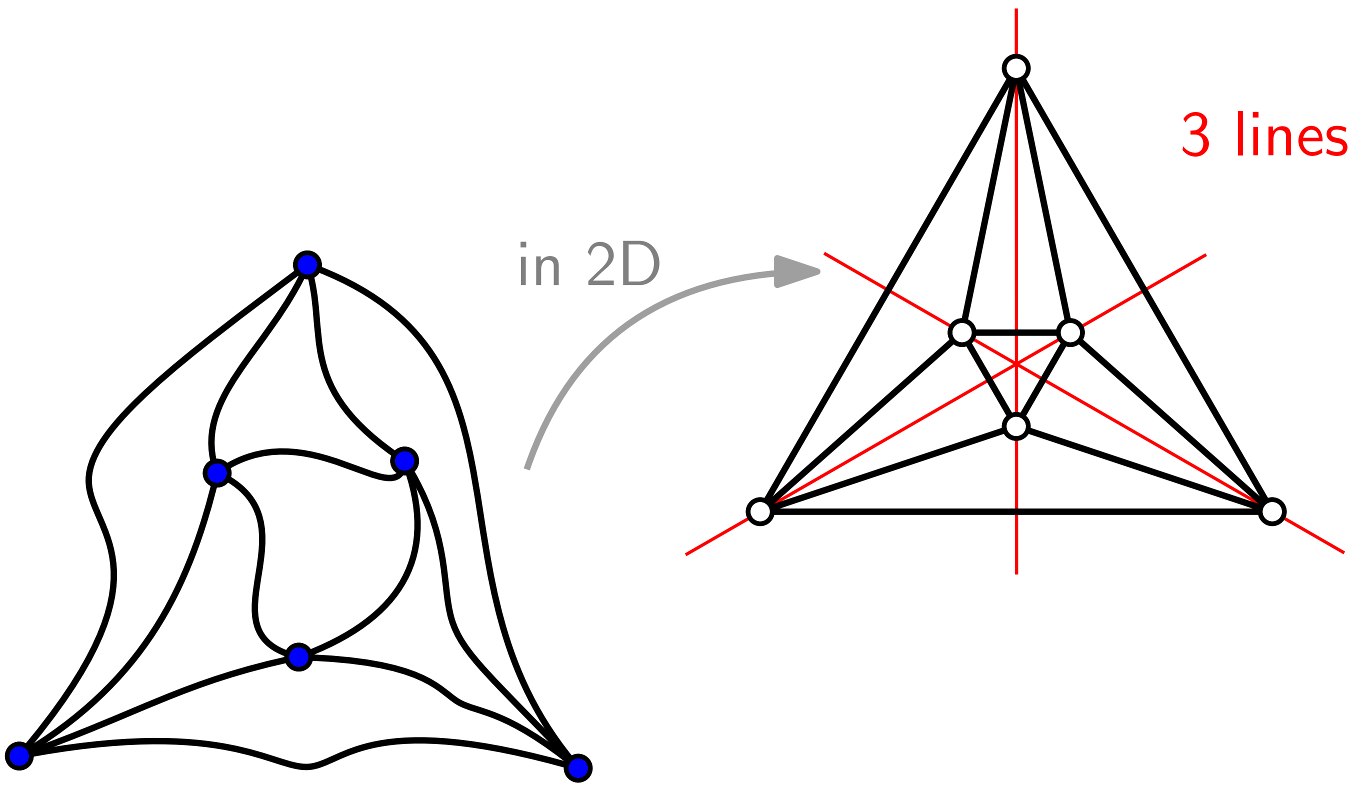

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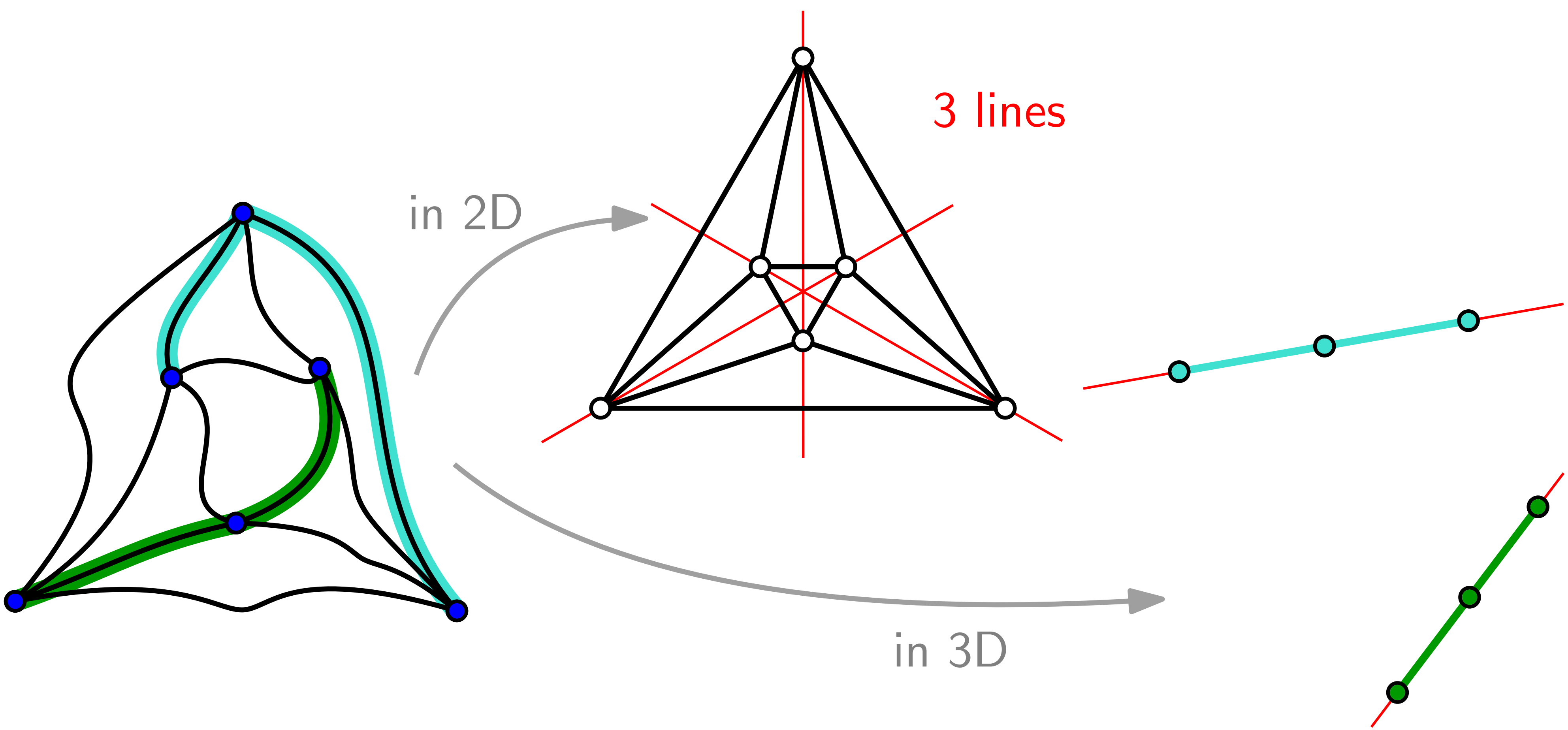

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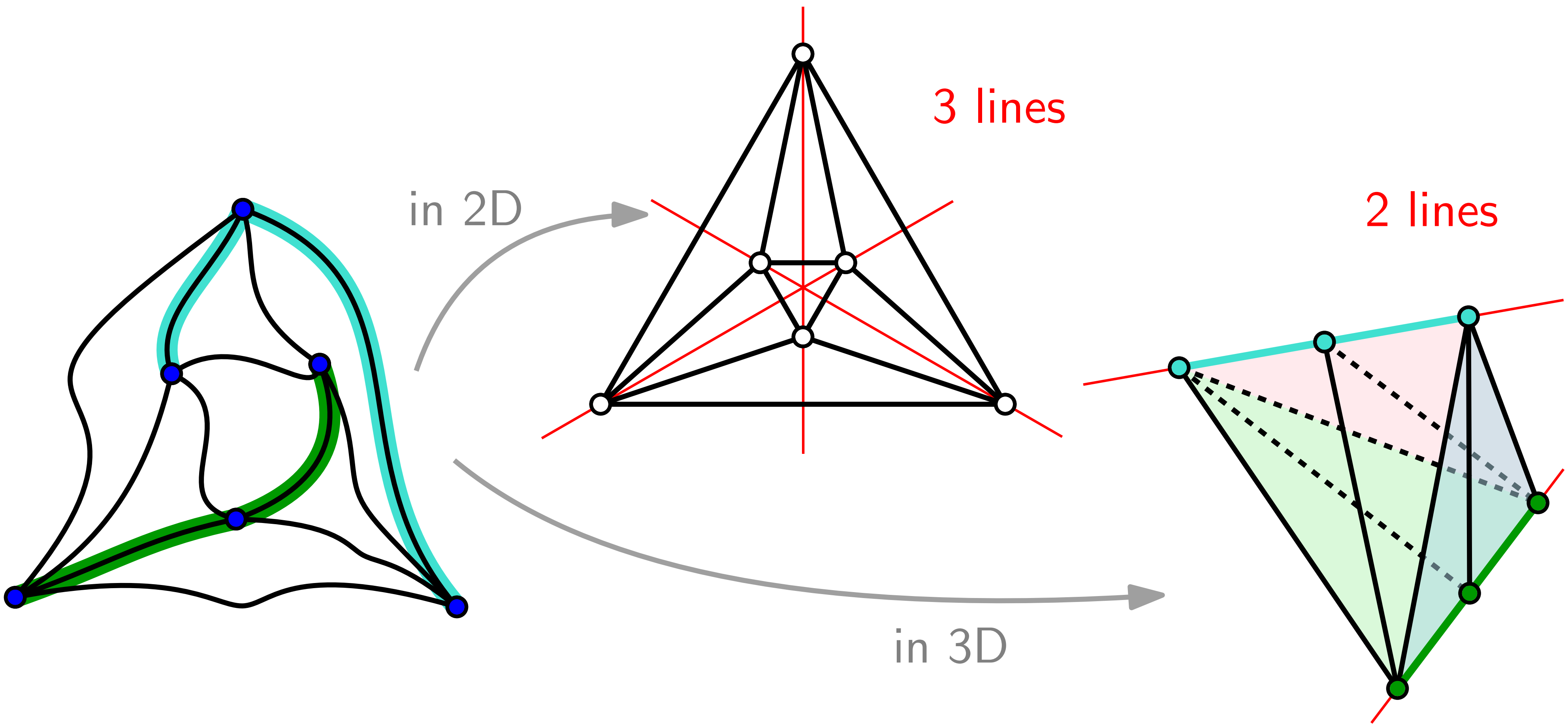

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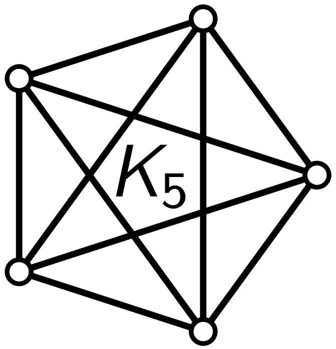

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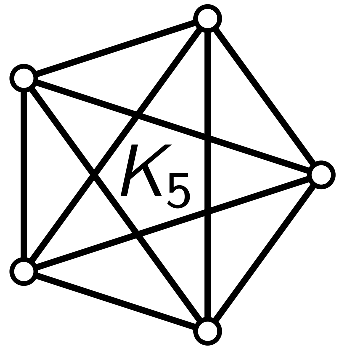

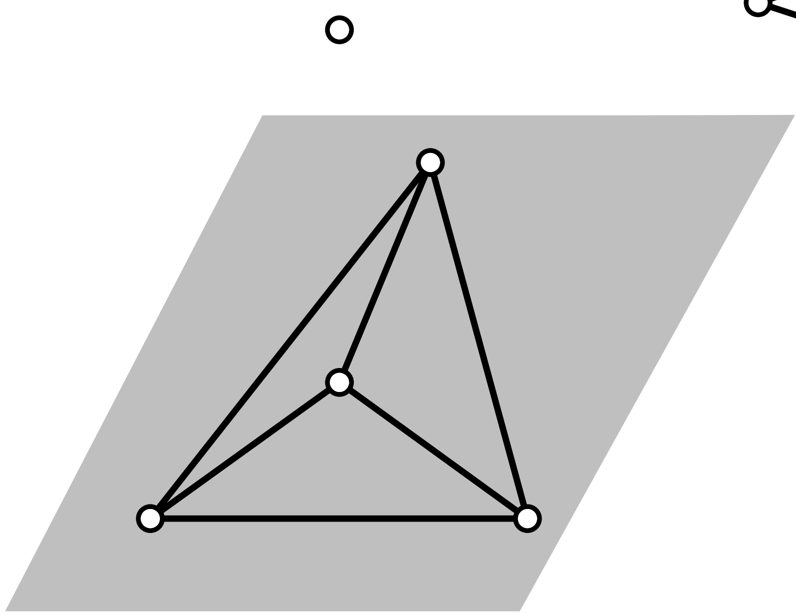

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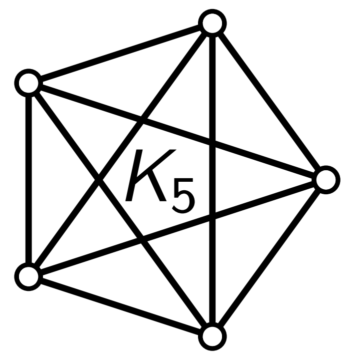

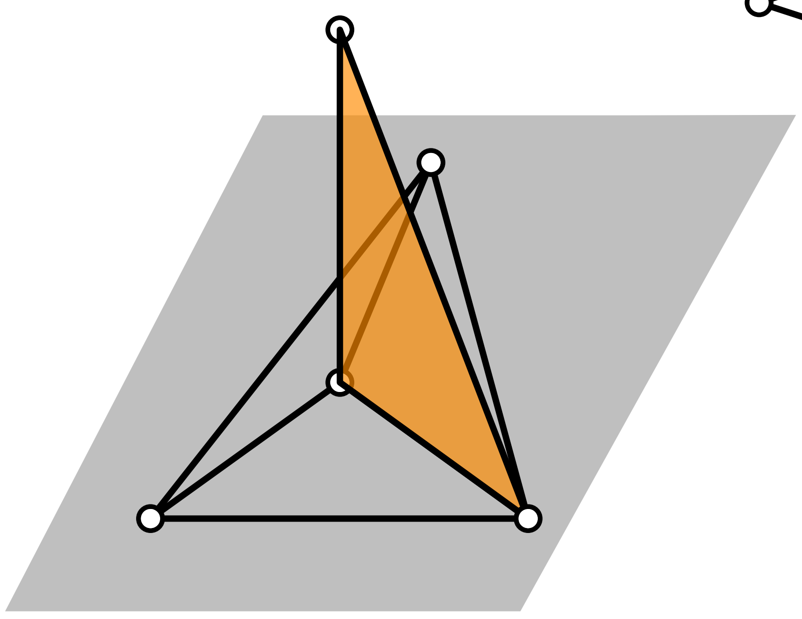

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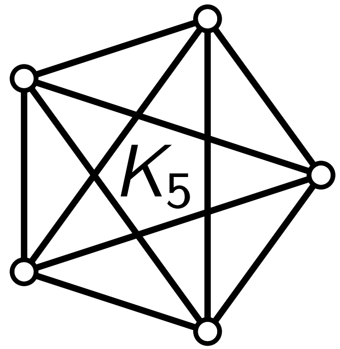

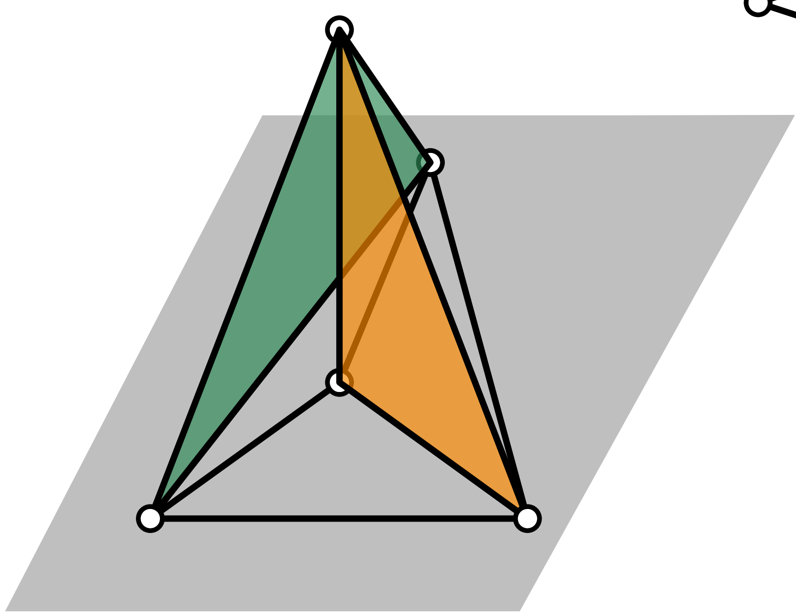

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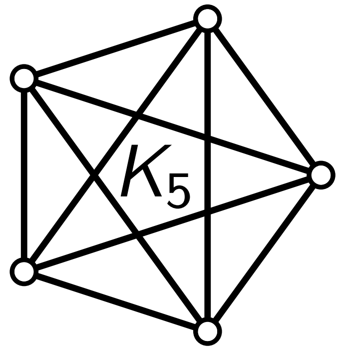

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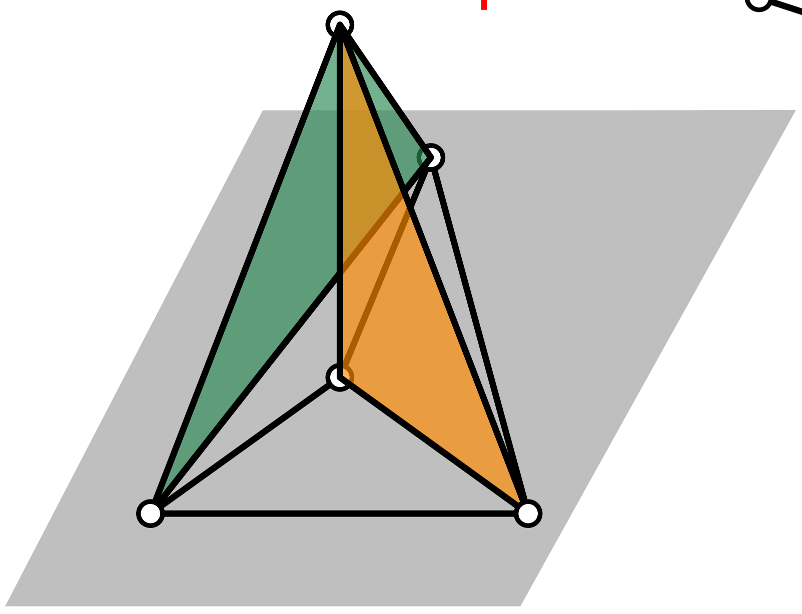

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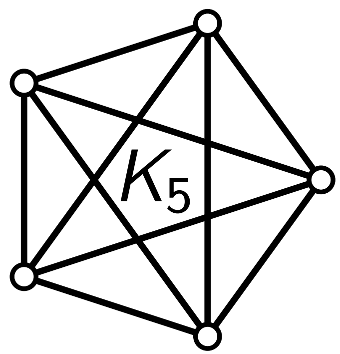

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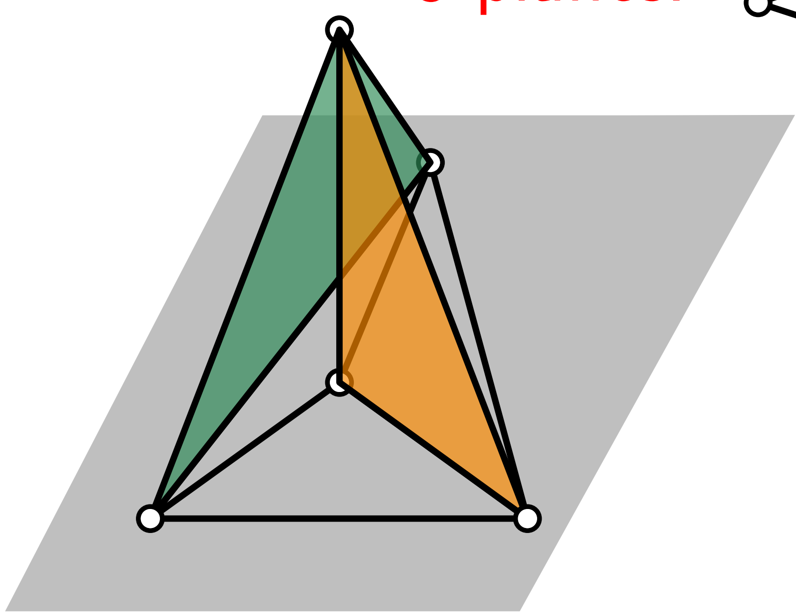

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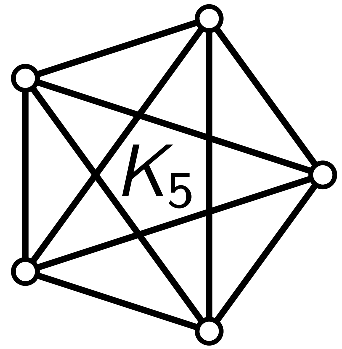

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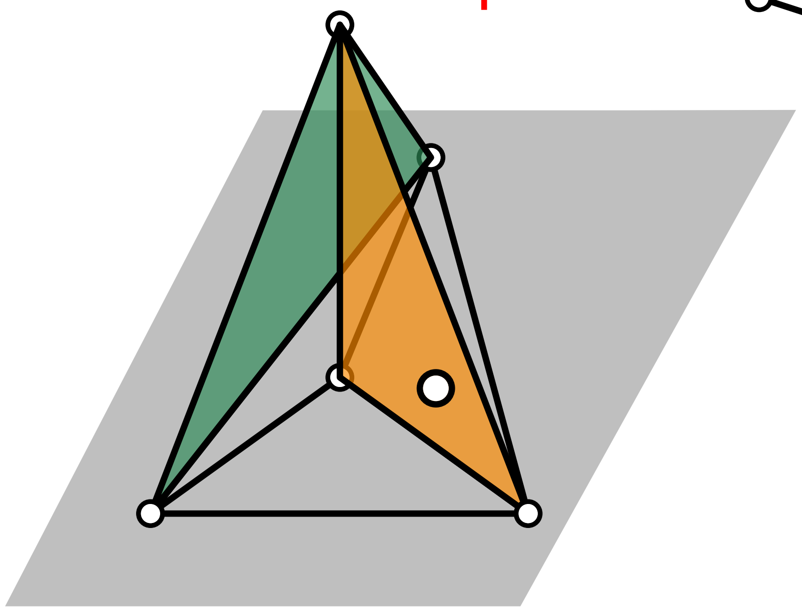

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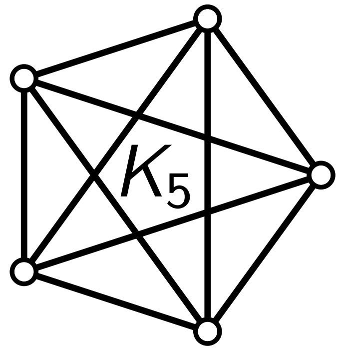

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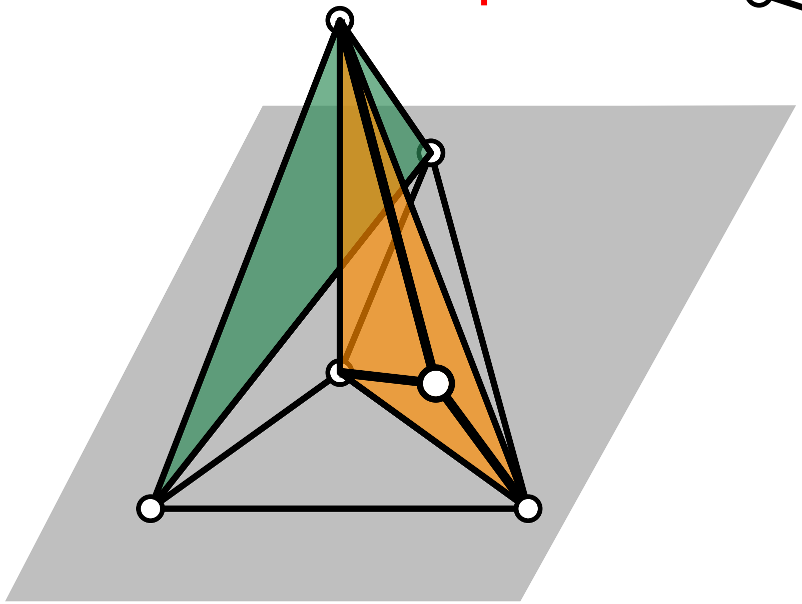

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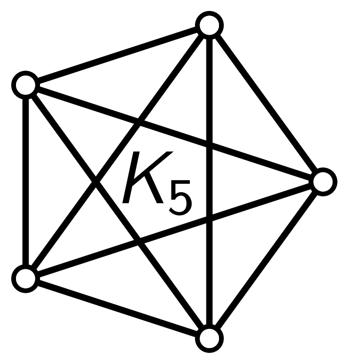

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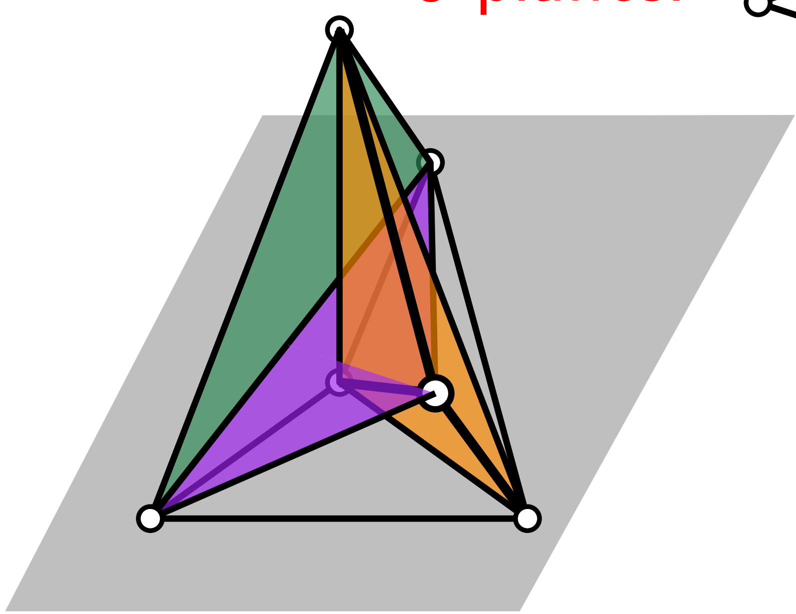

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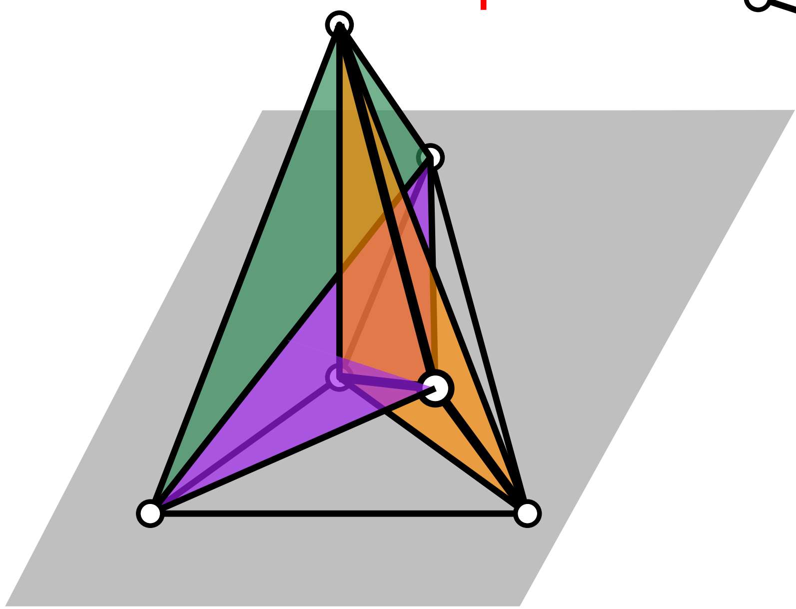

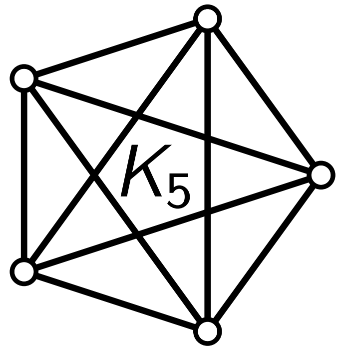

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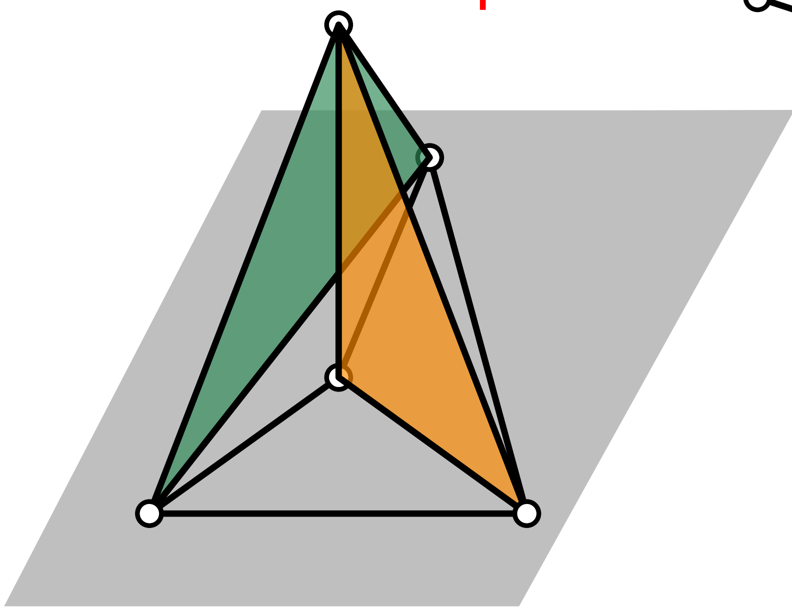

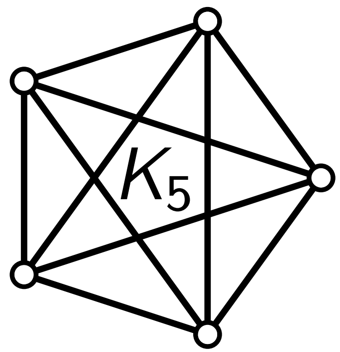

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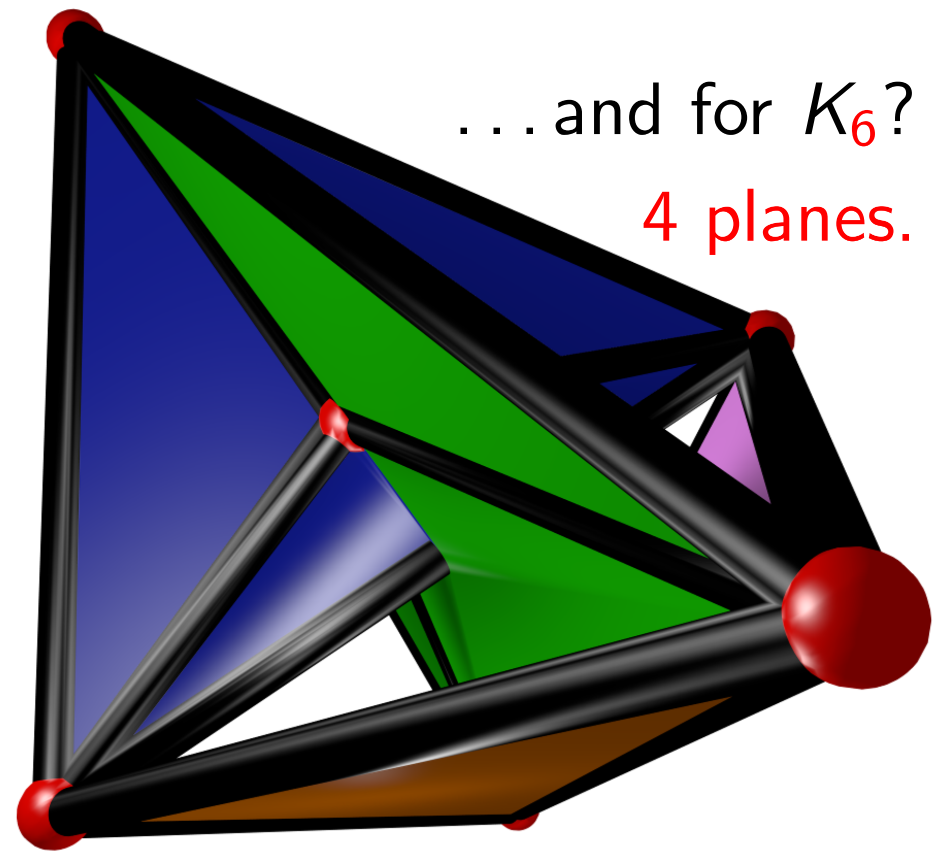

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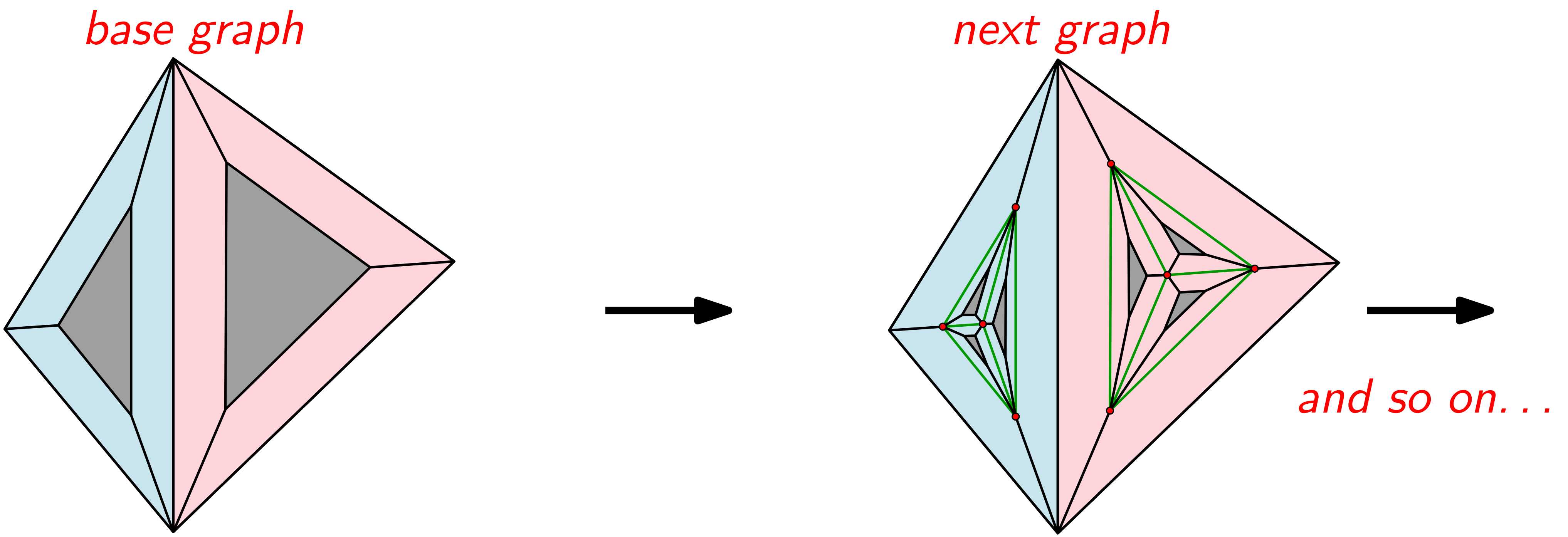

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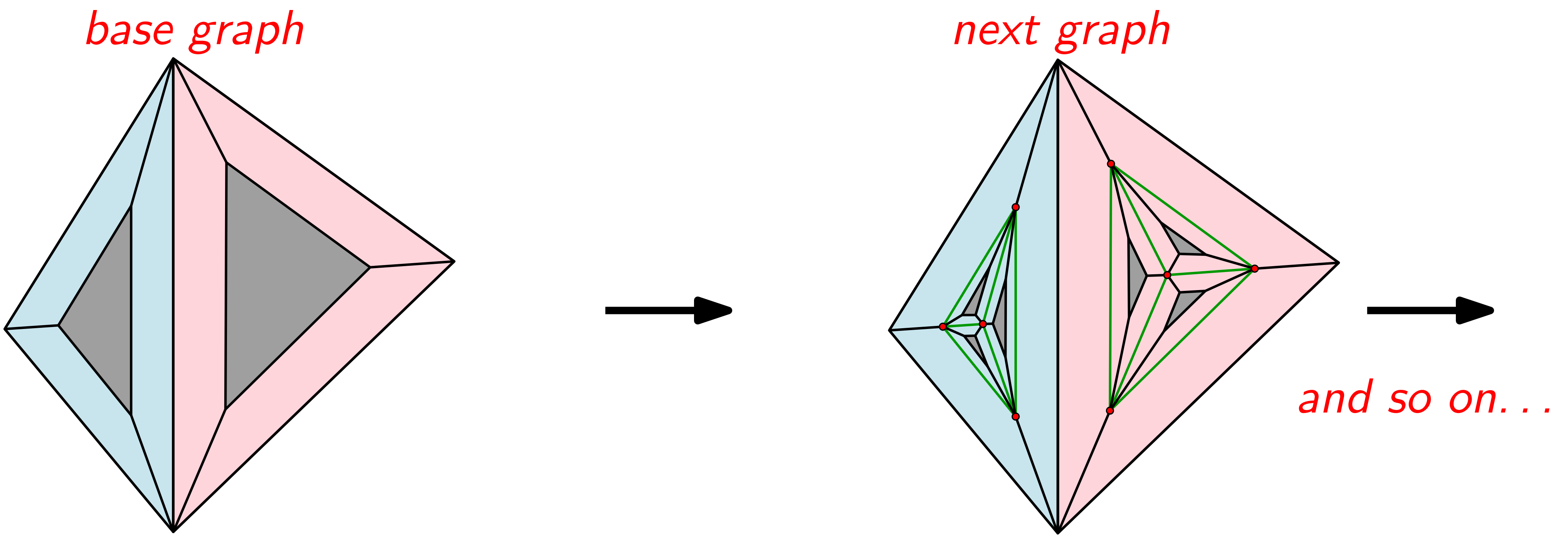

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[Firman, Straube, W. Poster @ GD'18]

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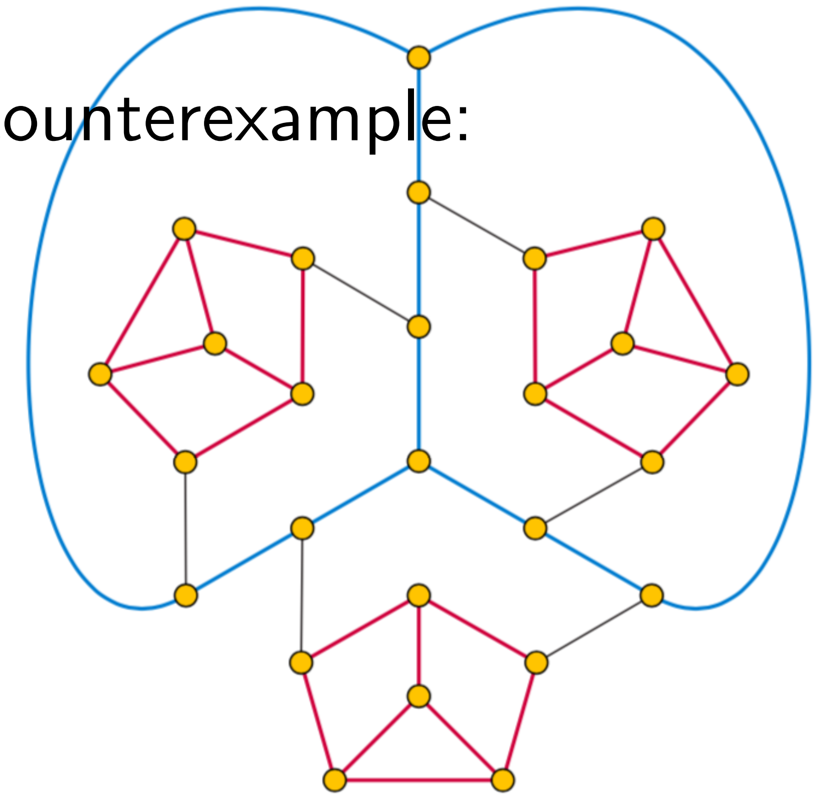

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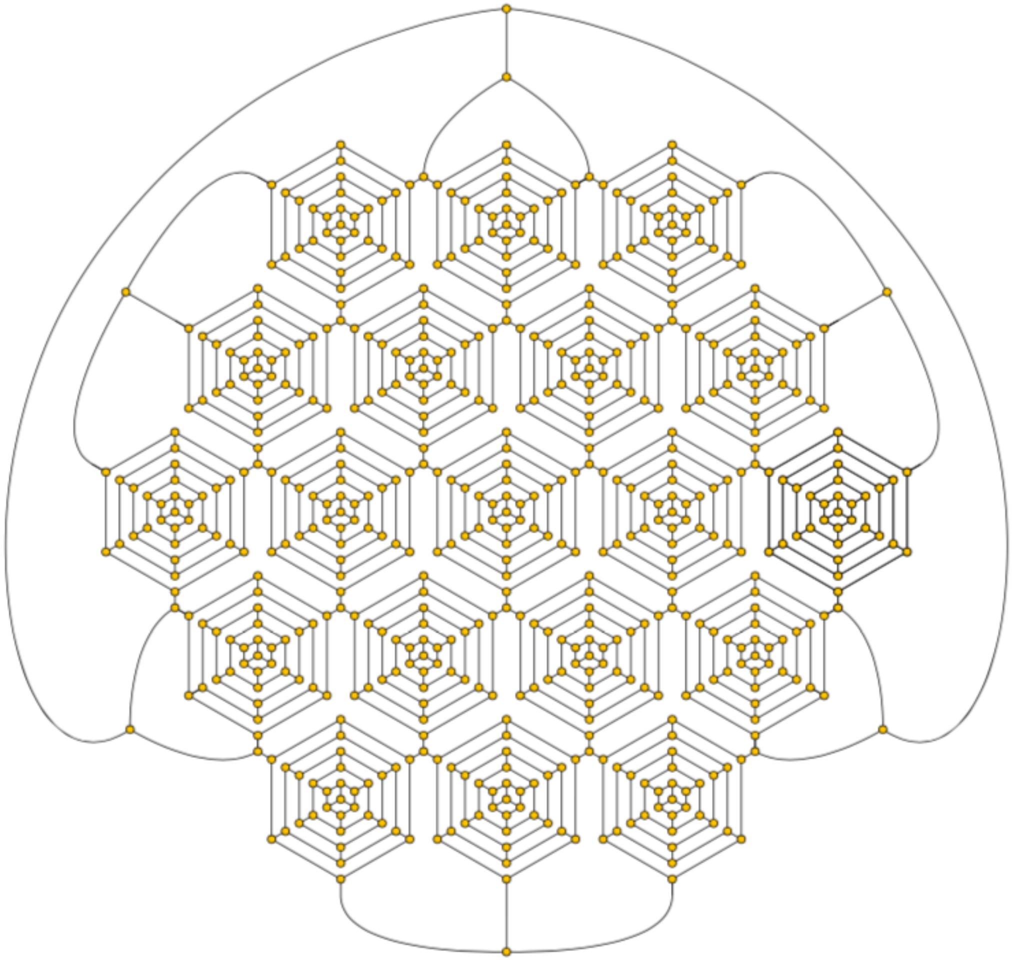

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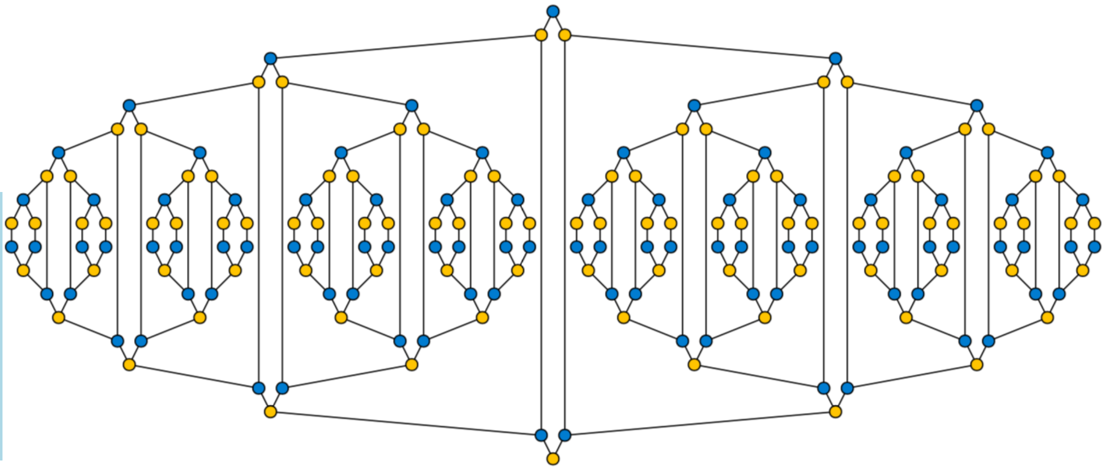

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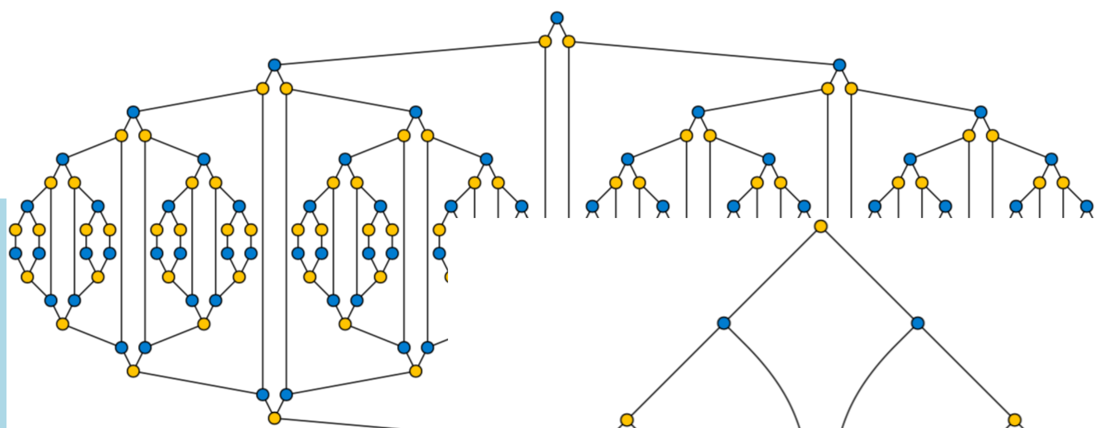

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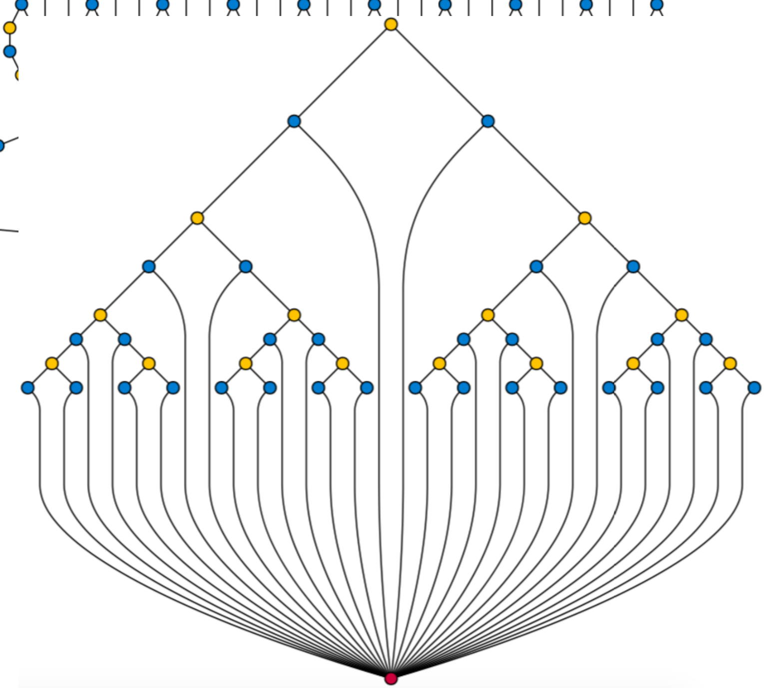

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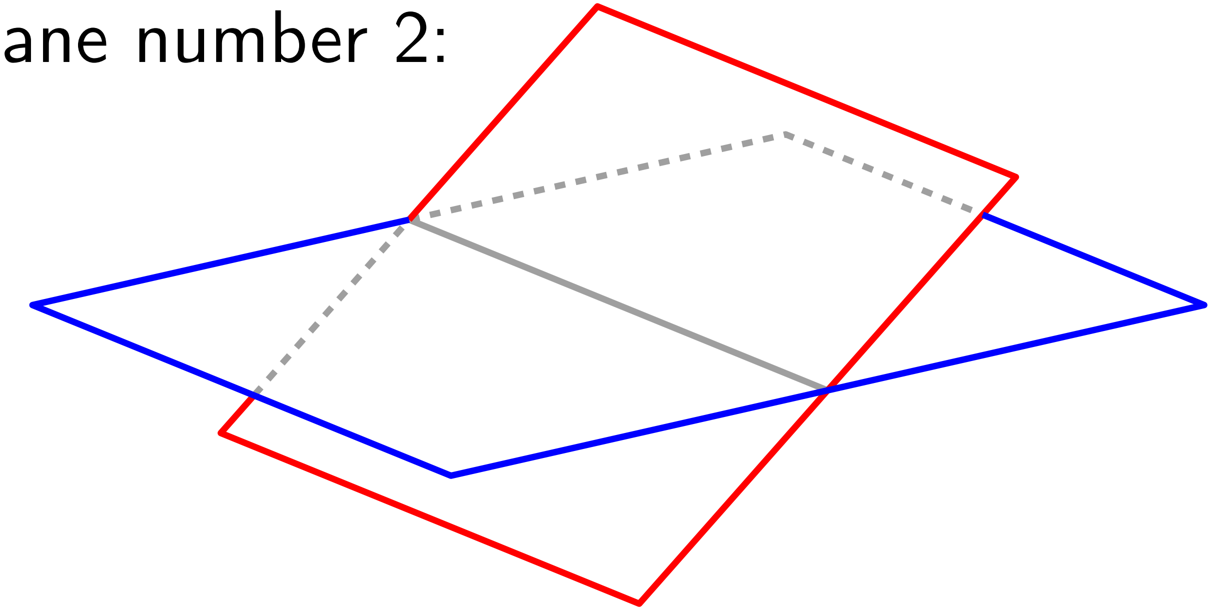

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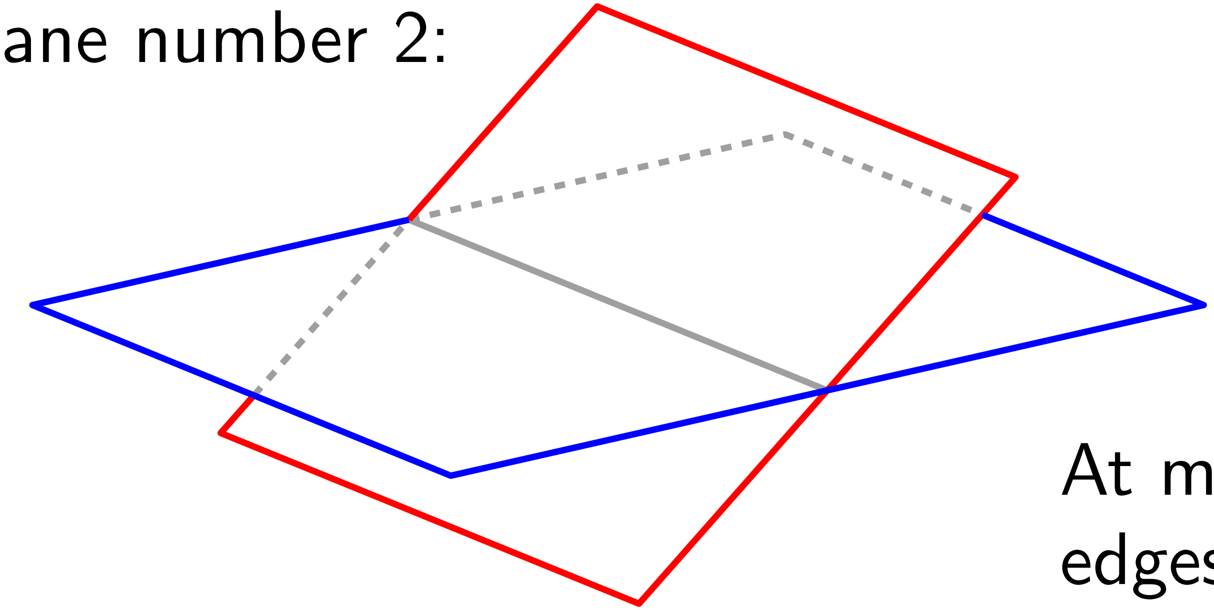

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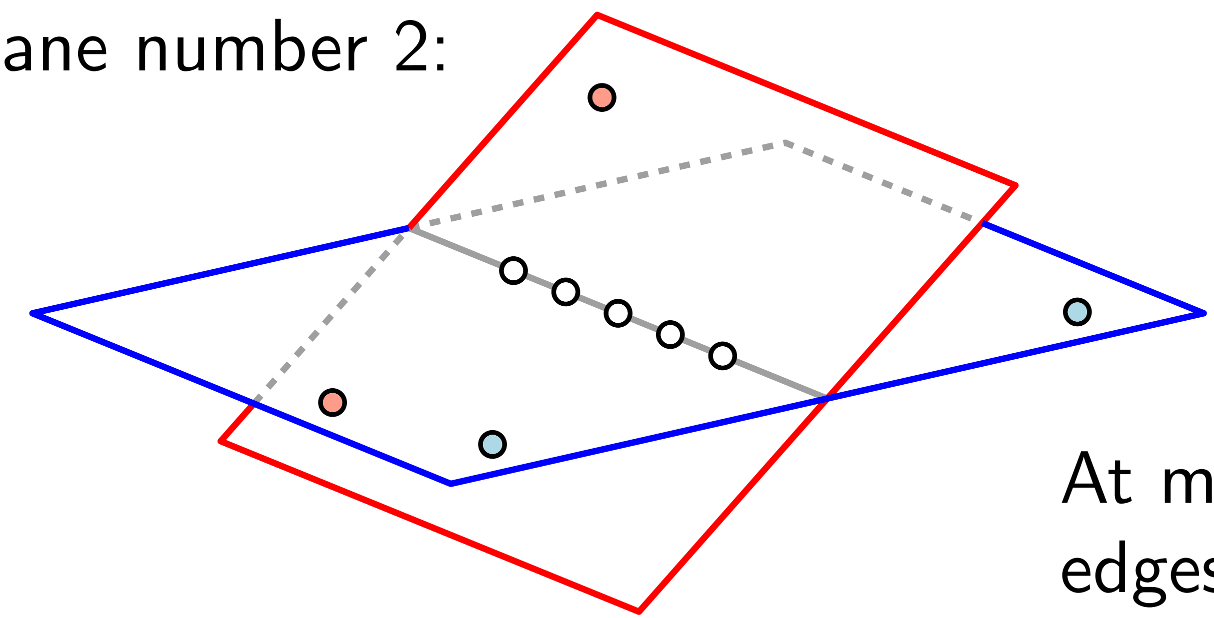

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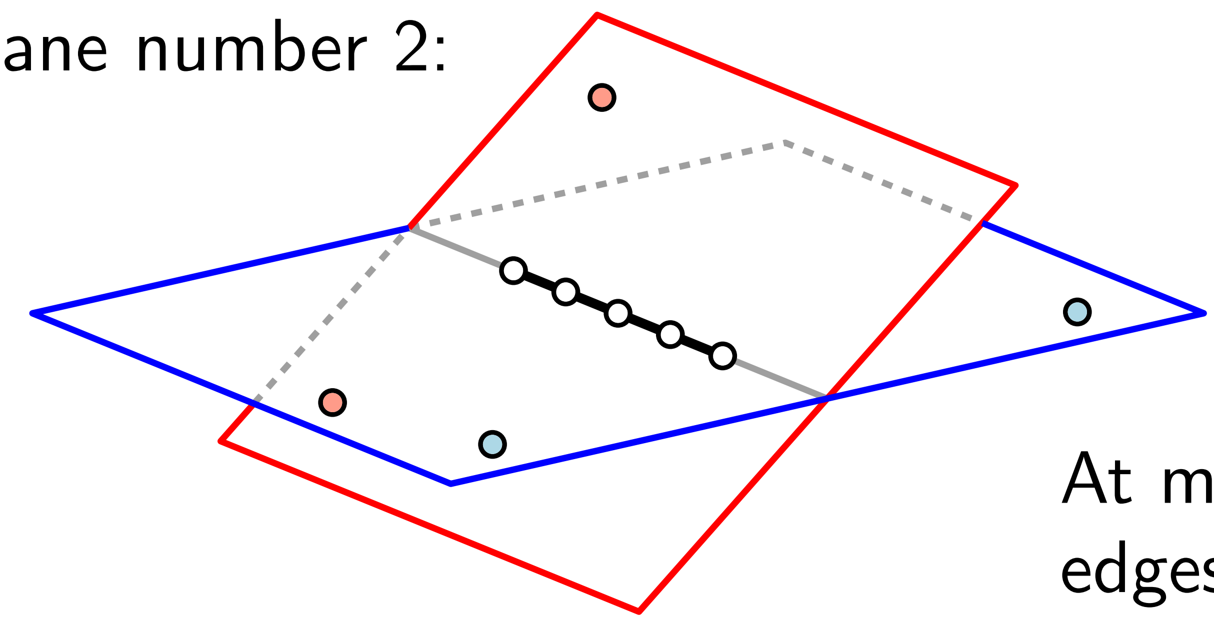

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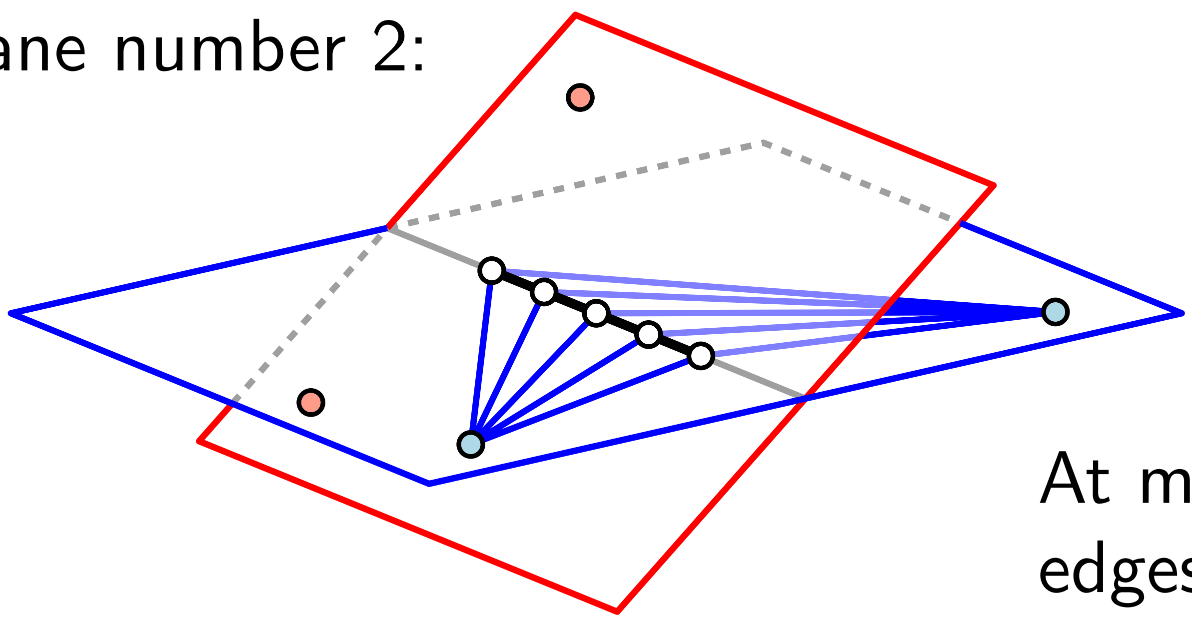

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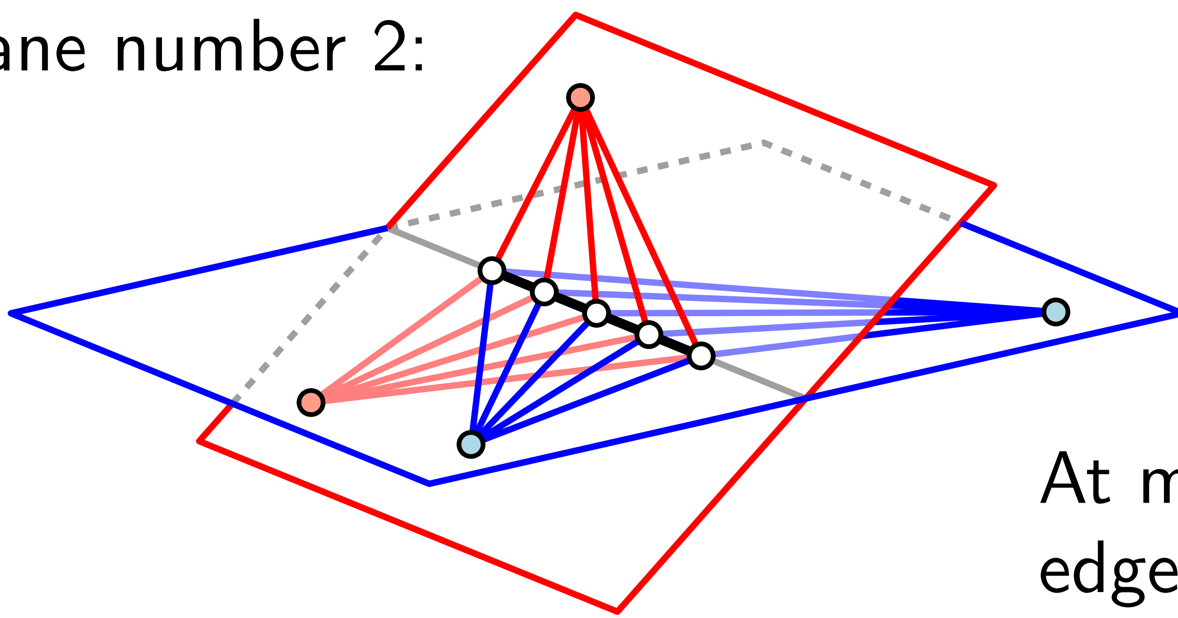

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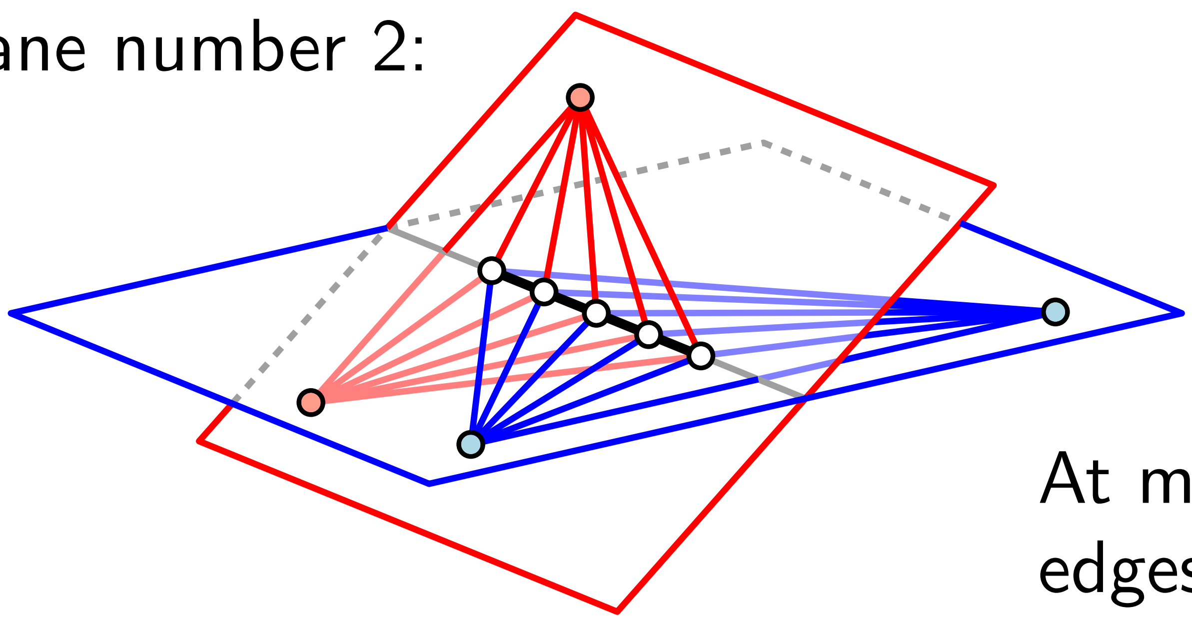

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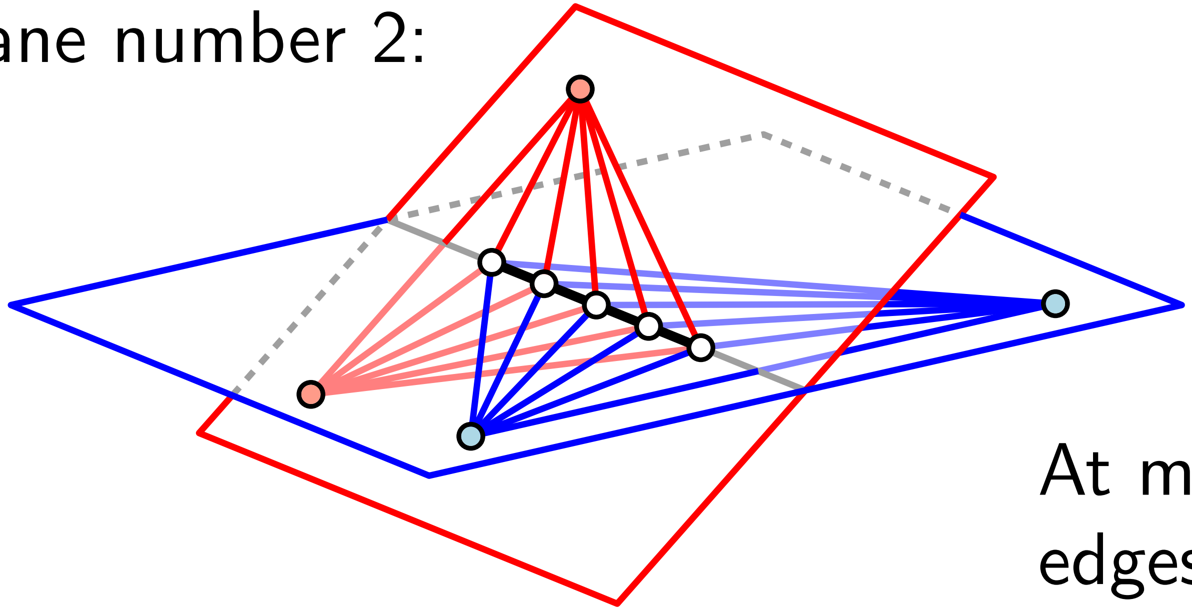
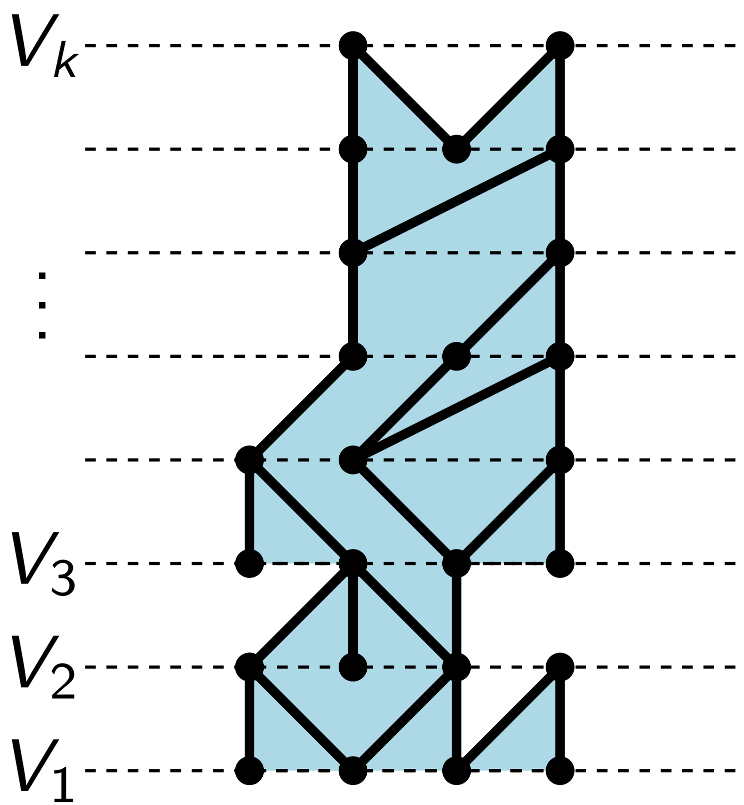


At most $5n - 19$ edges (if $n \geq 7$).

LevelPlanarity

A graph $G = (V, E)$ is *leveled-planar*
if V can be partitioned into $V_1, V_2, \ldots$
such that every edge connects vertices in consecutive sets and
G has a planar s-l drawing with $V_i \to \mathbb{R} \times \{i\}$ for $i = 1, 2, \ldots$



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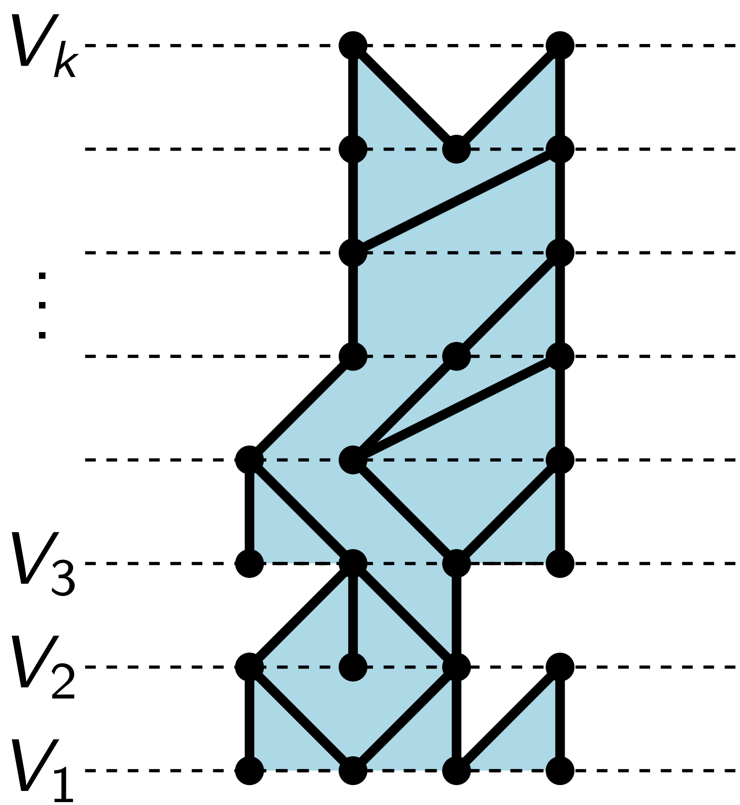

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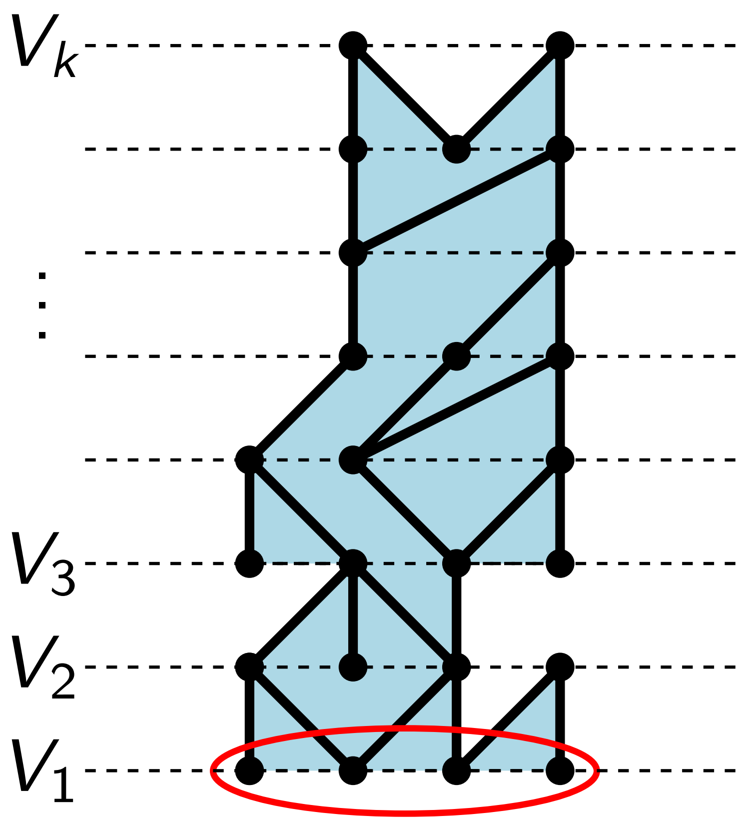

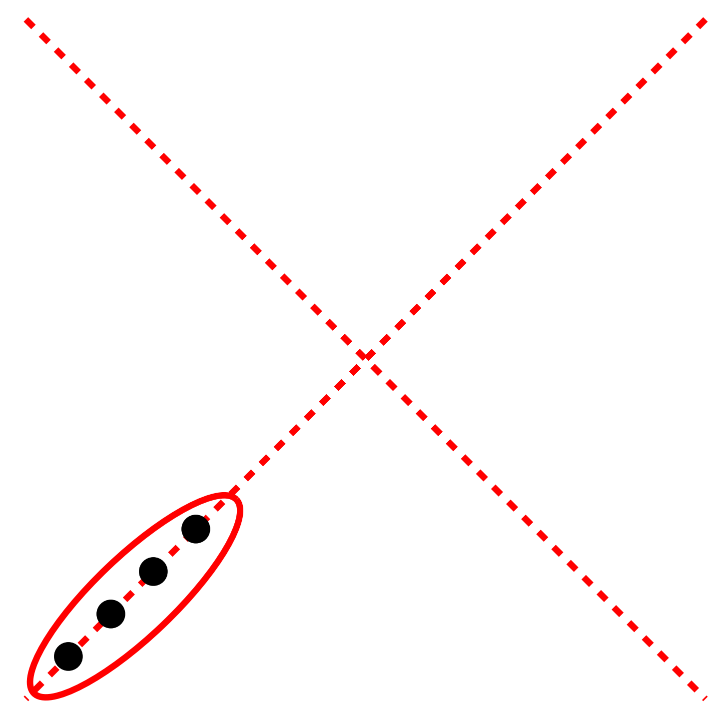

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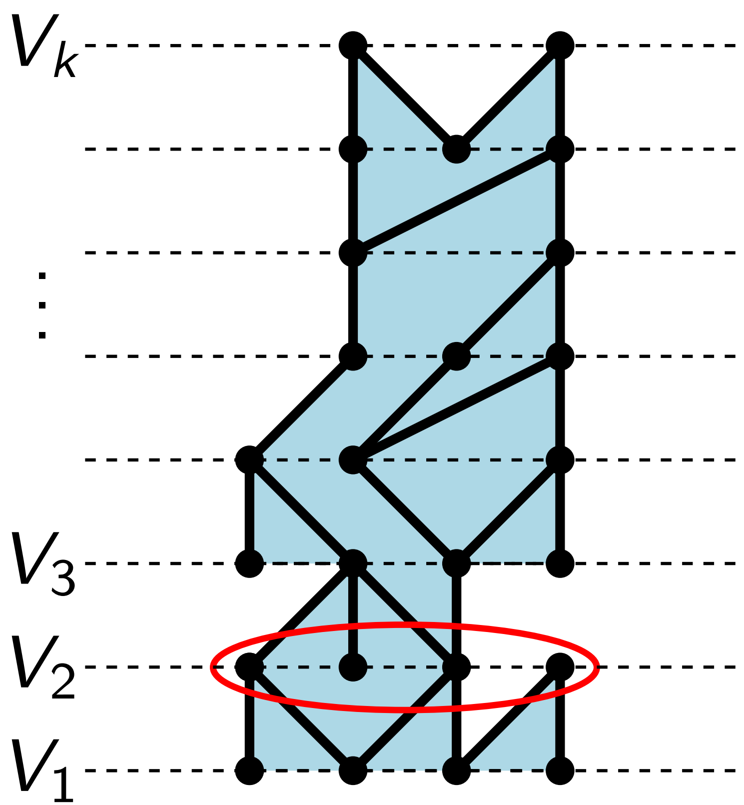

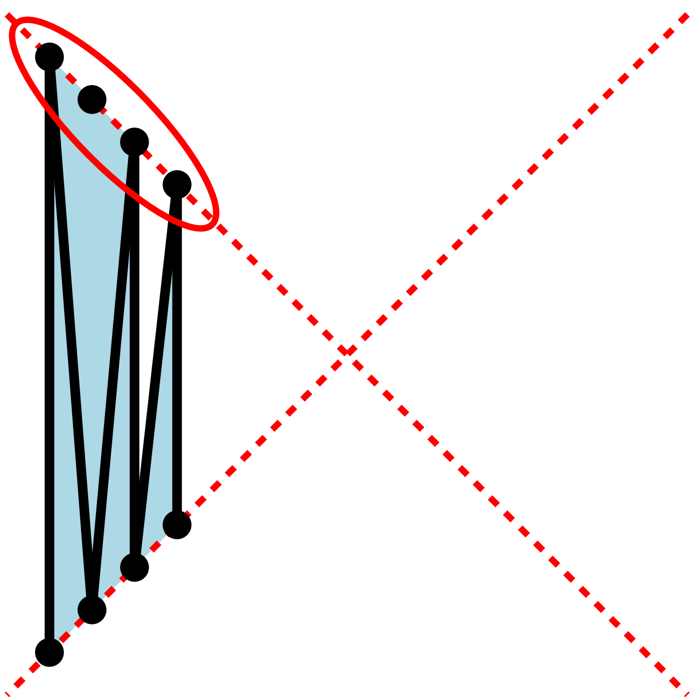

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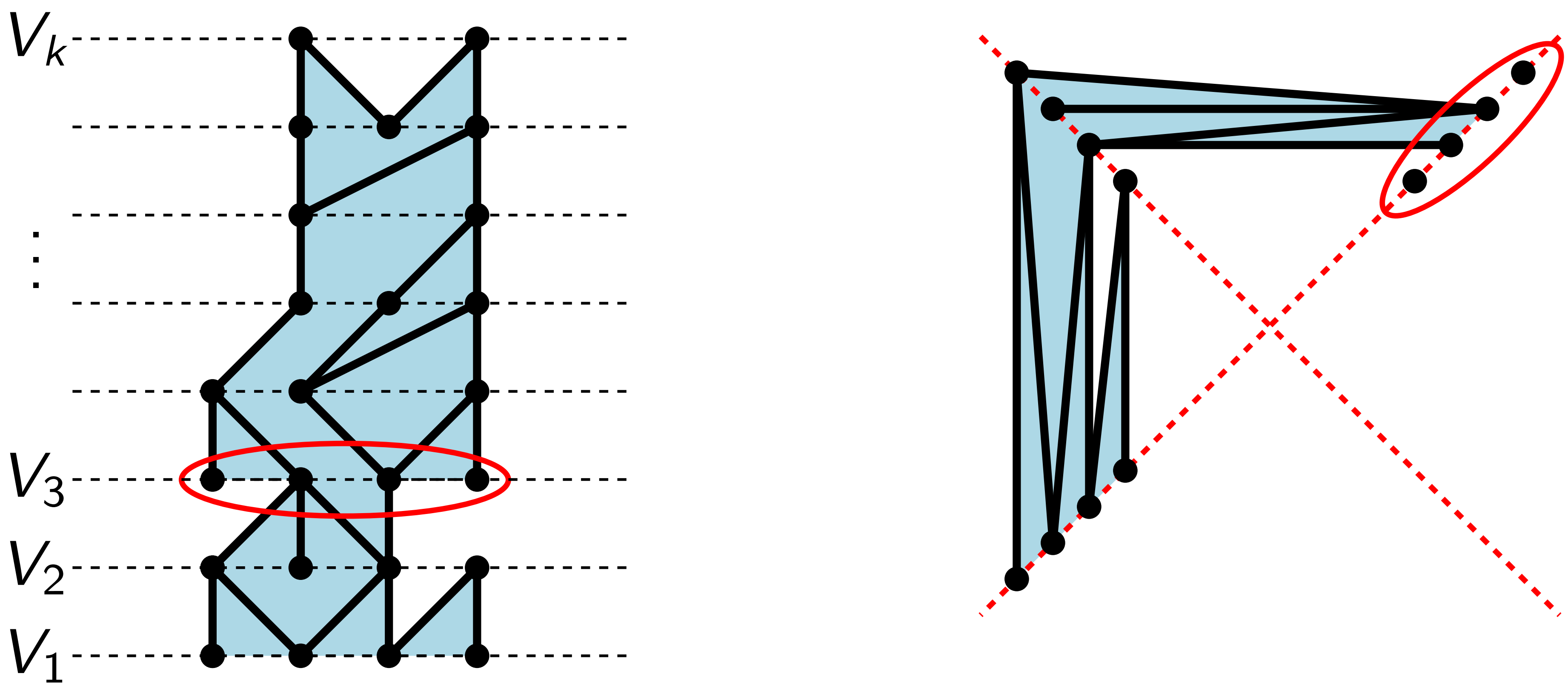

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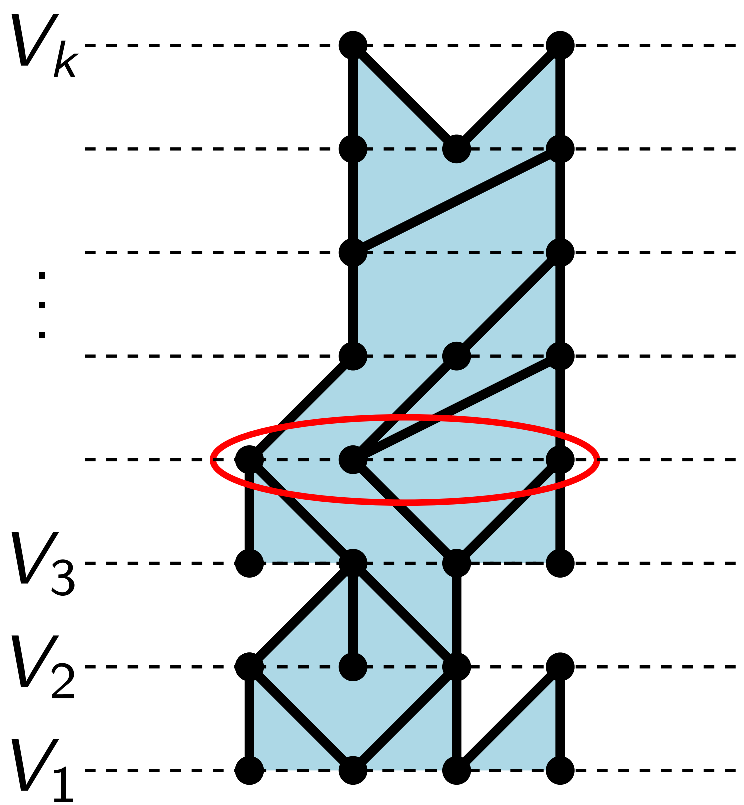

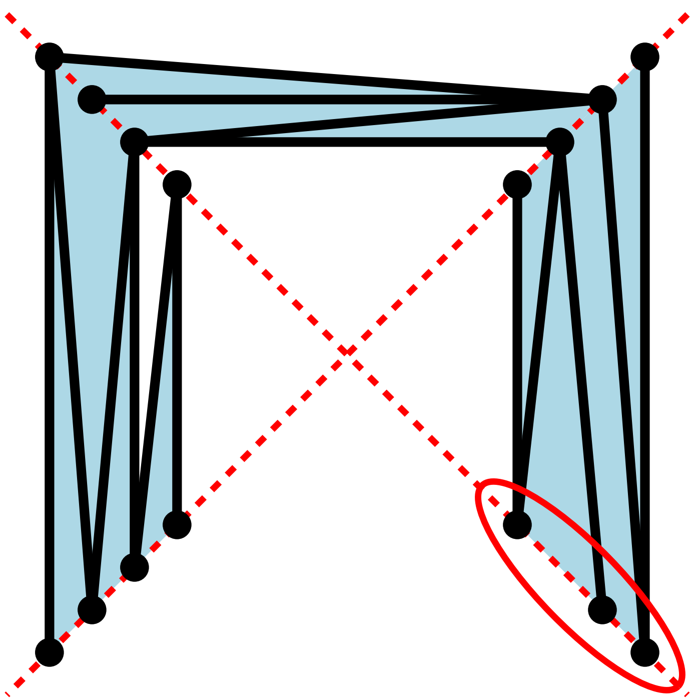

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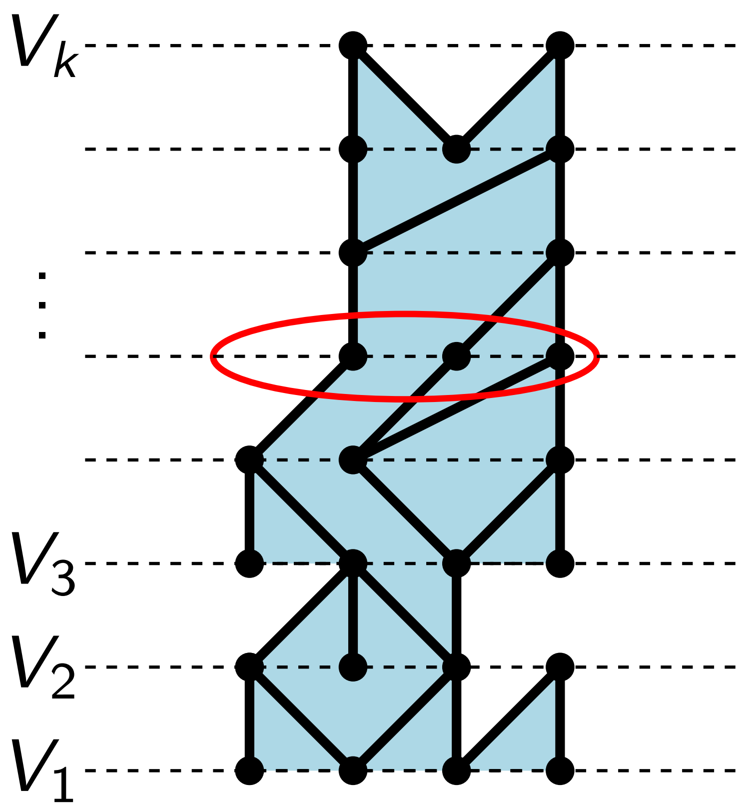

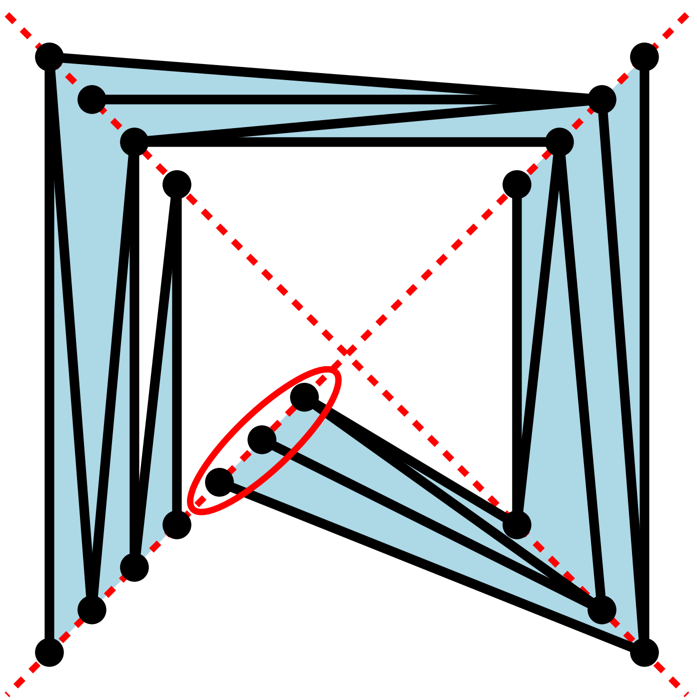

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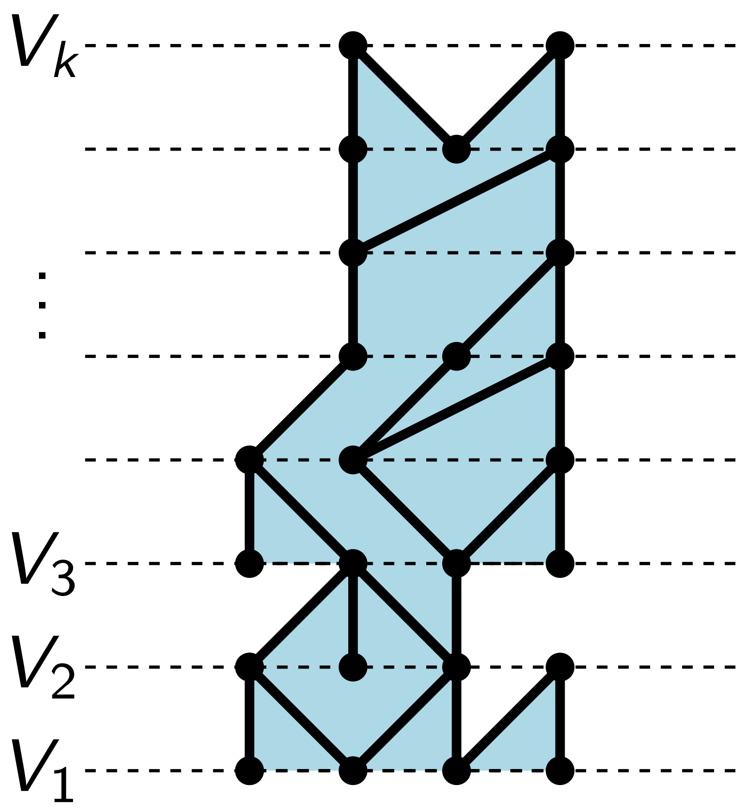

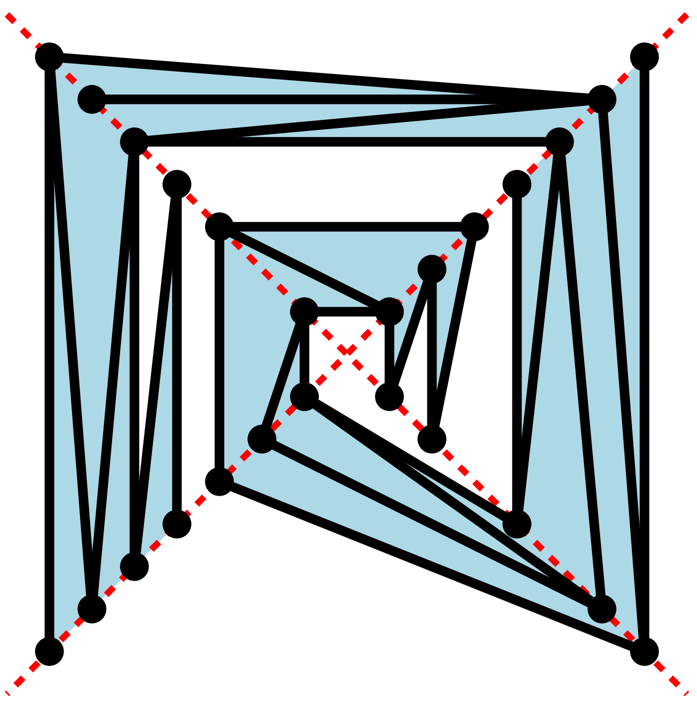

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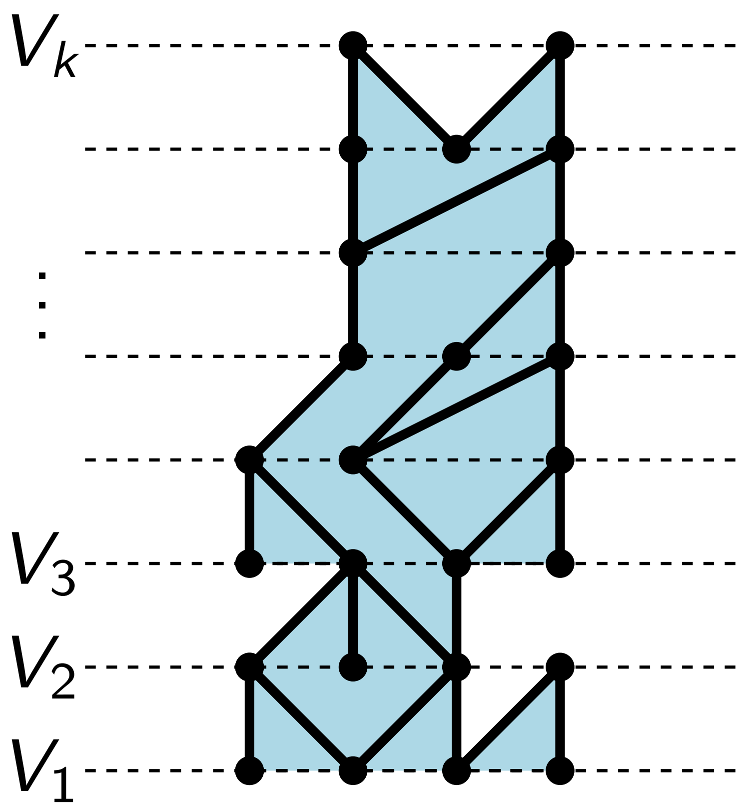

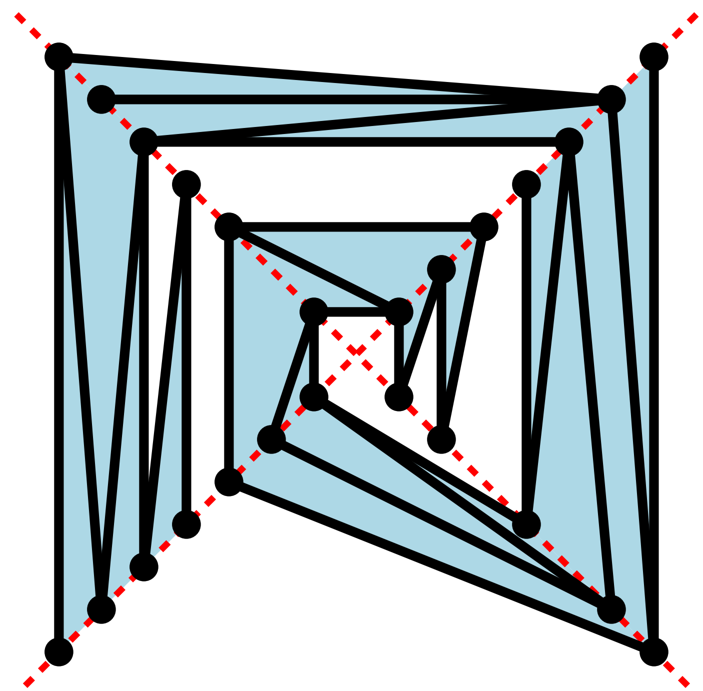

[Chaplick et al., Bannister et al., GD 2016]

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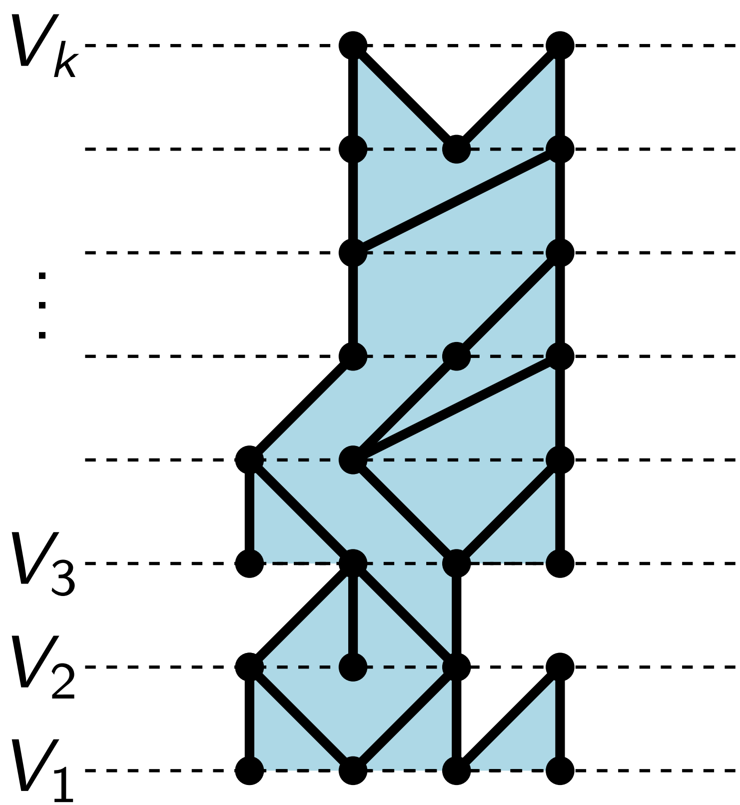

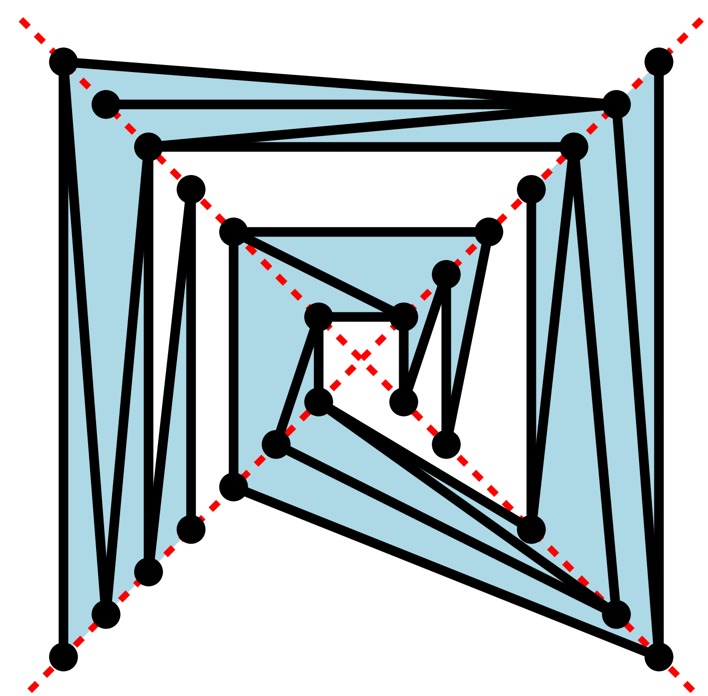

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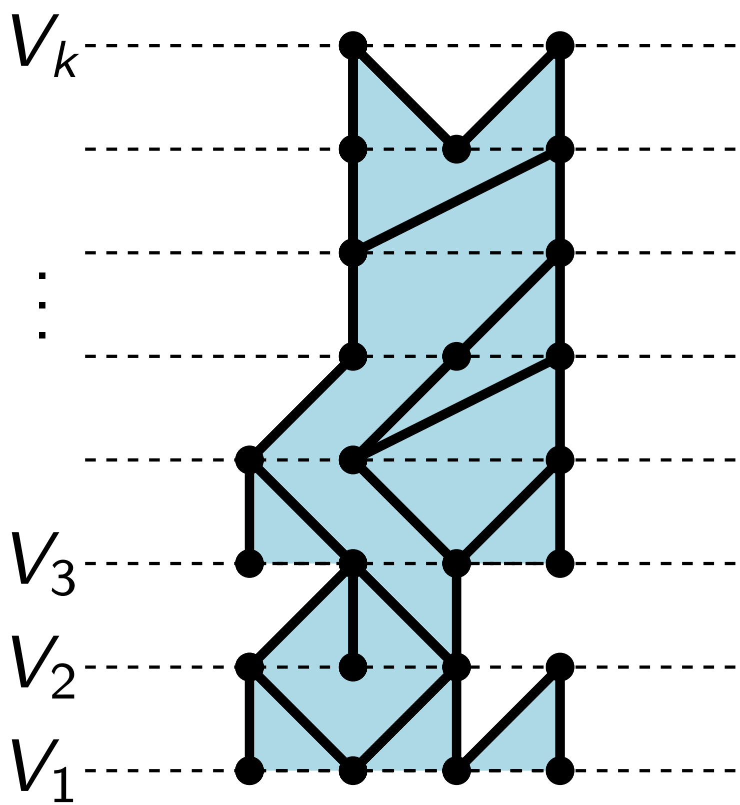
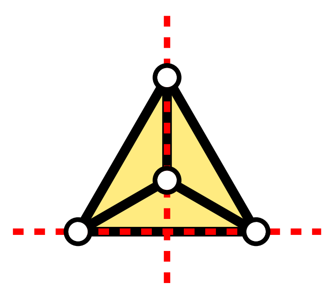
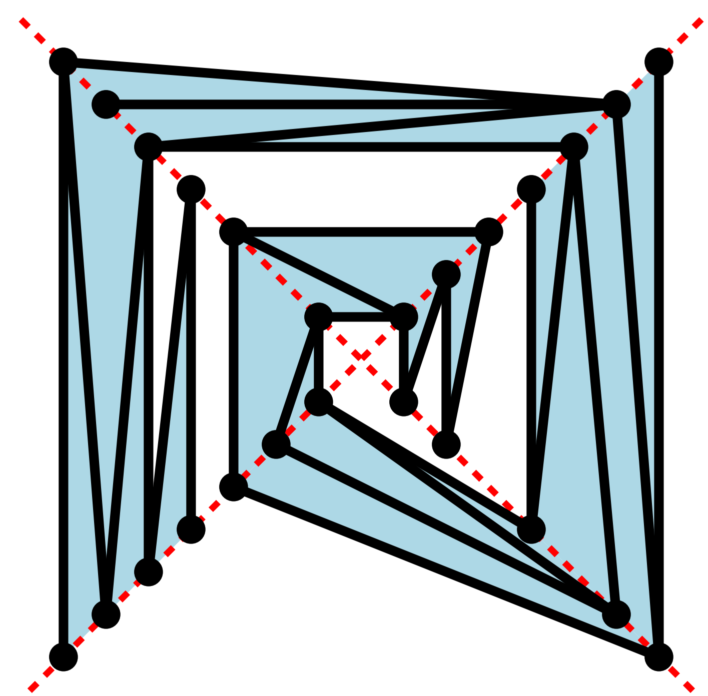


$\not\Leftarrow$

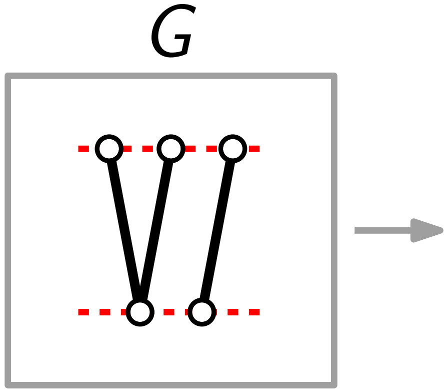
e.g., K_4:





[Chaplick et al., Bannister et al., GD 2016]

Transformation



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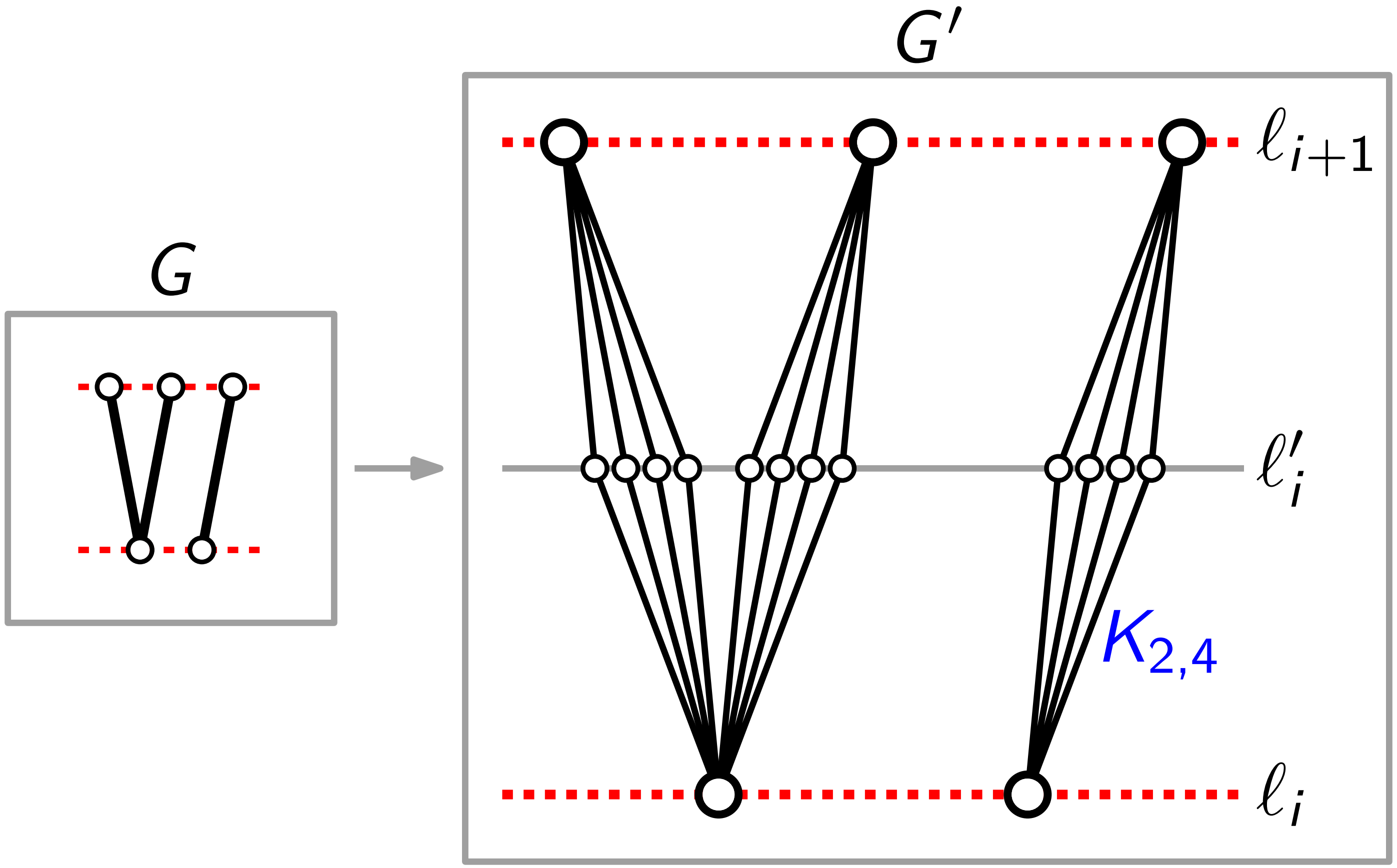

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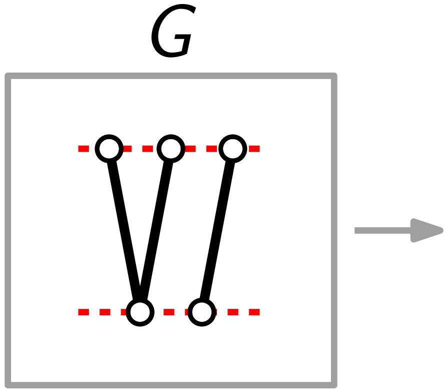

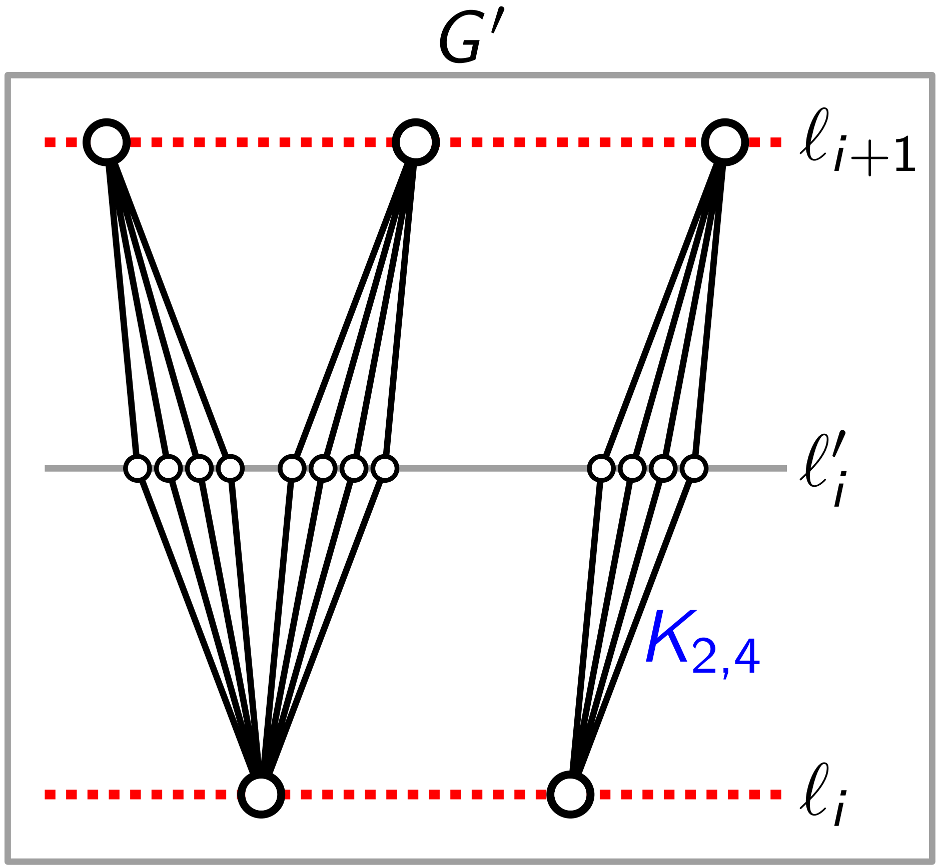

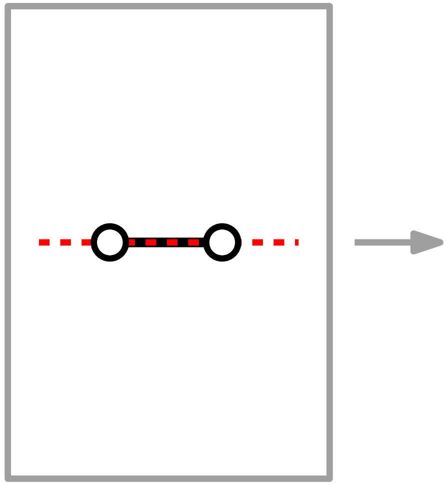

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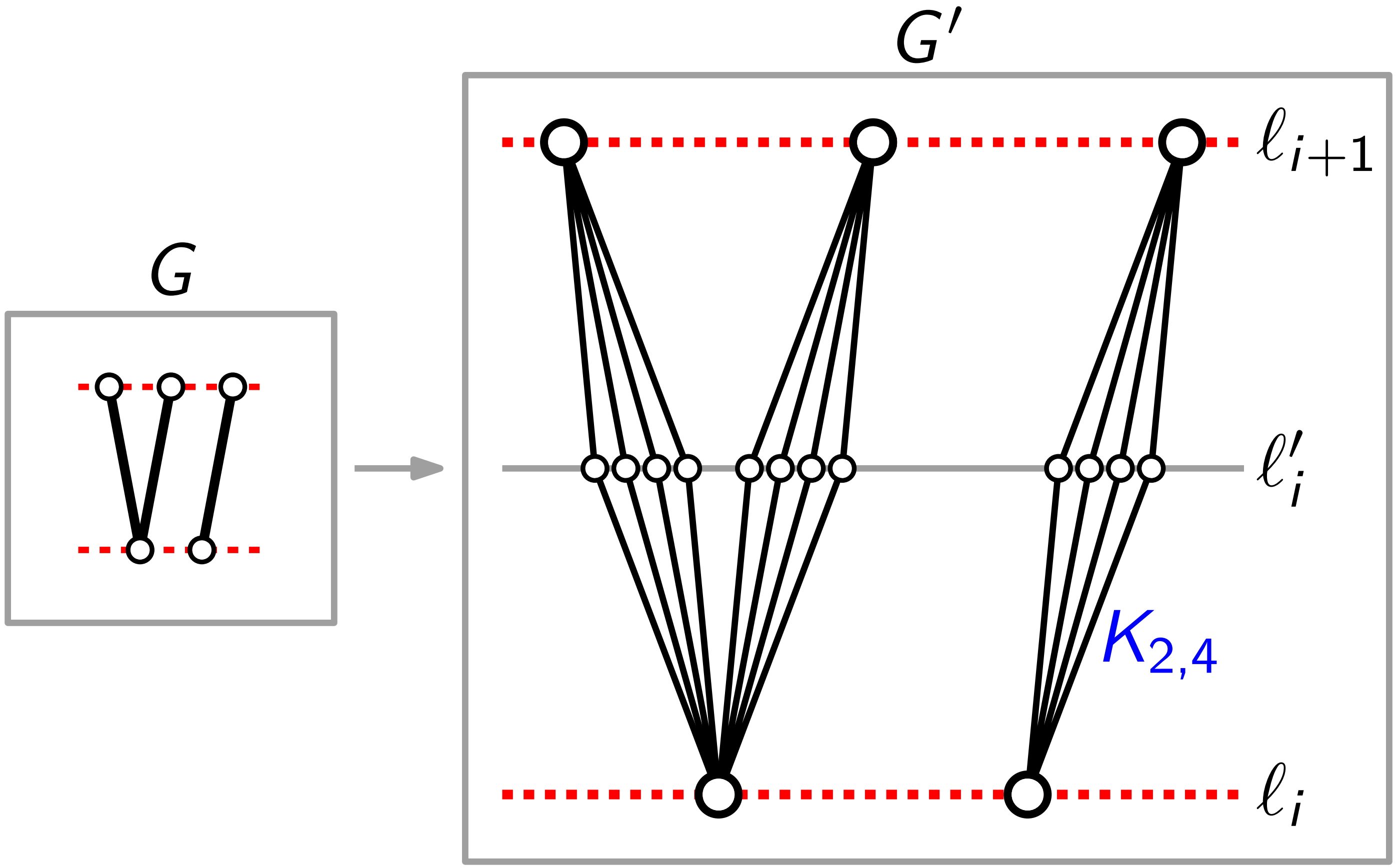

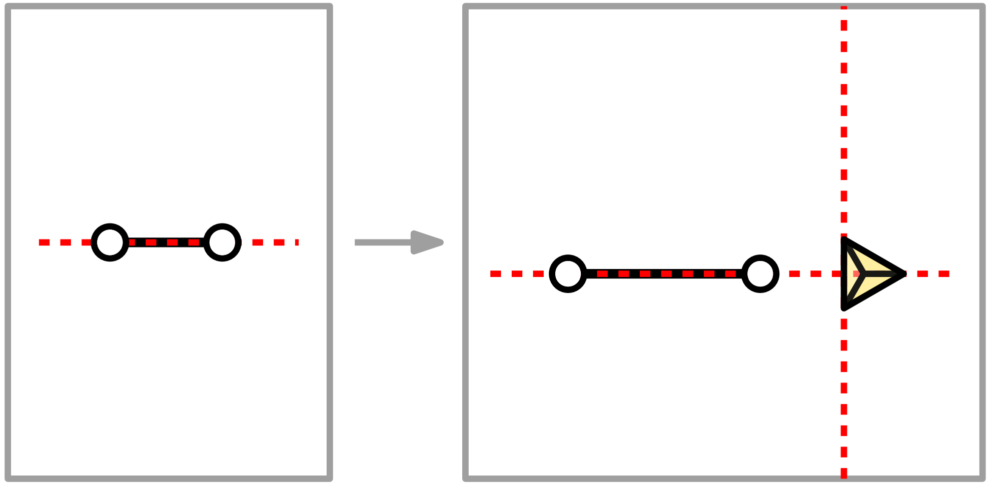

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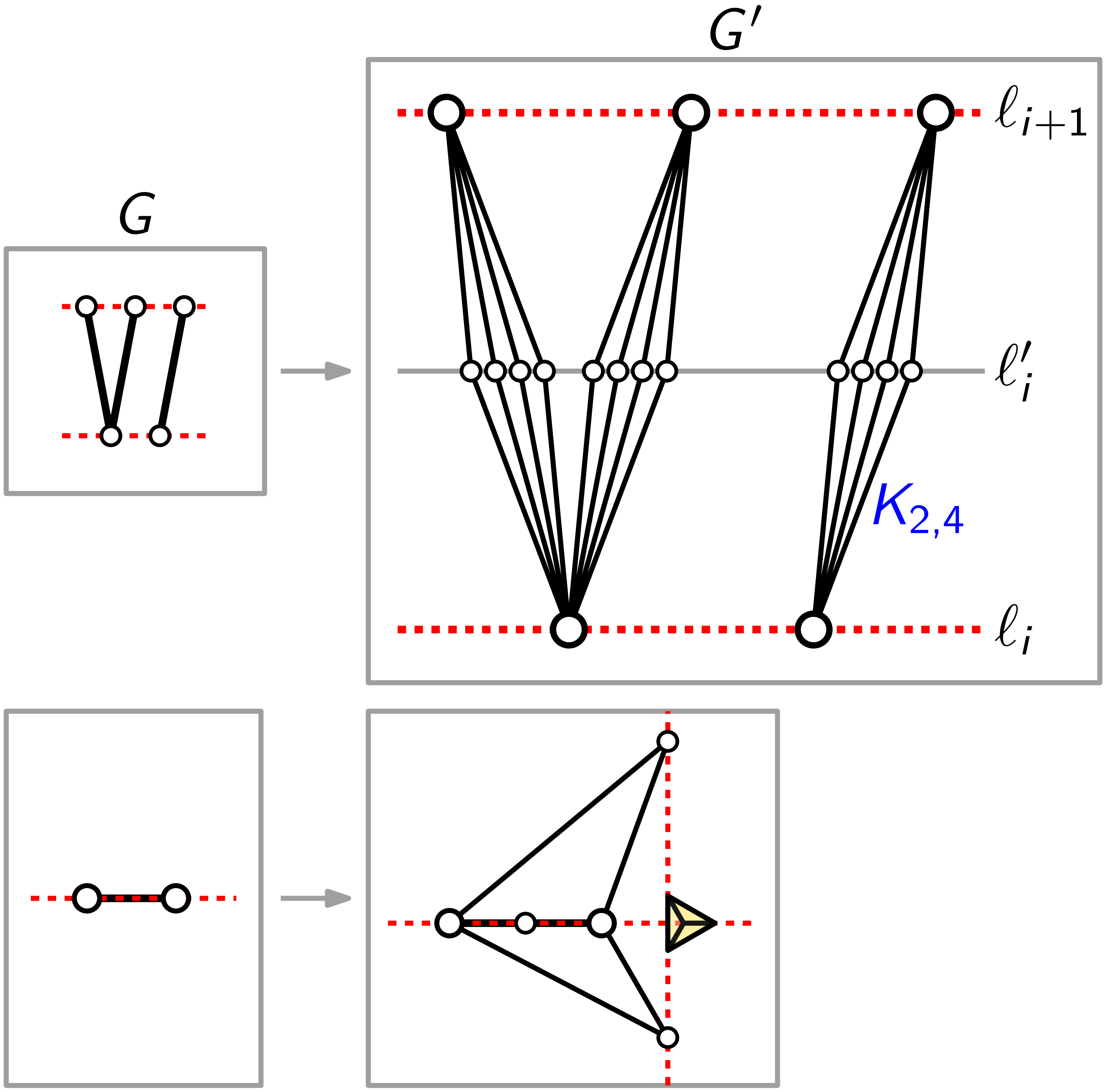

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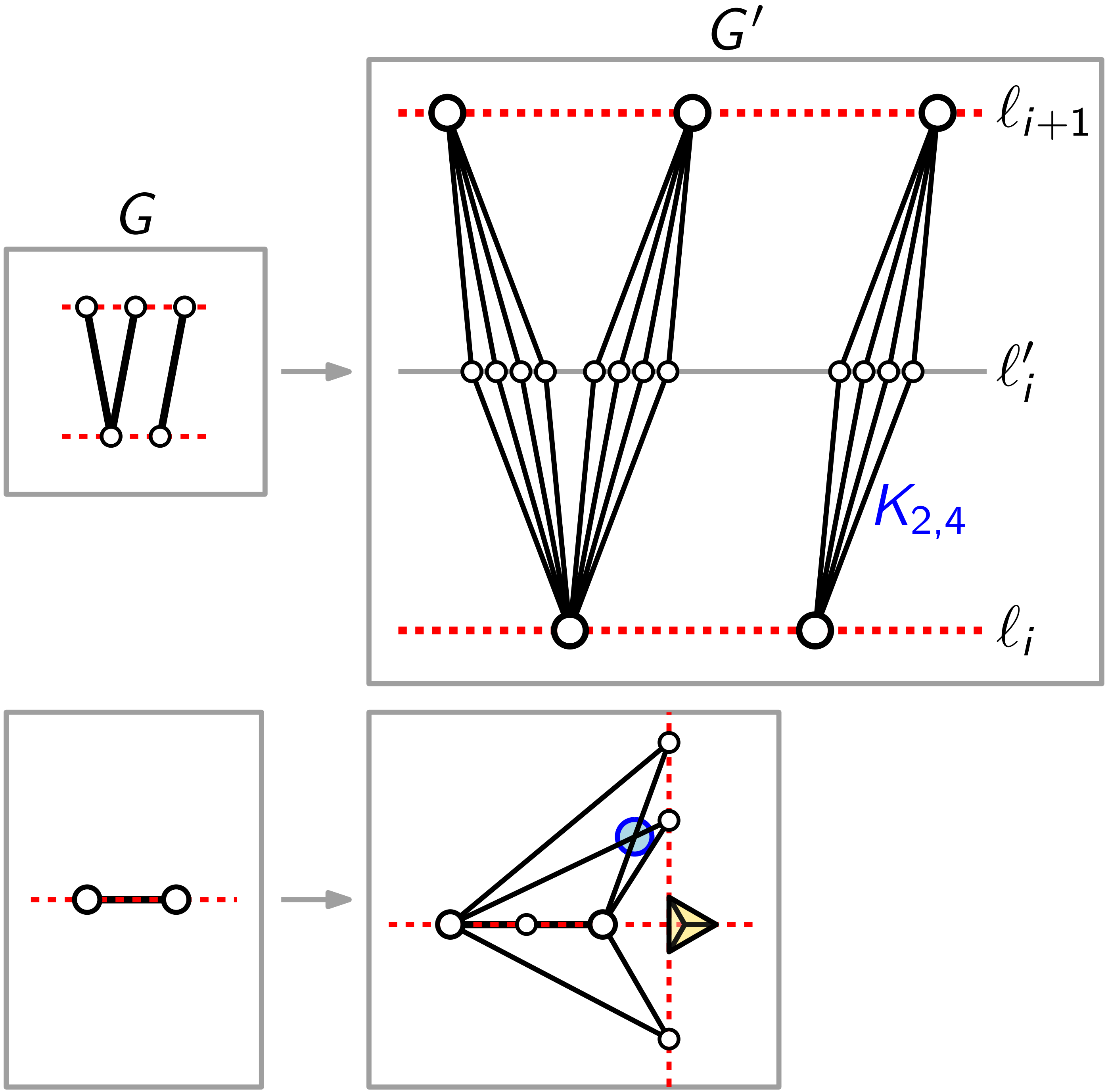

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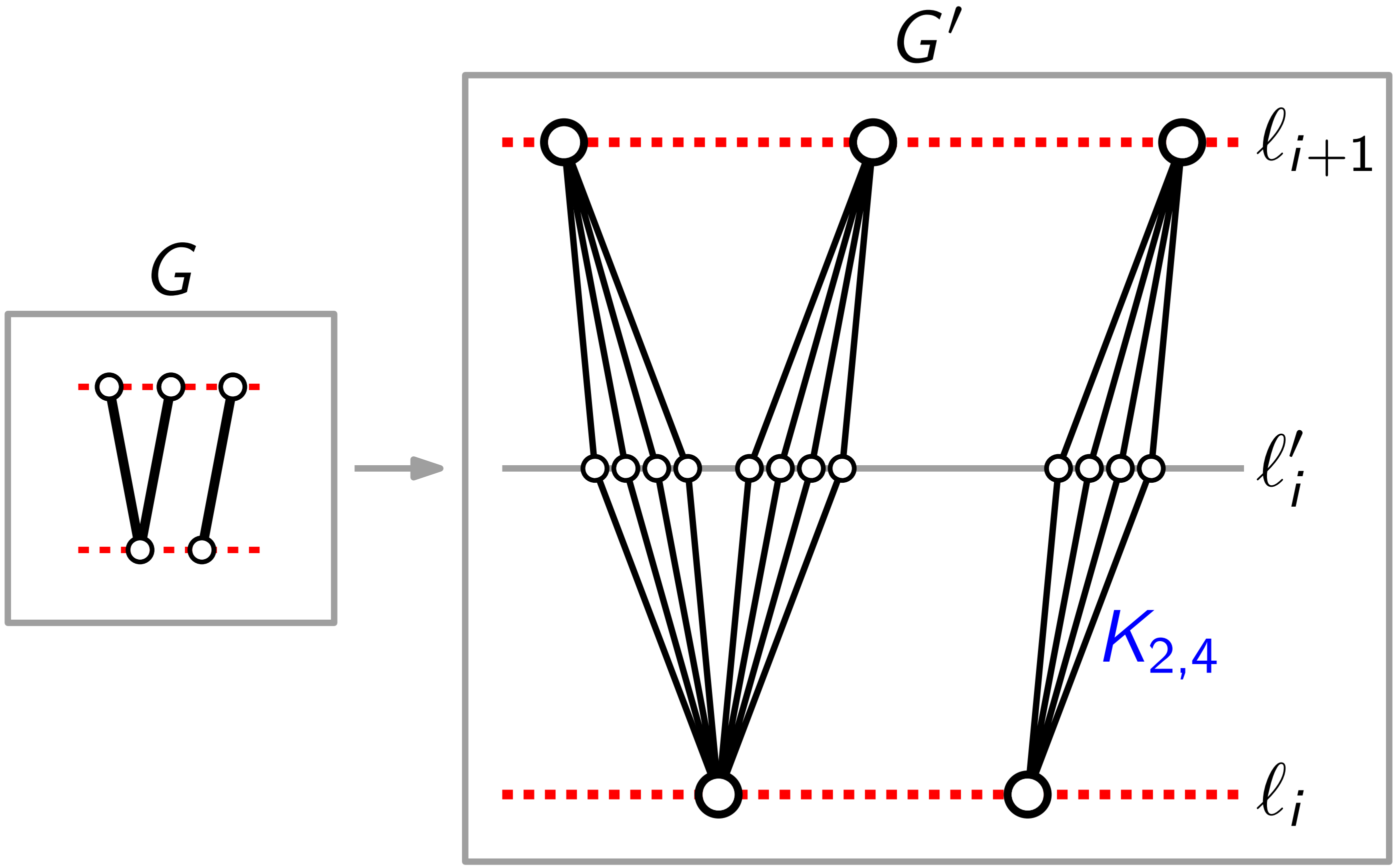

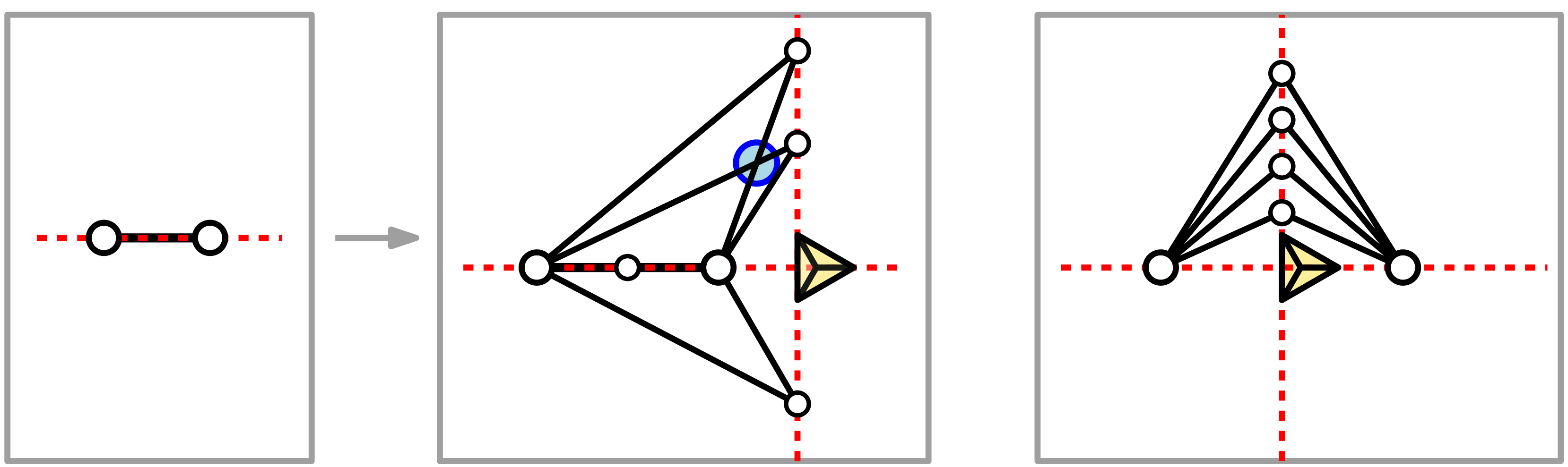

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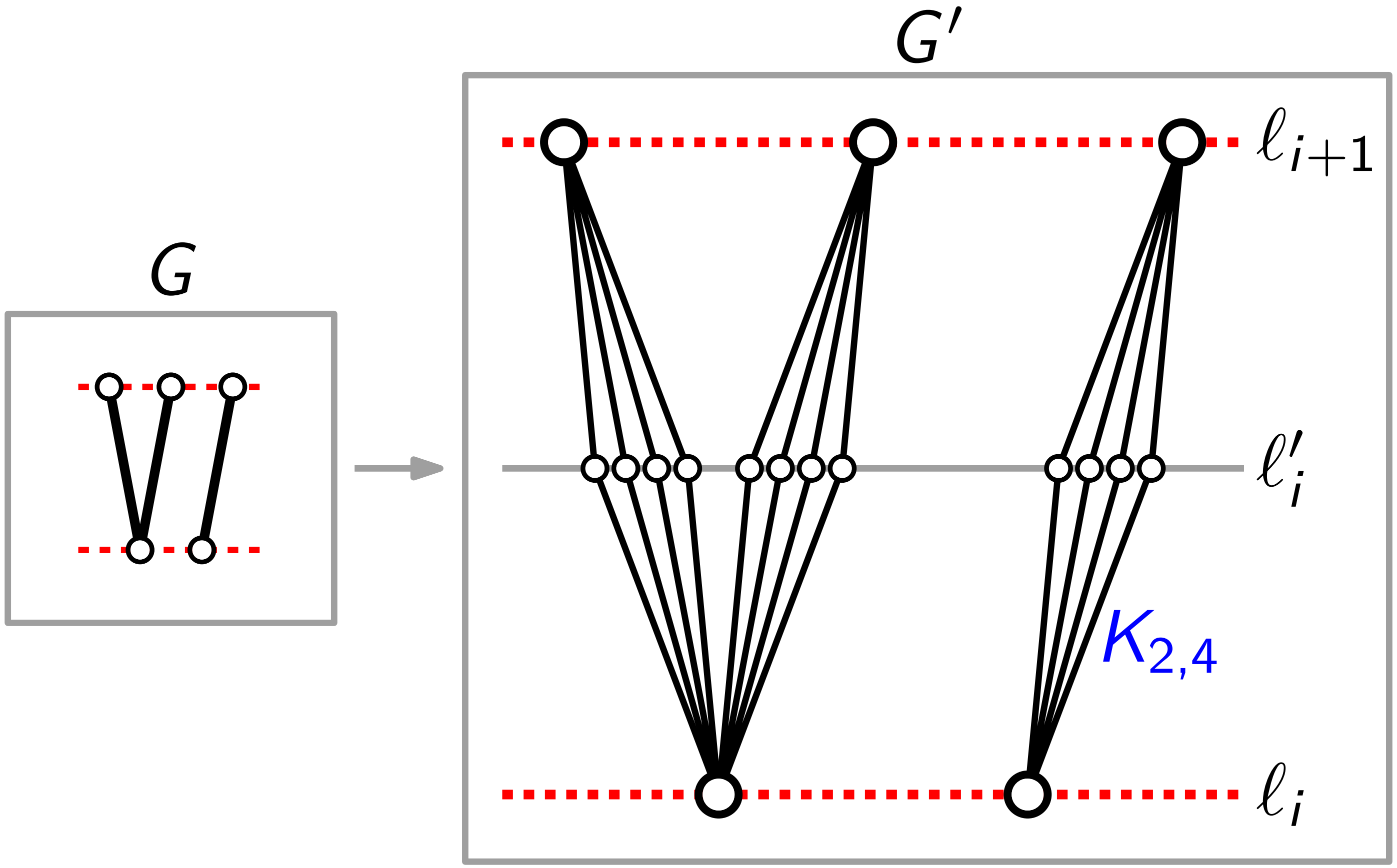

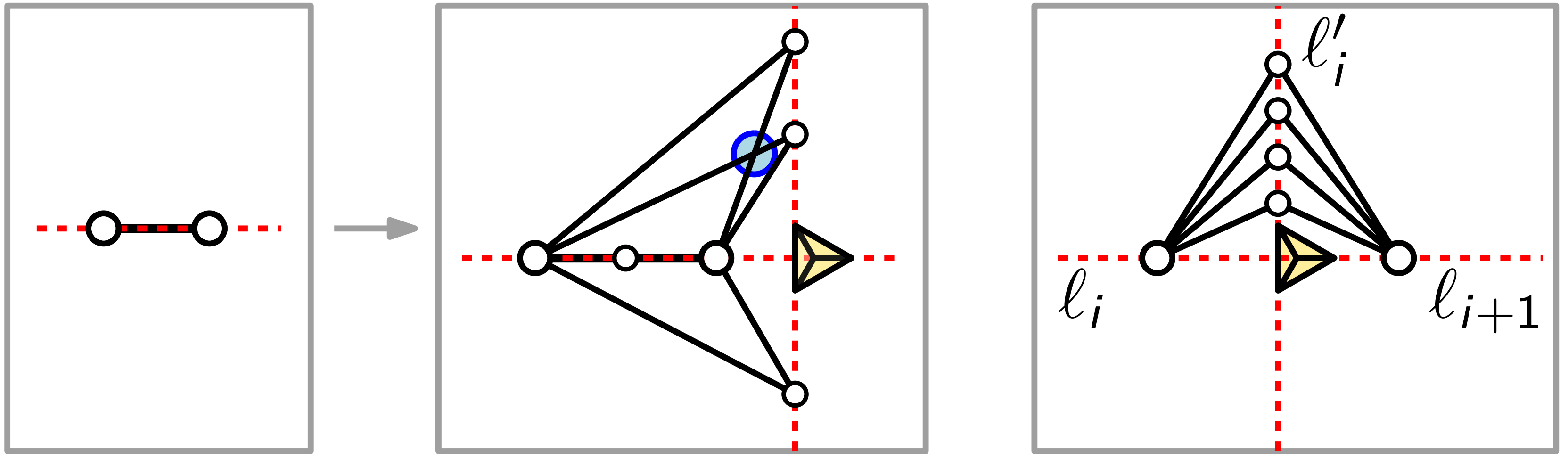

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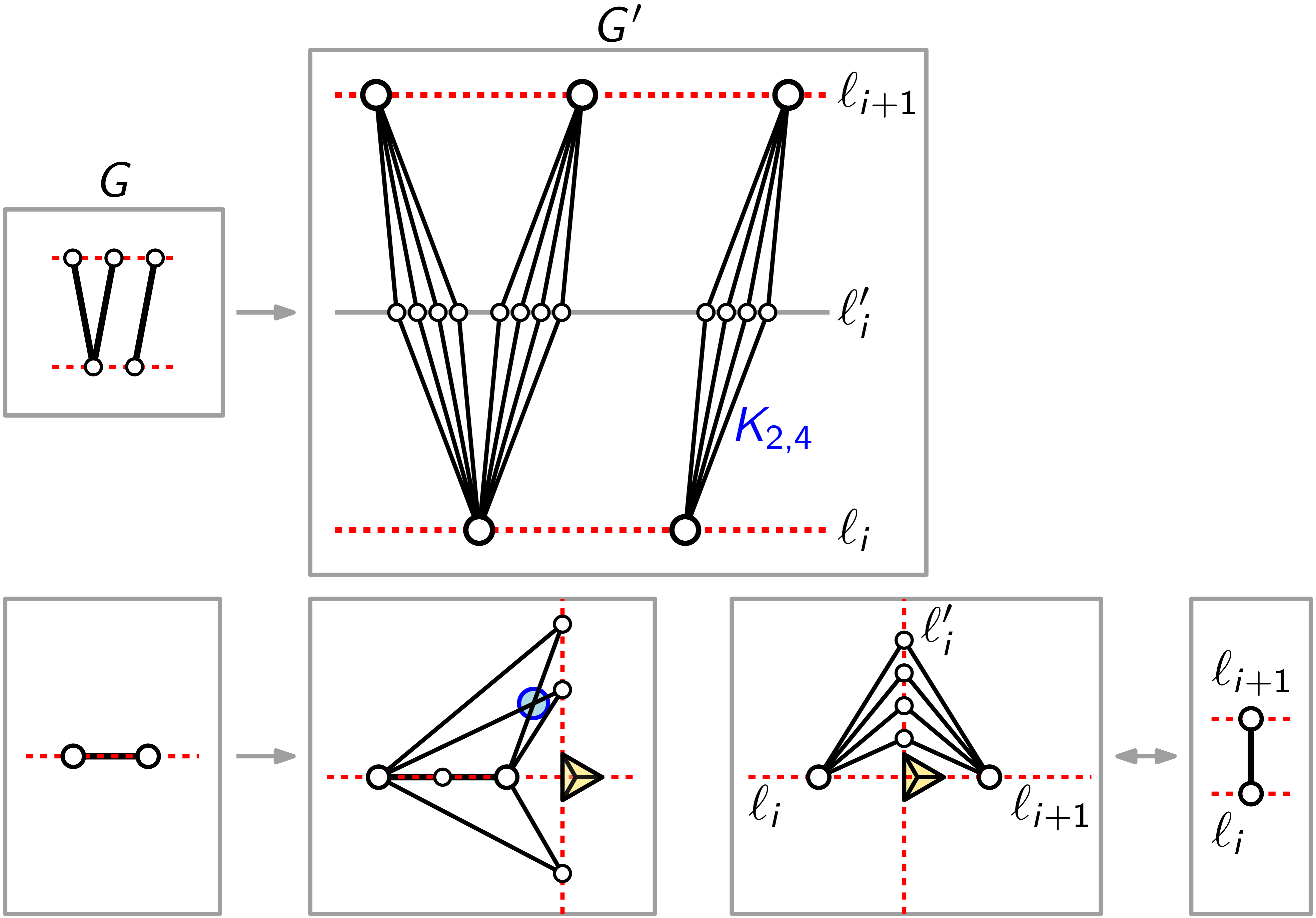

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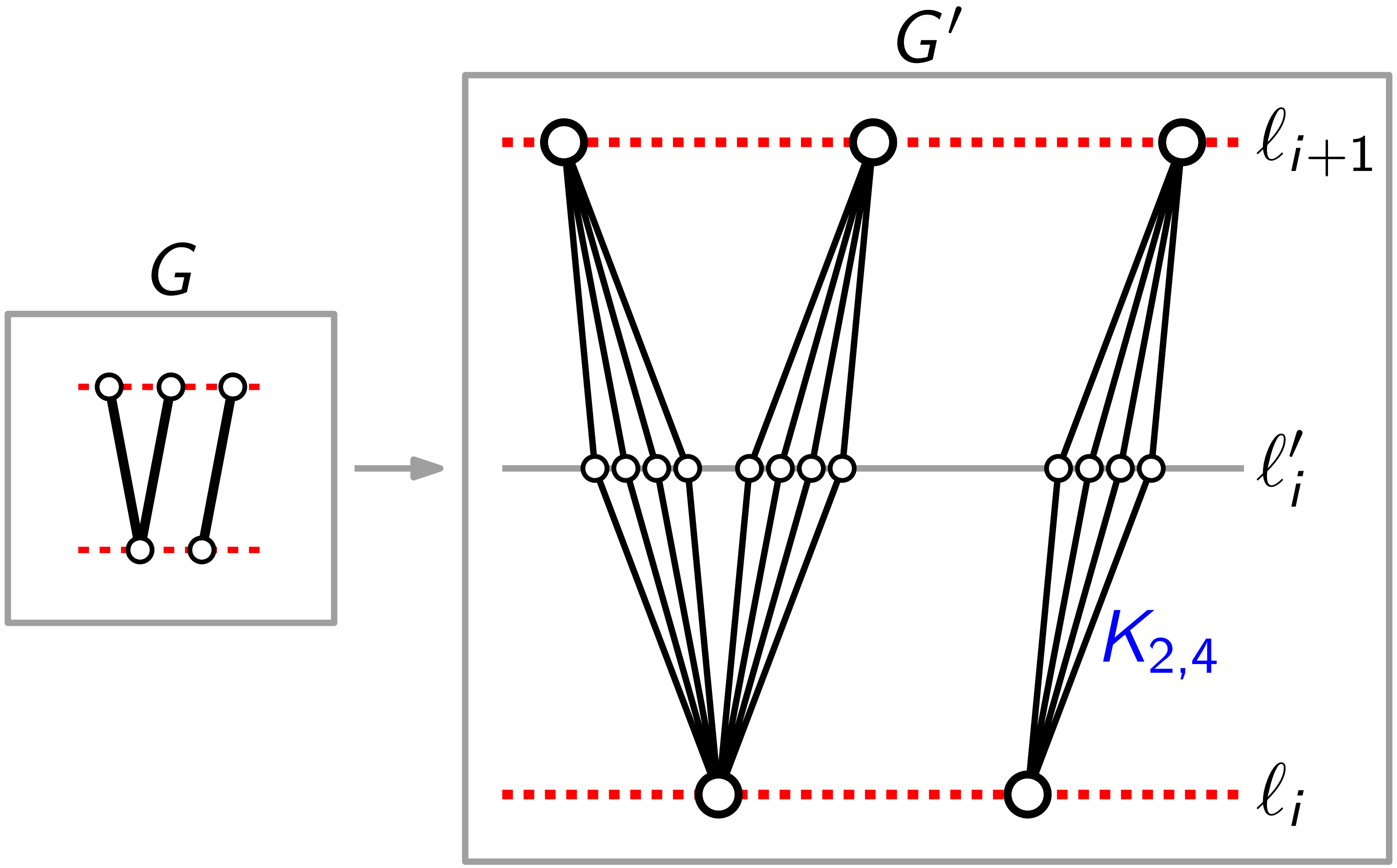
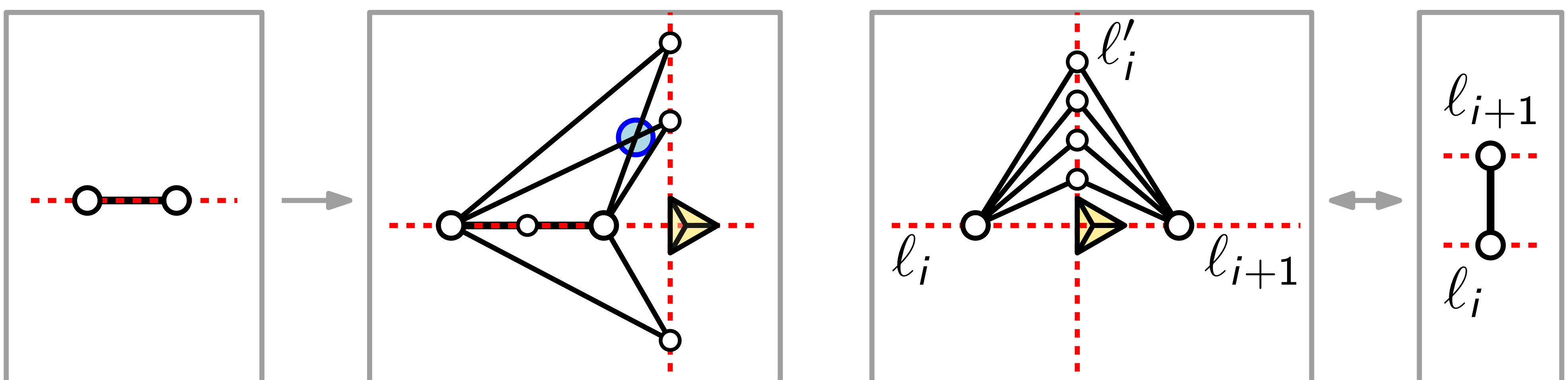
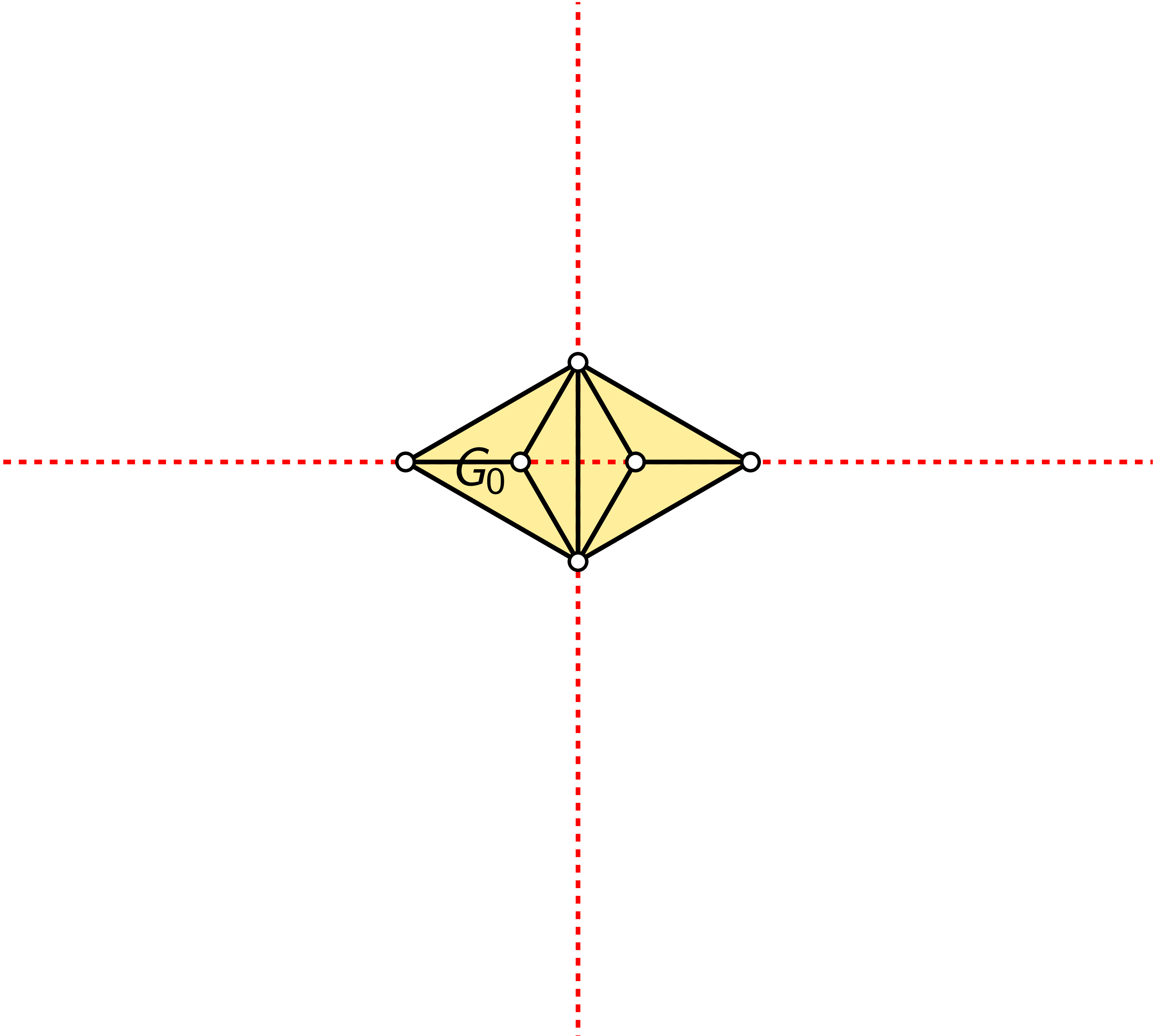


G' ensures that edges of G cannot be drawn *on* levels.



Reduction from LevelPlanarity



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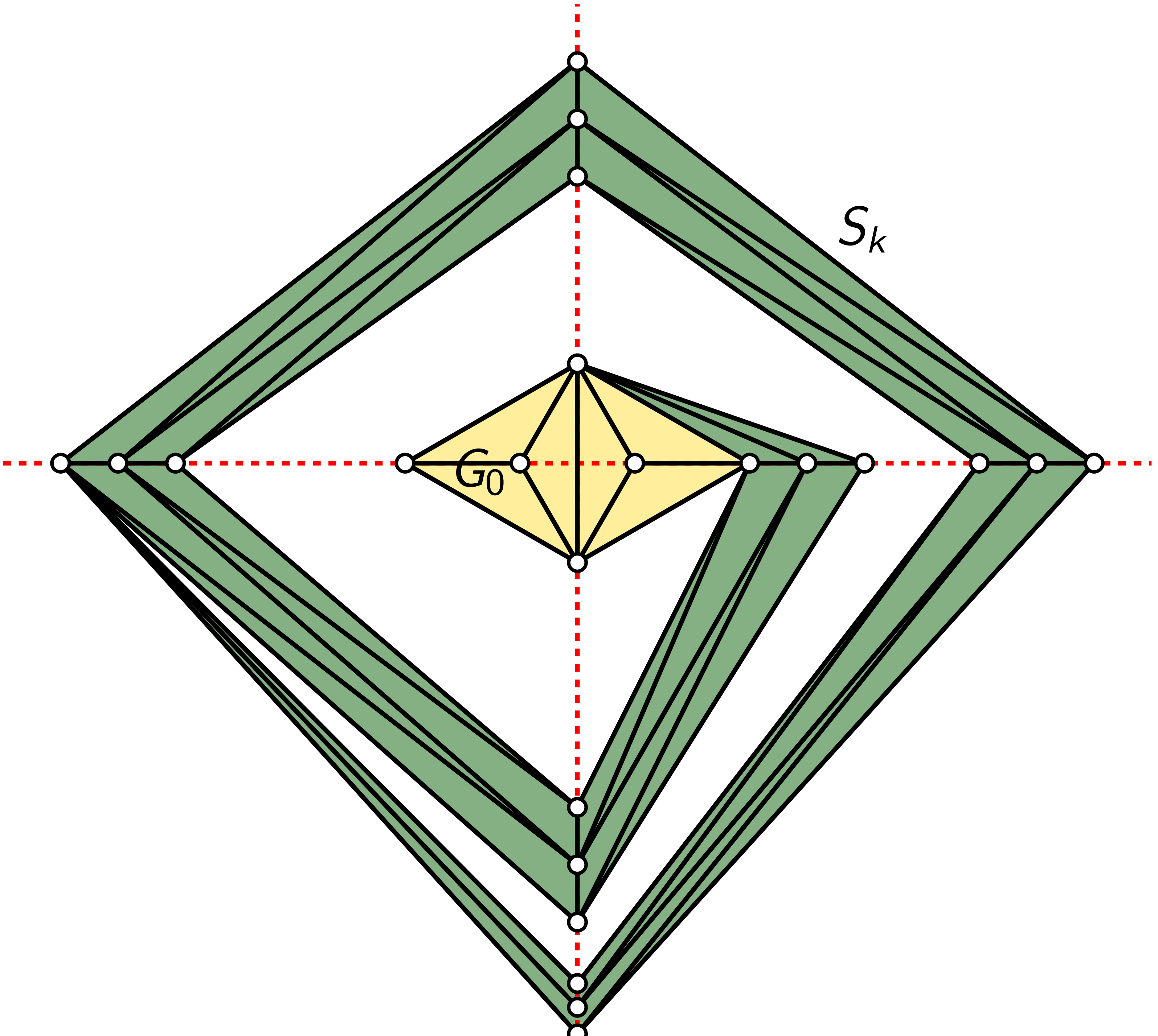

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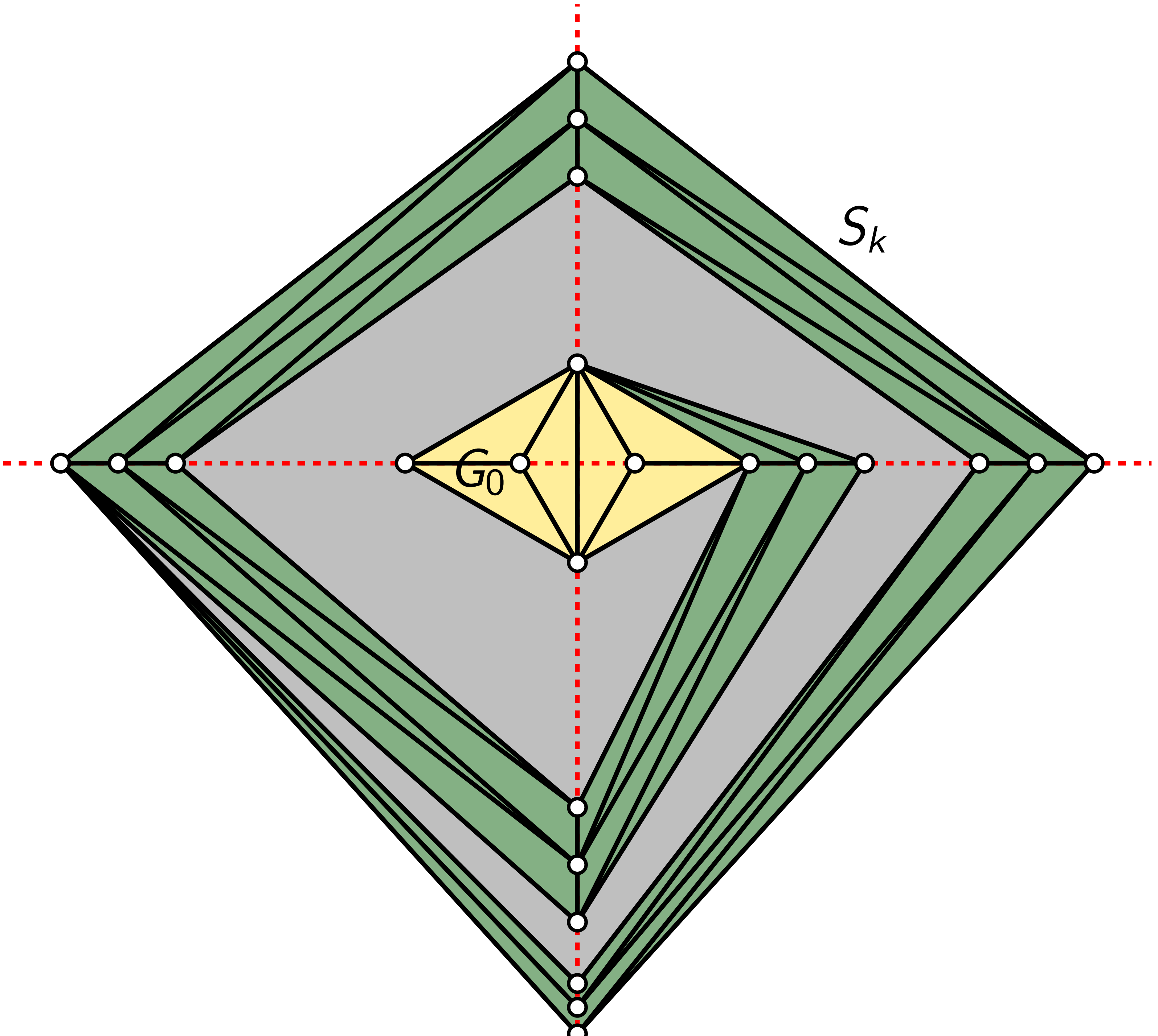

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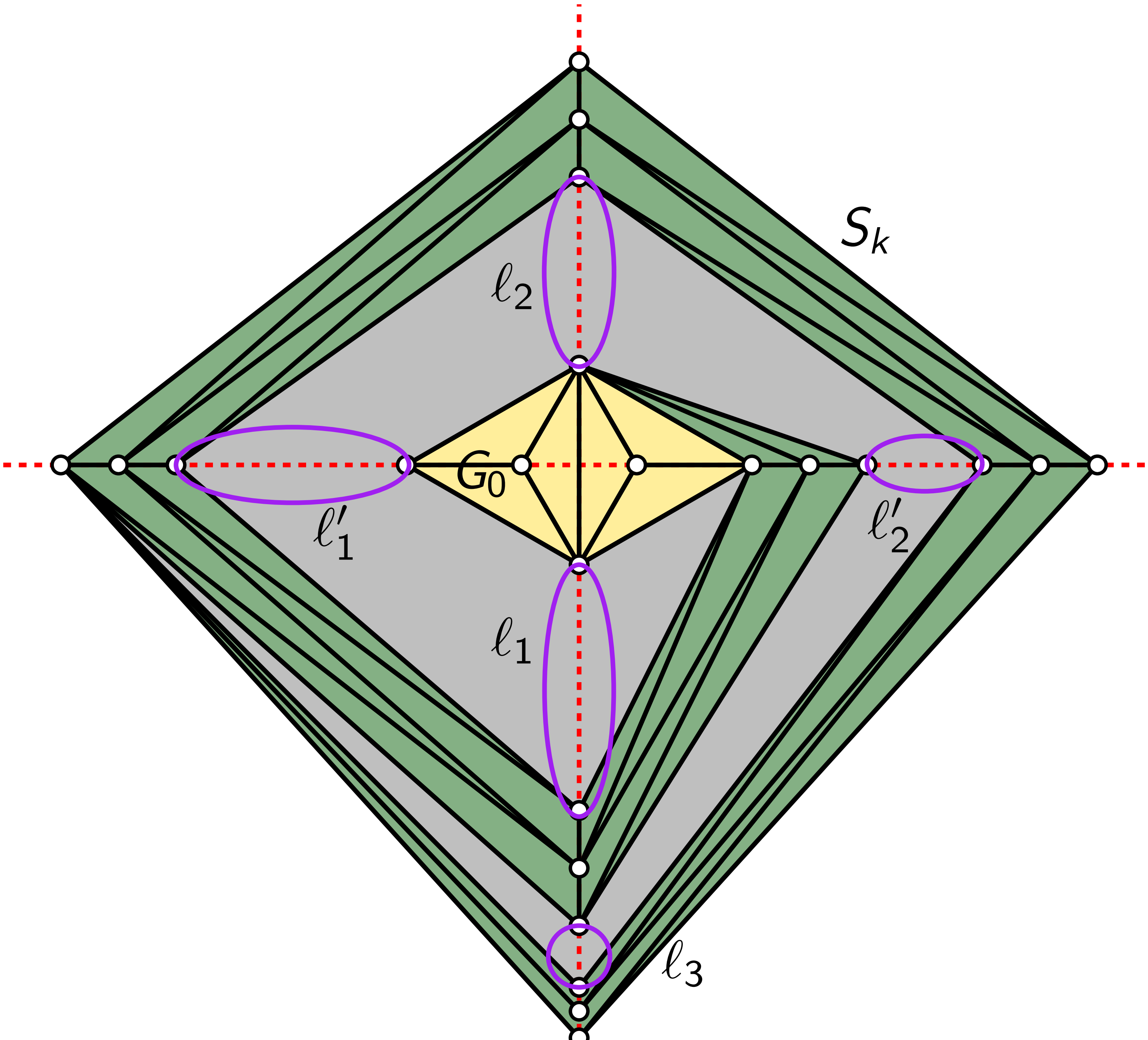

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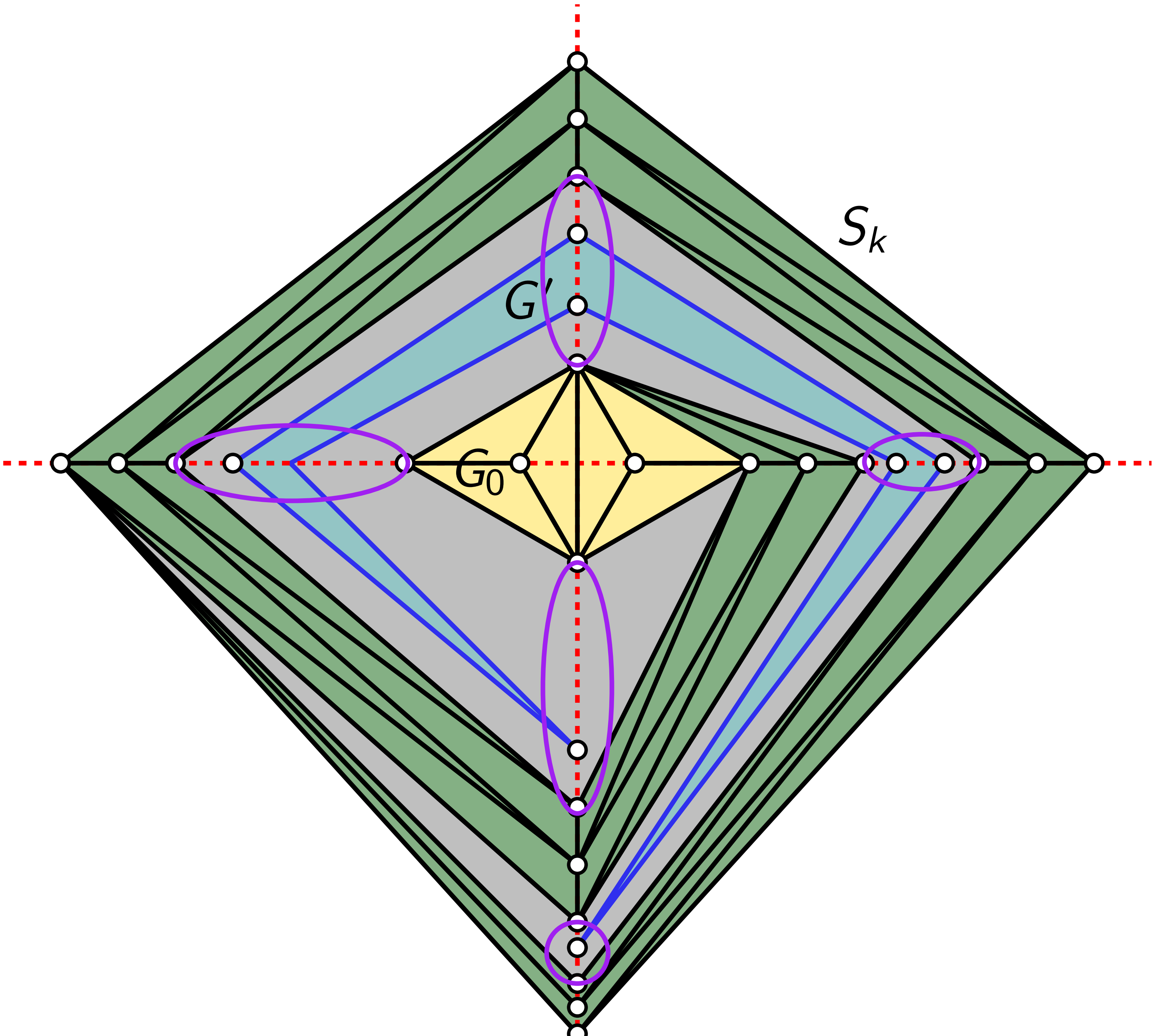

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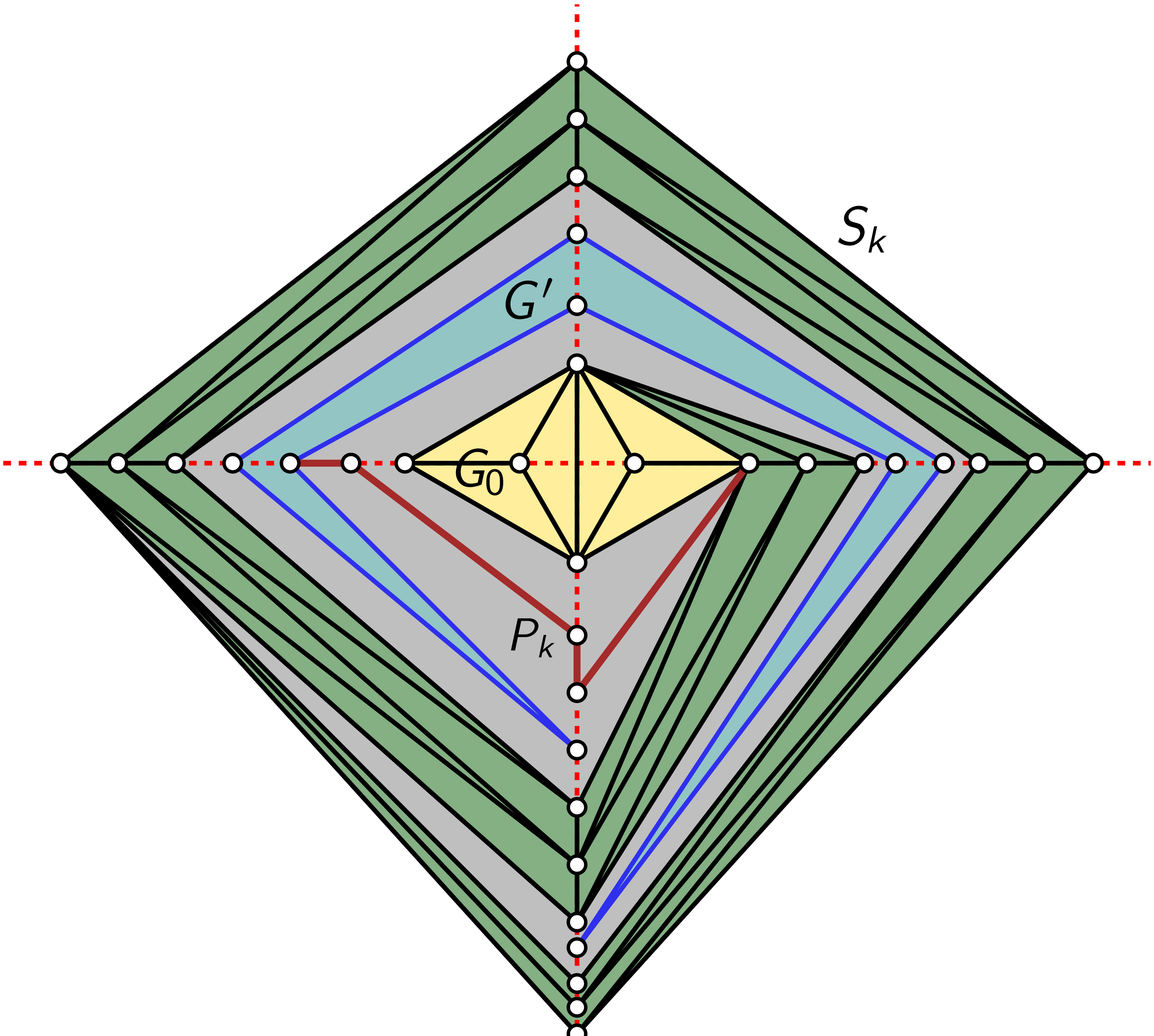

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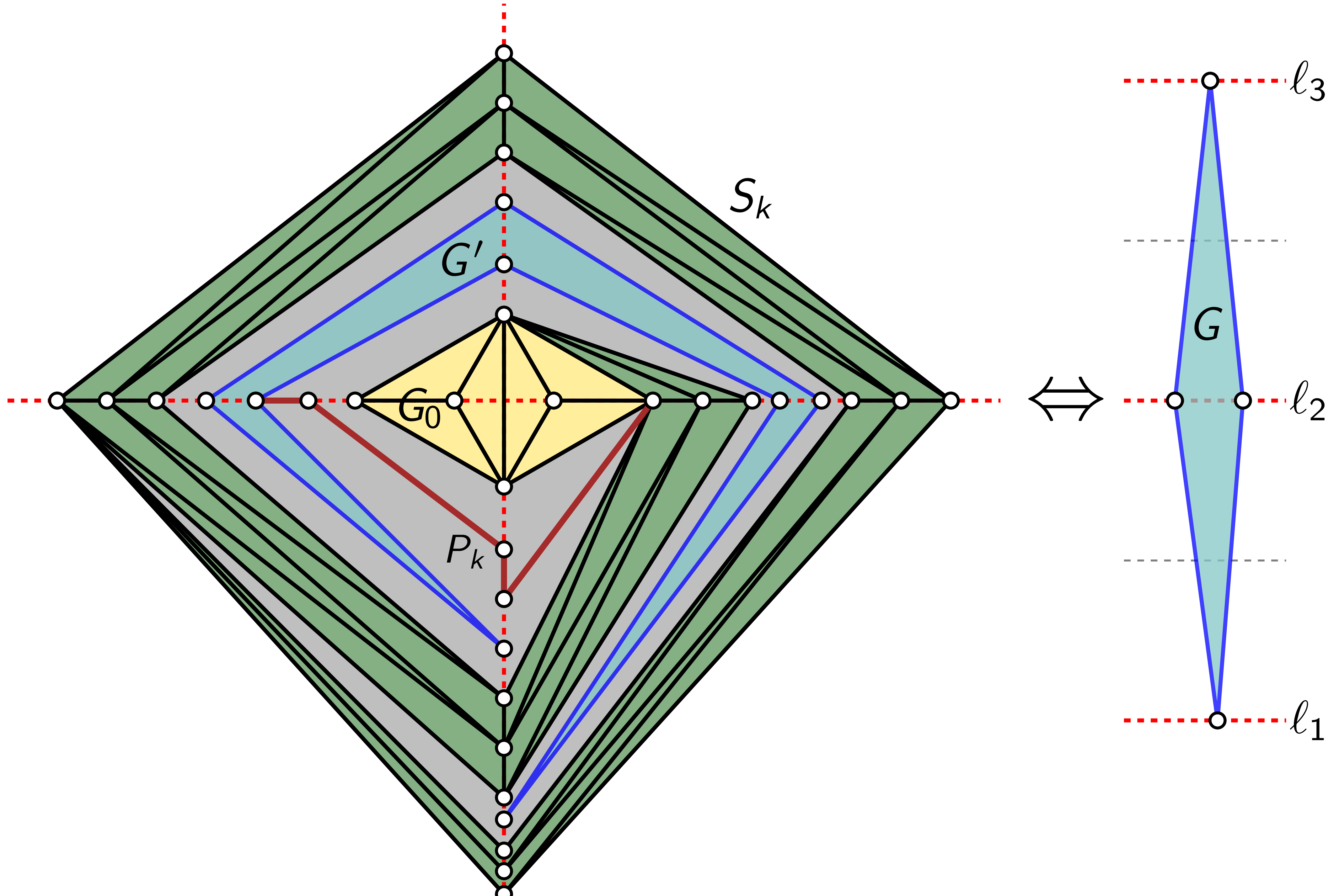

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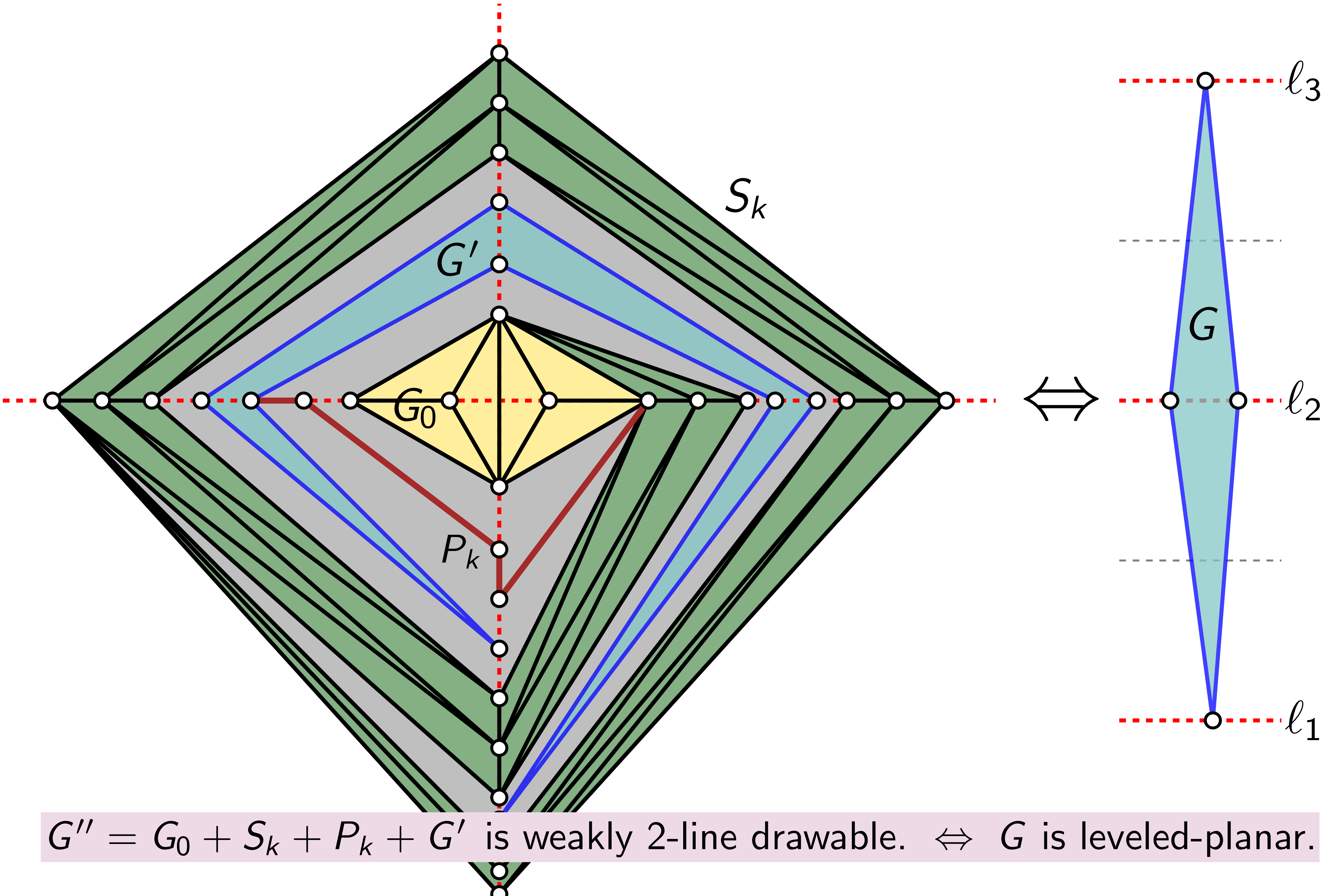

$G'' = G_0 + S_k + P_k + G'$ is weakly 2-line drawable. $\Leftrightarrow$ G is leveled-planar.

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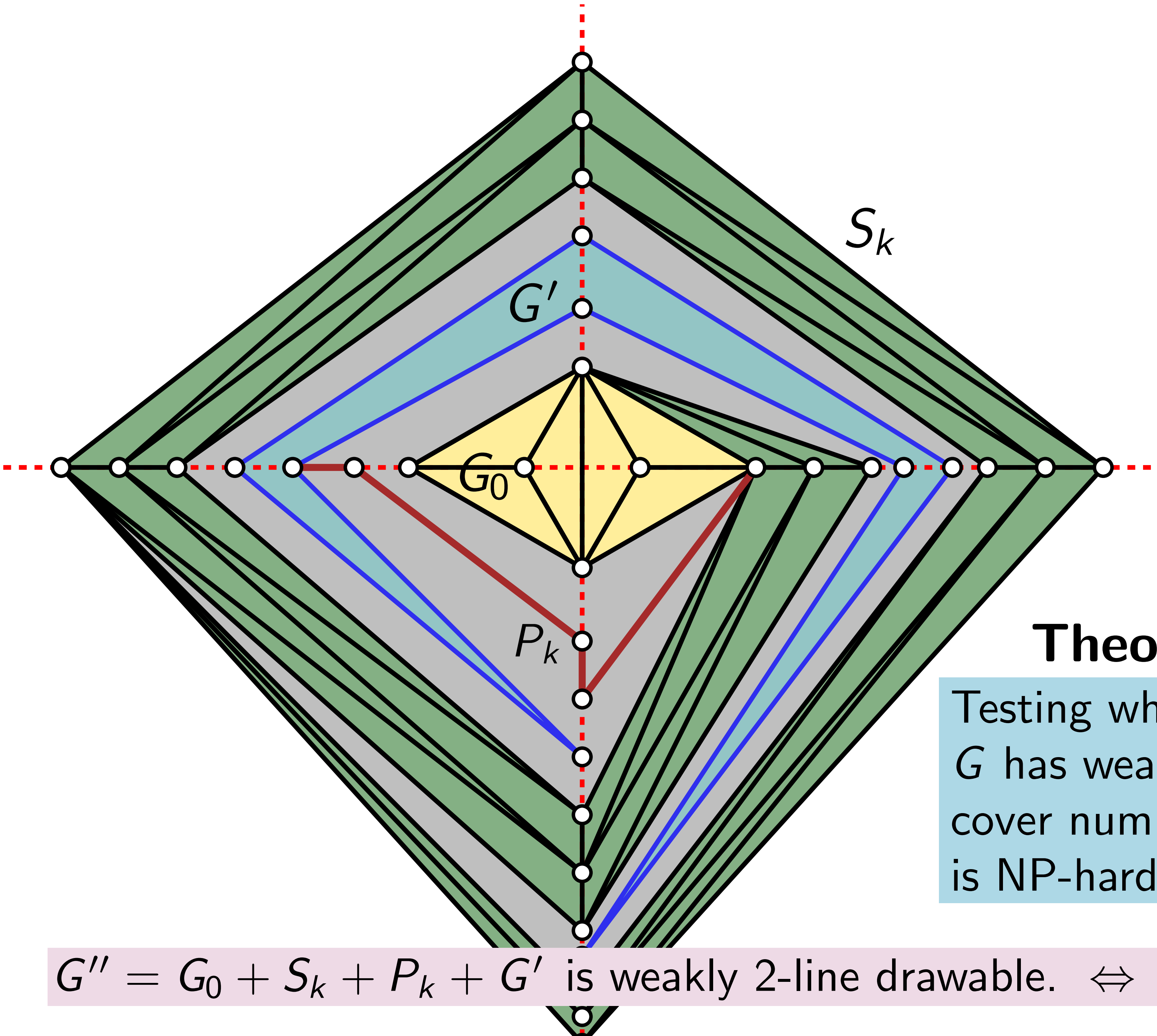

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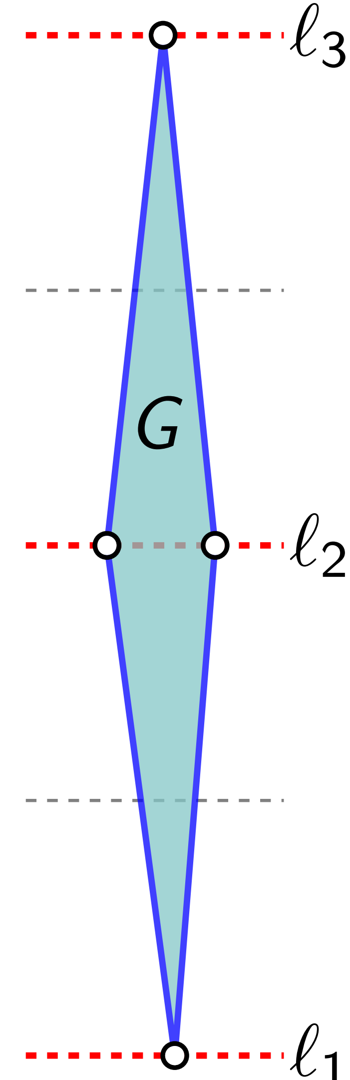

Theorem.
Testing whether G has weak line cover number 2 is NP-hard.

$G'' = G_0 + S_k + P_k + G'$ is weakly 2-line drawable. $\Leftrightarrow$ G is leveled-planar.

Our Results

- It is NP-hard to test whether a given graph G can be weakly covered by 2 lines.
  (Hence, weak line cover number is not in FPT.) ✓

- The weak line cover number of the universal stacked triangulation of depth d is $d + 1 \in \Theta(\log n)$.

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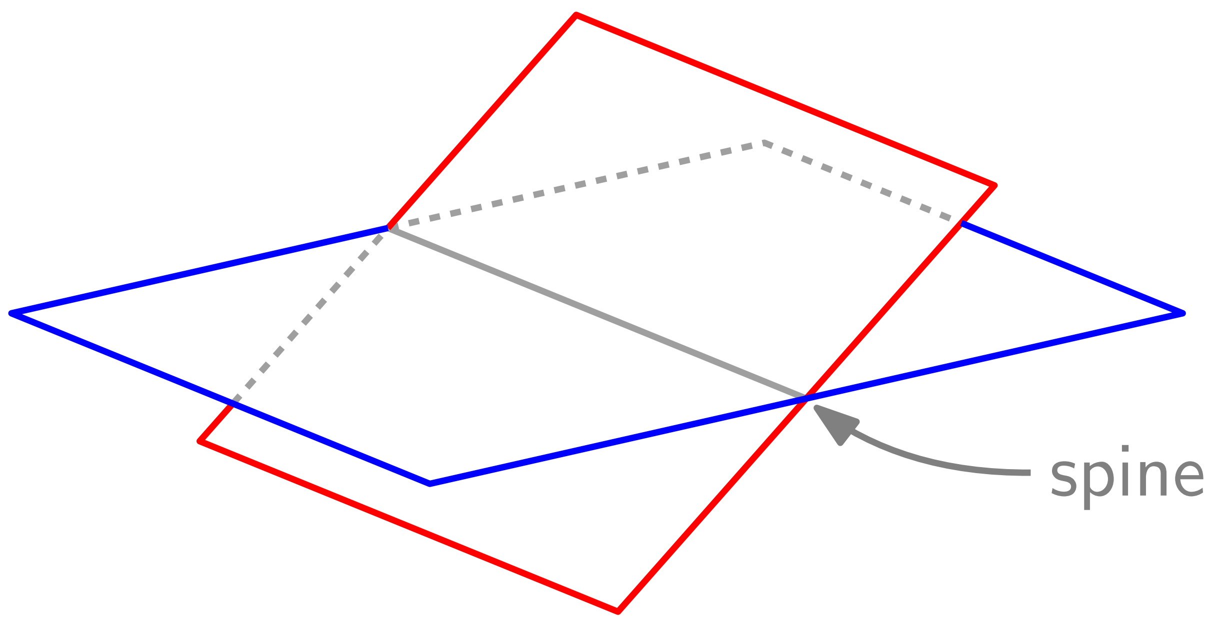

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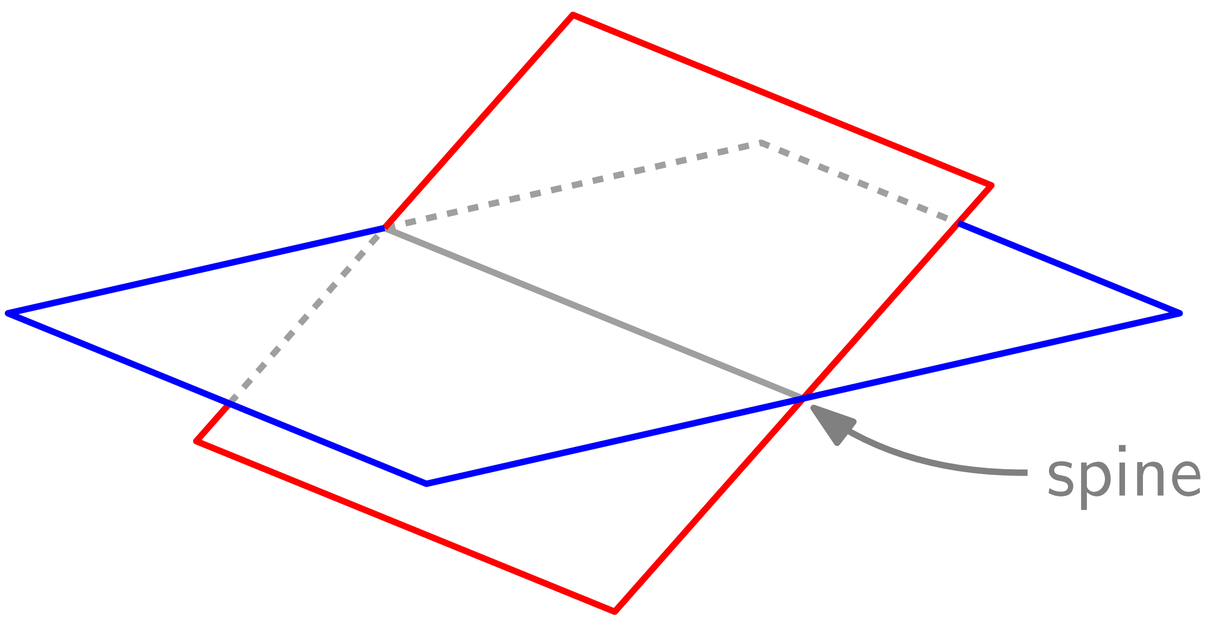

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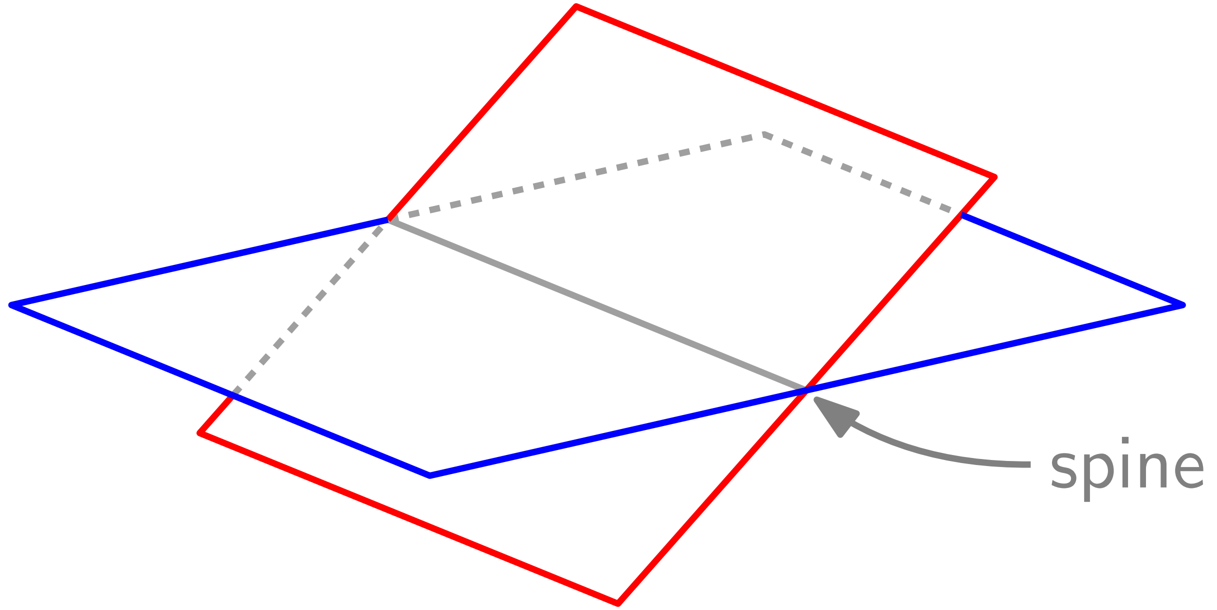
Let R be the subgraph of G in the red plane.



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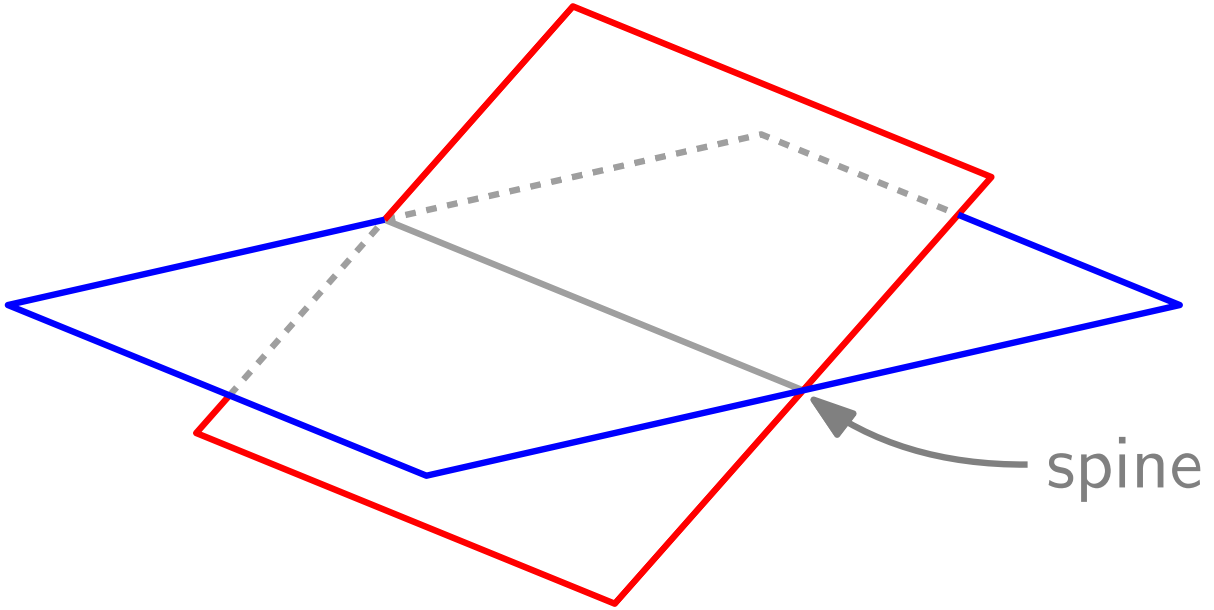

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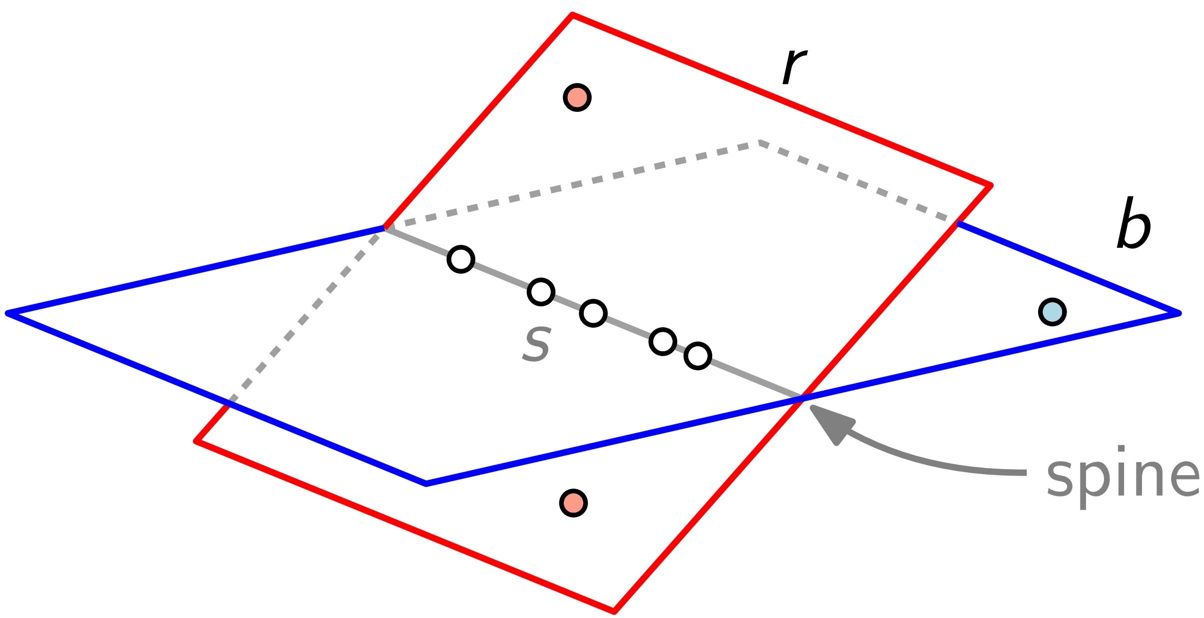

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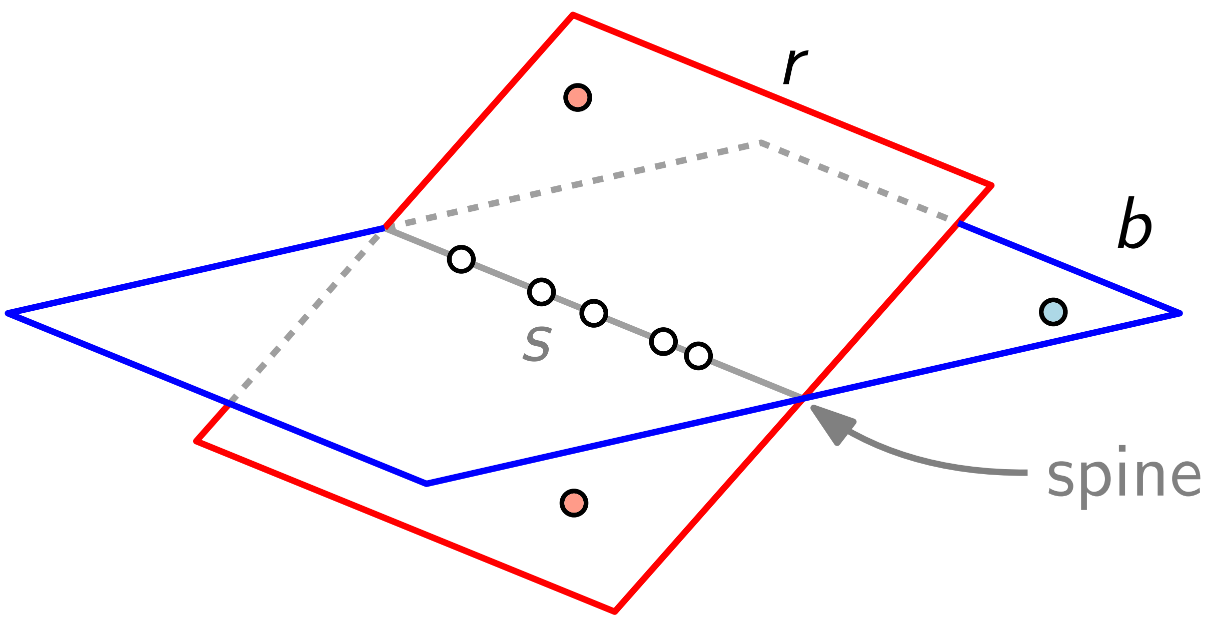

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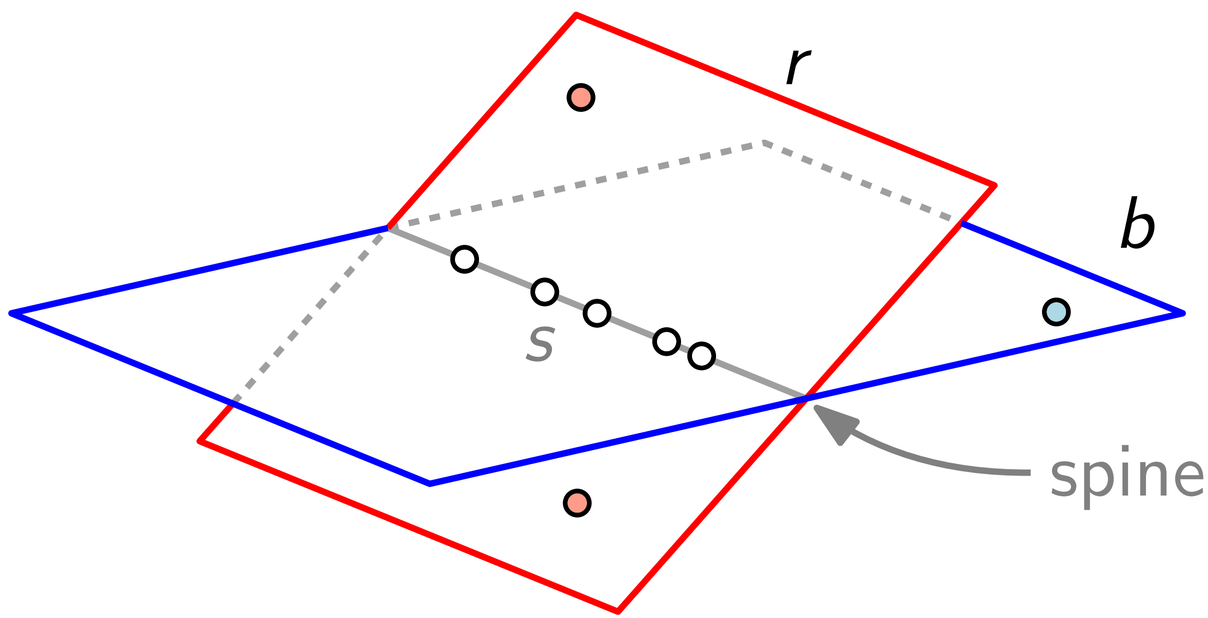

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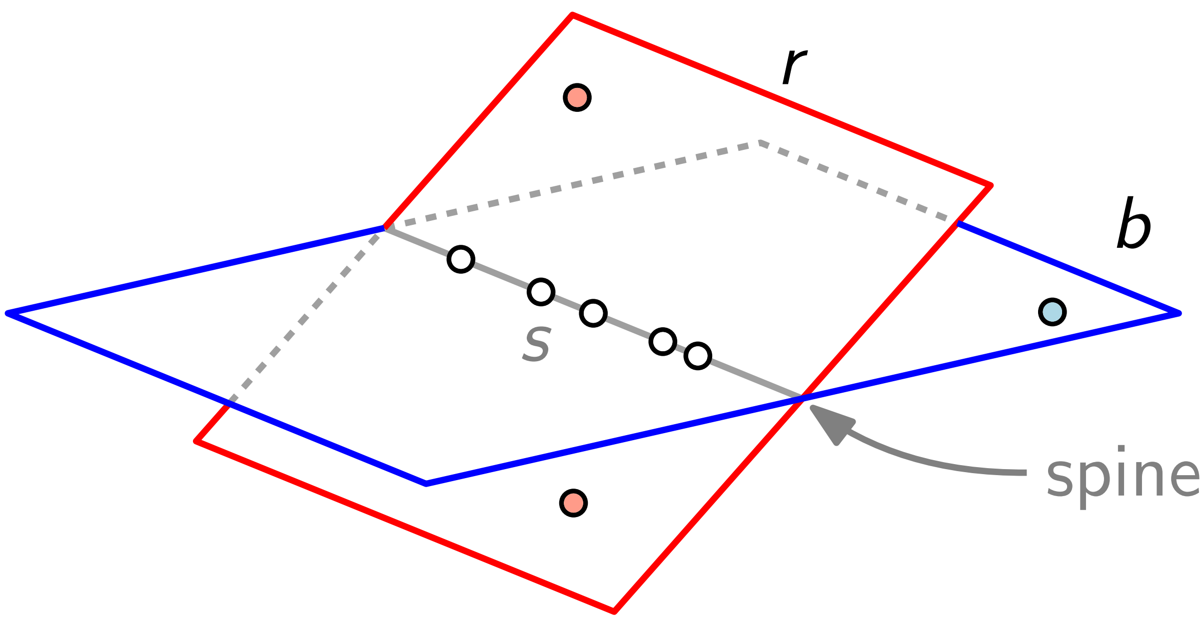

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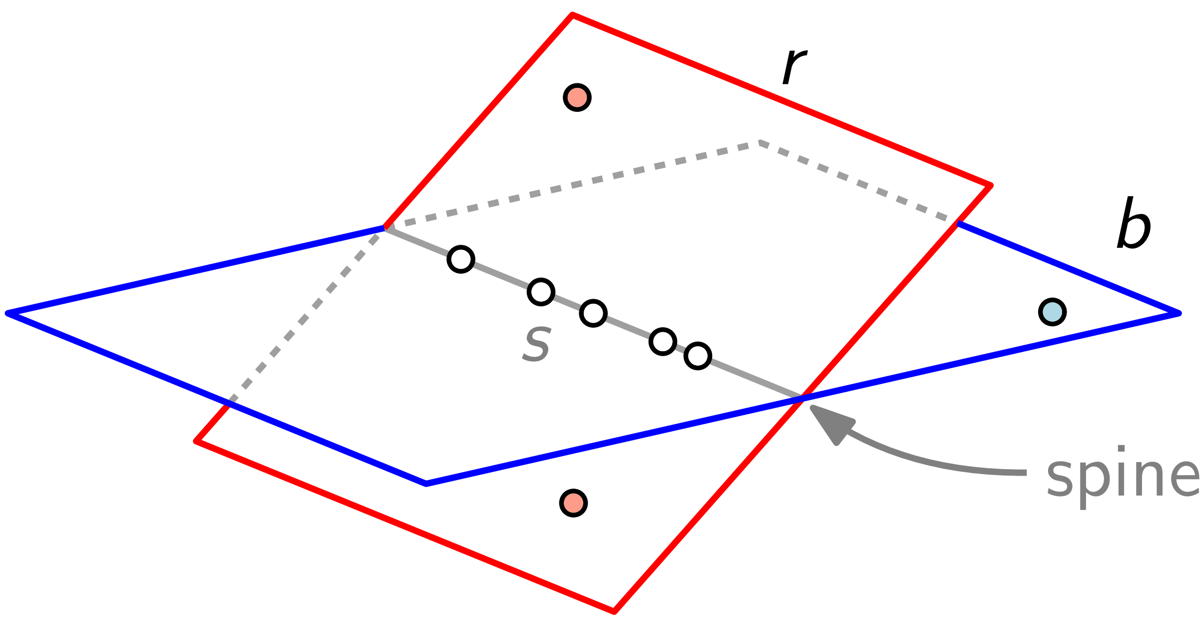

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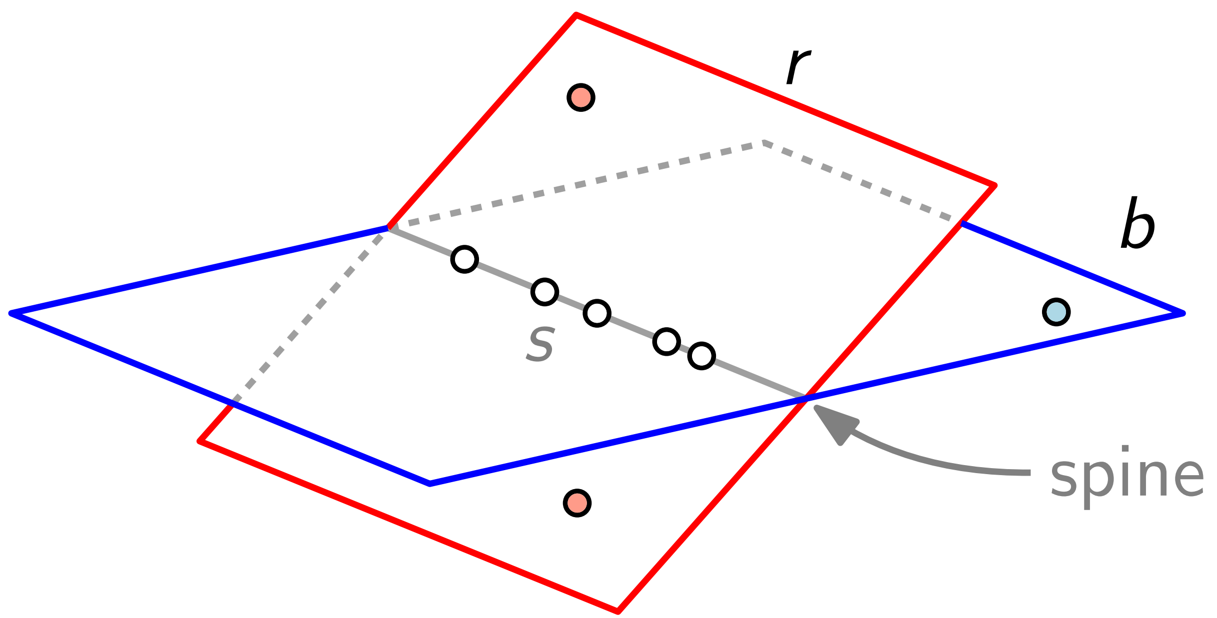

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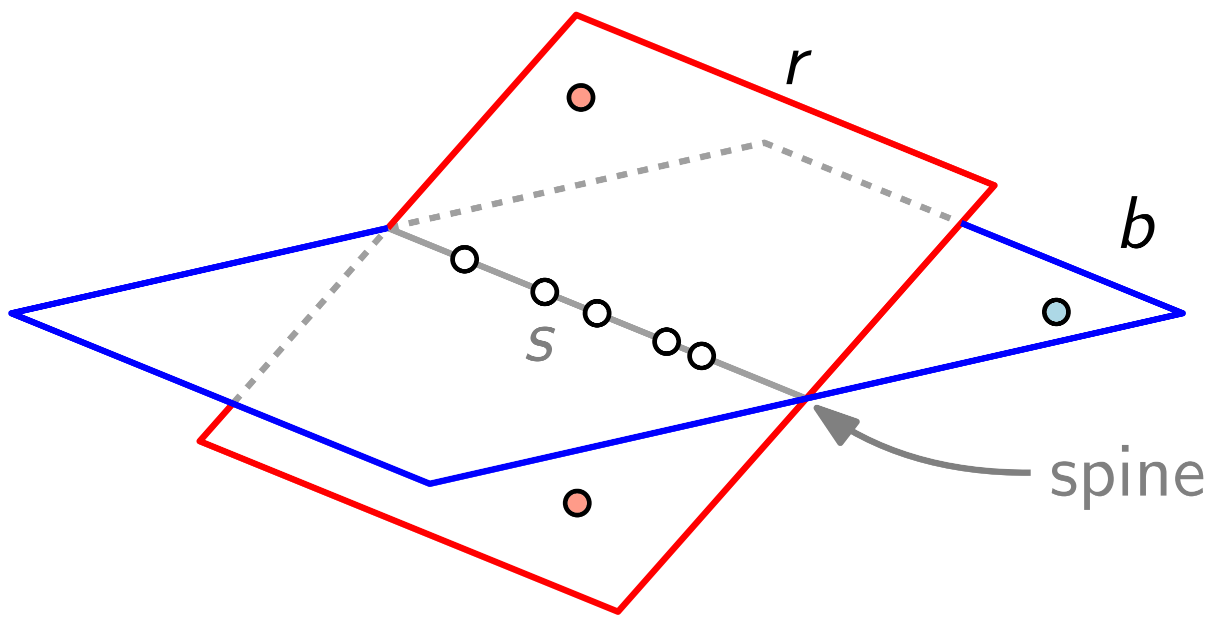

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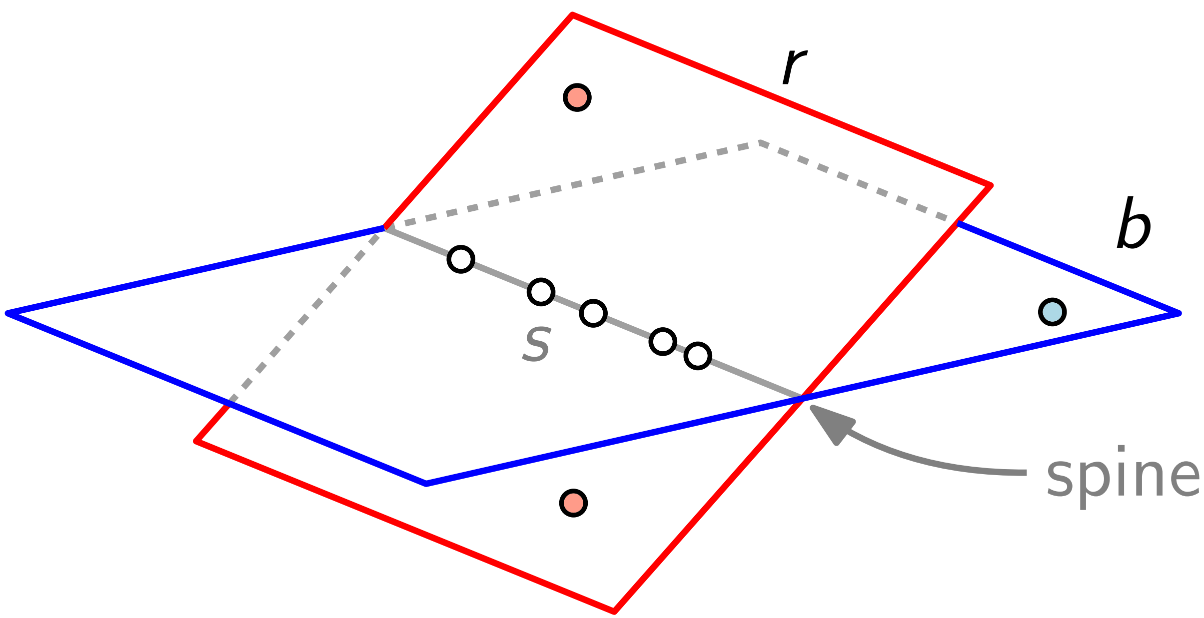

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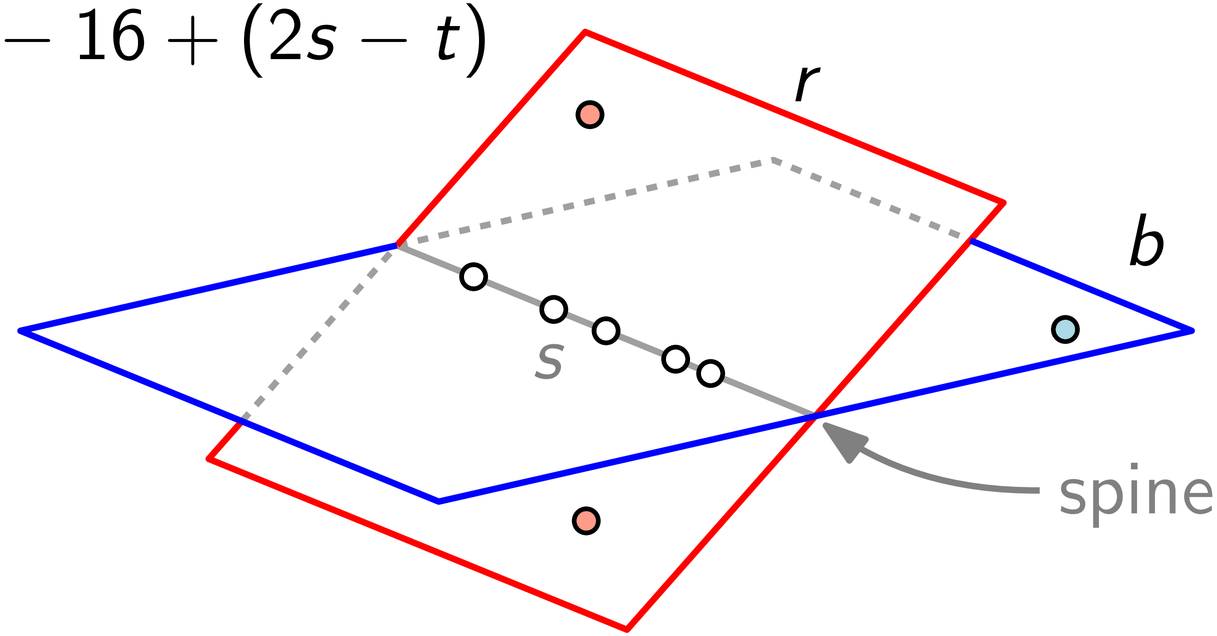

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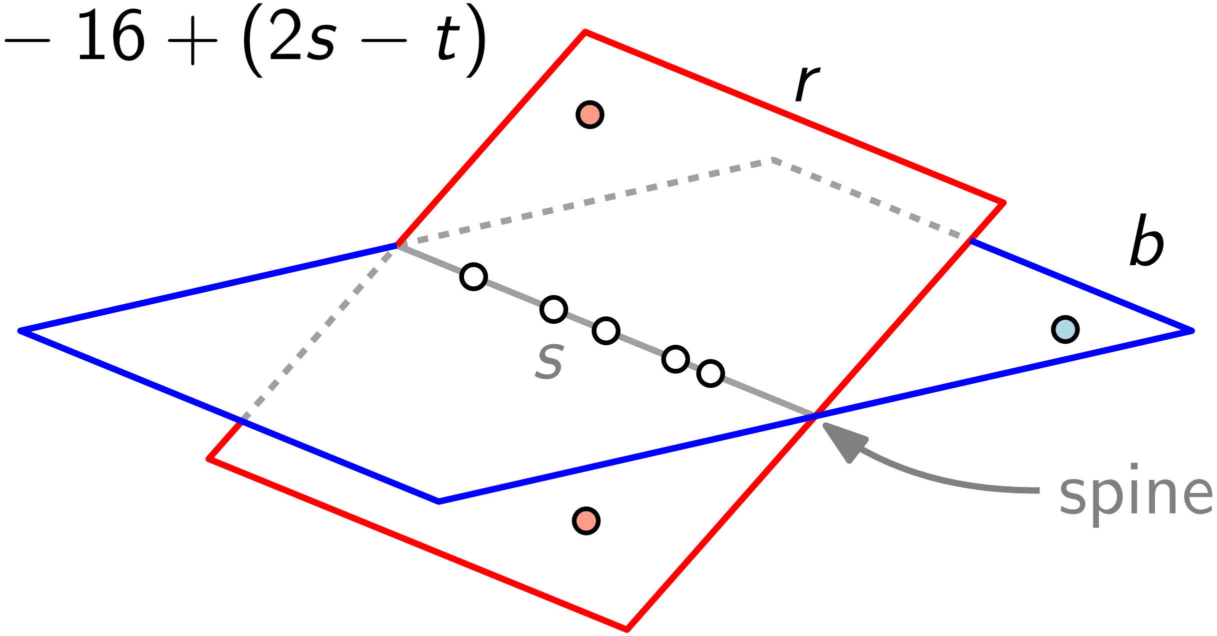

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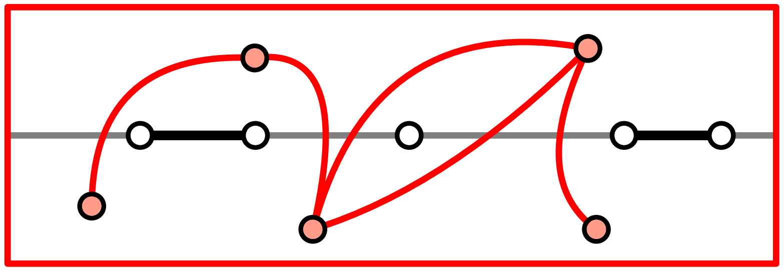

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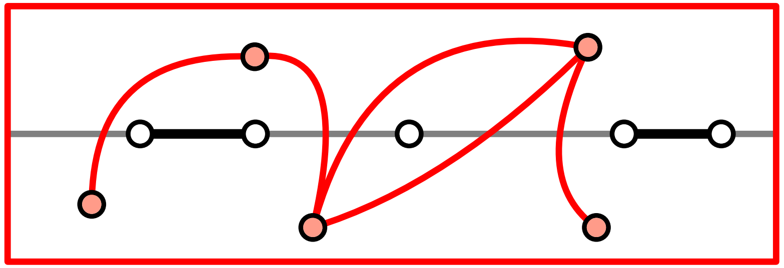

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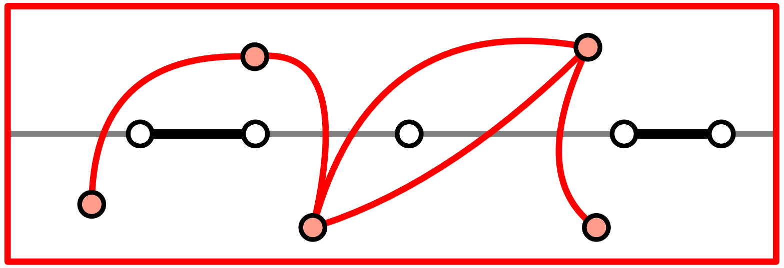

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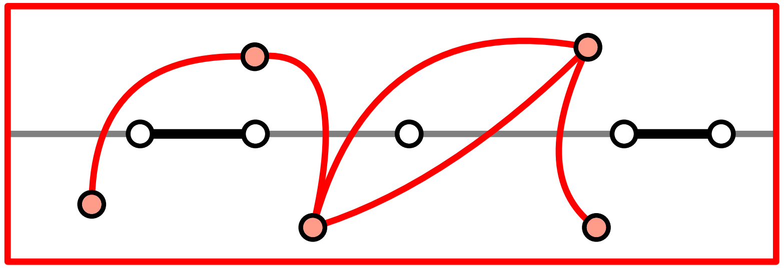

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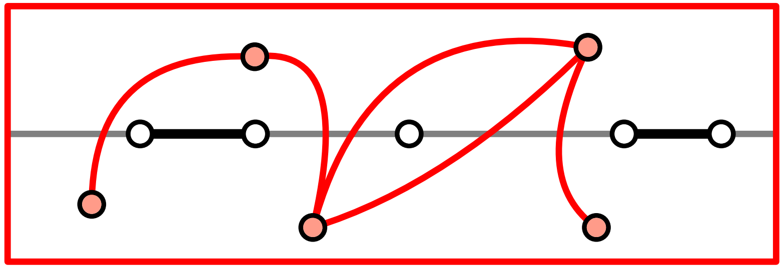

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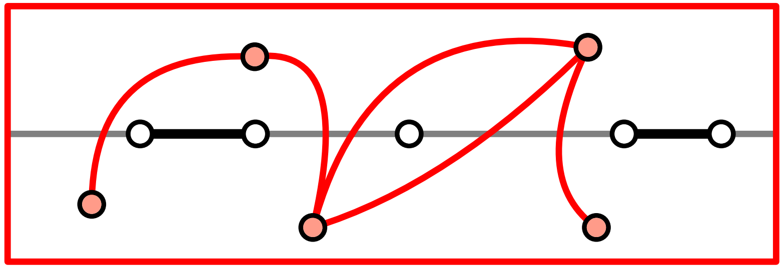

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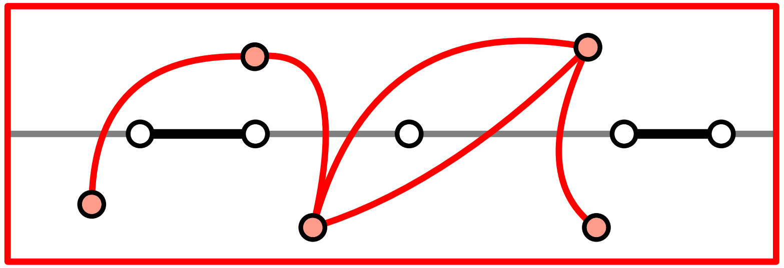

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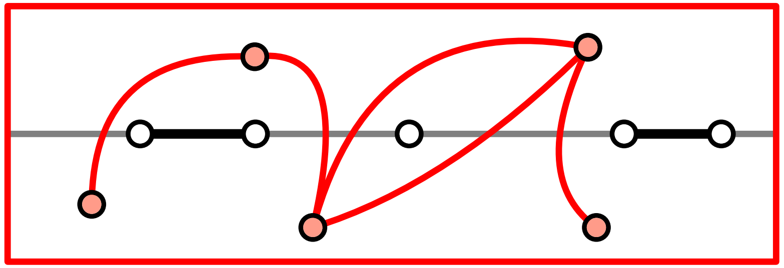

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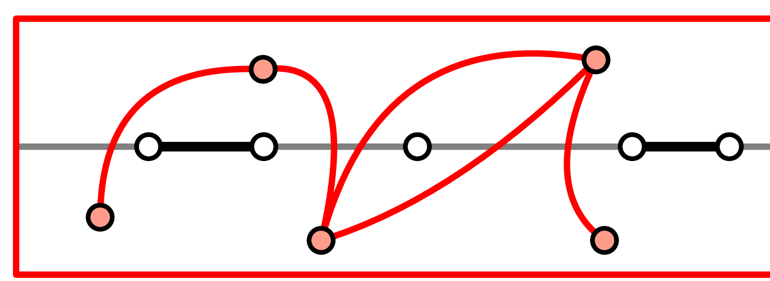

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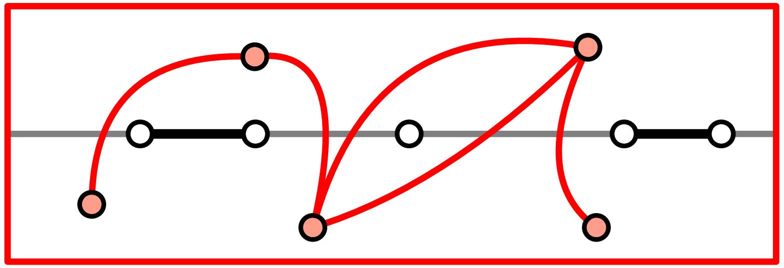

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- It is NP-hard to test whether a given graph G can be weakly covered by 2 lines.
  (Hence, weak line cover number is not in FPT.)

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- Open Problems

Open: Weak Line Cover Number

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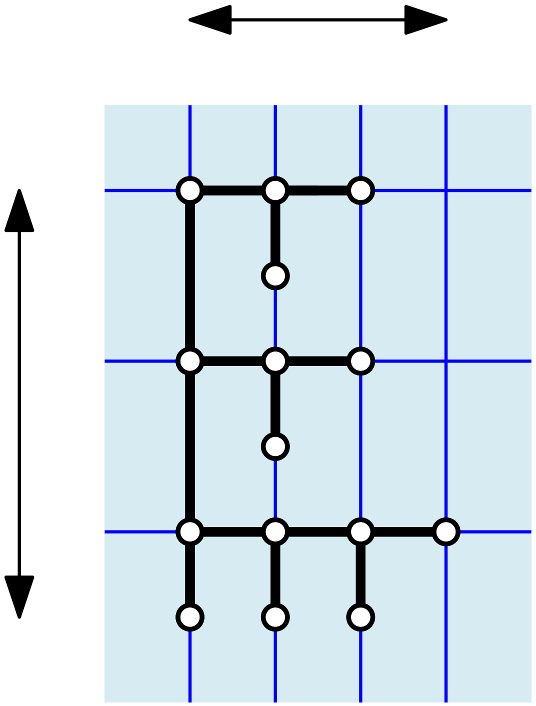
- Is deciding whether the weak line cover number is k in NP?

Open: Strong Line Covers for Binary Trees



- $O(n \log \log n)$ area

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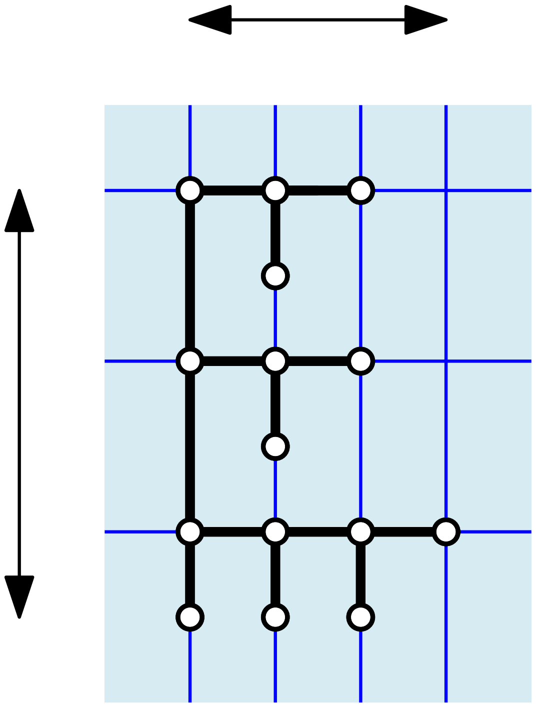

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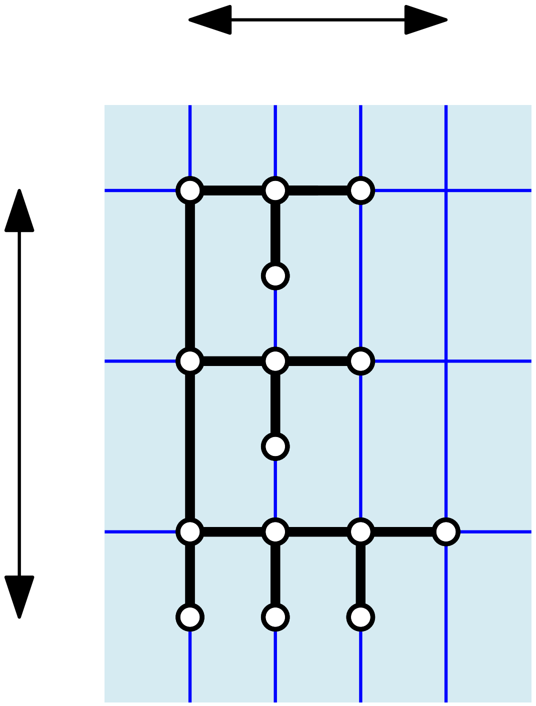

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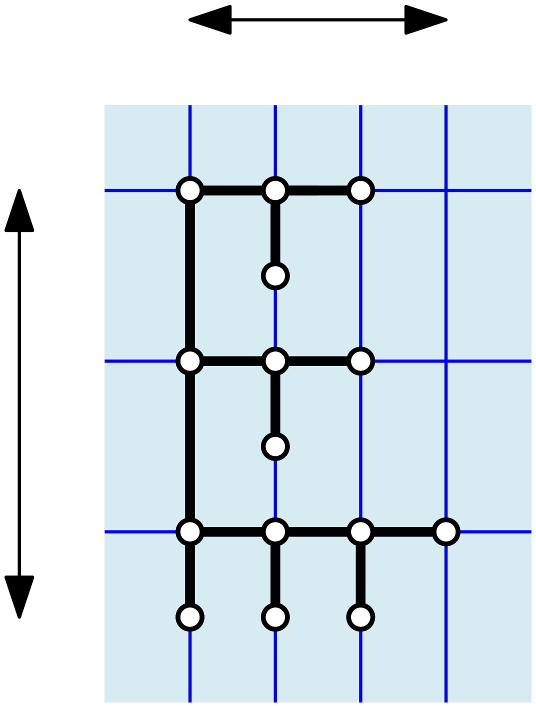

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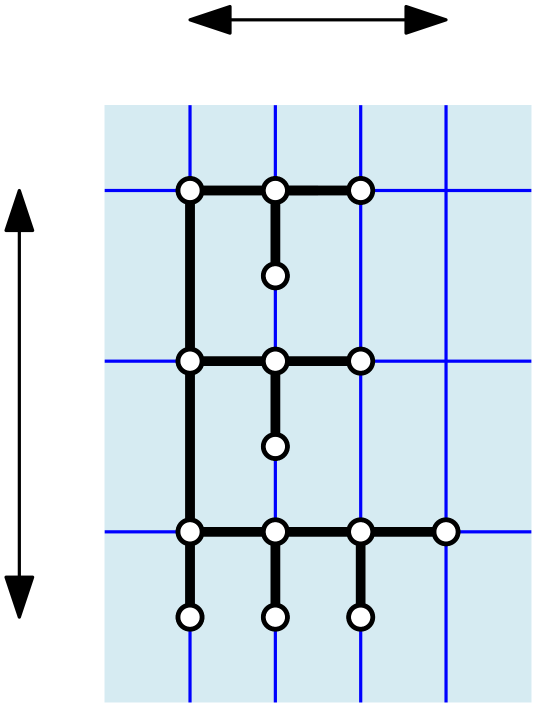

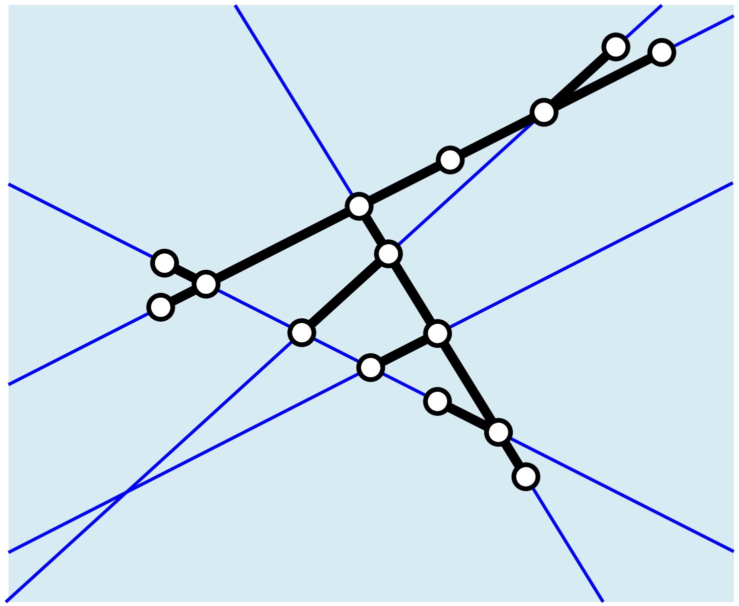

Do $O(\sqrt{n})$ lines suffice – if they can be arbitrary?

- $O(n \log\log n)$ area
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